

ADAMS OPERATIONS ON MONOMIAL RESOLUTIONS

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1. INTRODUCTION

In this note I give explicitly a monomial resolution for the unique complex 2-dimensional irreducible of D_8 and Q_8 . I observed that one could apply the Adams operations (in my examples here it is ψ^2) to morphisms. I wondered whether doing this gives a monomial chain complex. The answer to this is no in general but in my examples I rescaled the Adams operation sequence to give a chain complex. This chain complex will have an Euler characteristic which is the representation $\psi^2(\rho)$ - since this is a property of the Explicit Brauer Induction formulae. When we are in characteristic zero the operation $(-)^{(H,\lambda)}$ is exact so the Euler characteristic of the rescaled Adams operation complex has each $(-)^{(H,\lambda)}$ Euler characteristic equal to that of $\psi^2(\rho)$. For D_8 I completely checked this out and for Q_8 partially.

This raises questions:

Question 1.1. Is any of the above true when working with representations over a field of positive characteristic?

Question 1.2. Can I make, inductively using the restriction to subgroups and the Double Coset property, give explicitly models of the monomial resolutions for the 2-dimensional irreducibles of D_{2^n} and Q_{2^n} ?

2. MONOMIAL MORPHISMS

Write $[(K, \psi), g, (H, \phi)]$ for any triple consisting of $g \in G$, characters ϕ, ψ on subgroups $H, K \leq G$, respectively such that

$$(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

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which means that $K \leq g^{-1}Hg$ and that $\psi(k) = \phi(h)$ where $k = g^{-1}hg$ for $h \in H, k \in K$.

We have a well-defined left $k[G]$ -module homomorphism

$$[(K, \psi), g, (H, \phi)] : k[G] \otimes_{k[K]} k_\psi \longrightarrow k[G] \otimes_{k[H]} k_\phi$$

given by the formula $[(K, \psi), g, (H, \phi)](g' \otimes_{k[K]} v) = g'g^{-1} \otimes_{k[H]} v$. In the monomial category of G these two induced representations underlie $\underline{Ind}_K^G(k_\psi)$ and $\underline{Ind}_H^G(k_\phi)$ and

$$[(K, \psi), g, (H, \phi)] : \underline{Ind}_K^G(k_\psi) \longrightarrow \underline{Ind}_H^G(k_\phi)$$

is a typical monomial morphism between them.

Now suppose we raise each character to its n -th power to given $\underline{Ind}_K^G(k_{\psi^n})$ and $\underline{Ind}_H^G(k_{\phi^n})$ then we have a morphism

$$[(K, \psi^n), g, (H, \phi^n)] : \underline{Ind}_K^G(k_{\psi^n}) \longrightarrow \underline{Ind}_H^G(k_{\phi^n})$$

given by the same formula.

Therefore - although it requires a proof - changing each object and each morphism in a monomial resolution should at least give rise to a chain complex in the monomial category. Since the Adams operation $\psi^n(V)$ of a representation V is generally only a virtual representation we would not expect the above change to give a monomial resolution of anything.

Question 2.1. What are the results which do describe what happens?

3. A $\mathbb{C}[D_8]$ -MONOMIAL RESOLUTION

The dihedral group of order eight is given by

$$D_8 = \langle x, y \mid x^4 = 1 = y^2, yxy = x^3 \rangle.$$

The subgroups and the one-dimensional complex characters on them are

$$\begin{aligned} D_8 &: 1, \chi_1, \chi_2, \chi_1\chi_2 \\ &\quad \chi_1(x) = -1, \chi_1(y) = 1, \chi_2(x) = 1, \chi_2(y) = -1 \\ \langle x \rangle &: 1, \phi, \phi^2, \phi^3 \\ &\quad \phi(x) = i = \sqrt{-1} \\ \langle x^2, y \rangle &: 1, \tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_1\tilde{\chi}_2 \\ &\quad \tilde{\chi}_1(x^2) = -1, \tilde{\chi}_1(y) = 1, \tilde{\chi}_2(x^2) = 1, \tilde{\chi}_2(y) = -1 \\ \langle x^2, xy \rangle &: 1, \hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_1\hat{\chi}_2 \\ &\quad \hat{\chi}_1(x^2) = -1, \hat{\chi}_1(xy) = 1, \hat{\chi}_2(x^2) = 1, \hat{\chi}_2(xy) = -1 \\ \langle x^2 \rangle &: 1, \chi \\ \langle 1 \rangle &: 1 \end{aligned}$$

Now we shall construct a monomial resolution for $\rho = \text{Ind}_{\langle x \rangle}^{D_8}(\phi)$.

The conjugacy classes in D_8 are $1, x, x^2, y, xy$. The character values of ρ on these are respectively $2, 0, -2, 0, 0$. Therefore the non-zero $\rho^{(H,\lambda)}$'s are:

$$\begin{aligned}\rho^{(\langle x \rangle, \phi)} &= \phi, \quad \rho^{(\langle x \rangle, \phi^3)} = \phi^3 \\ \rho^{(\langle x^2, y \rangle, \tilde{\chi}_1)} &= \tilde{\chi}_1, \quad \rho^{(\langle x^2, y \rangle, \tilde{\chi}_1 \tilde{\chi}_2)} = \tilde{\chi}_1 \tilde{\chi}_2 \\ \rho^{(\langle x^2, xy \rangle, \hat{\chi}_1)} &= \hat{\chi}_1, \quad \rho^{(\langle x^2, xy \rangle, \hat{\chi}_1 \hat{\chi}_2)} = \hat{\chi}_1 \hat{\chi}_2 \\ \rho^{(\langle x^2 \rangle, \chi)} &= 2\chi \\ \rho^{(\langle 1 \rangle, 1)} &= 2.\end{aligned}$$

Set $G = D_8$. Let $\underline{\text{Ind}}_H^G(\lambda)$ denote the G -line bundle given by $\text{Ind}_H^G(\lambda)$ with lines given by $g \otimes_H \mathbb{C}_\lambda$ as g runs through coset representatives of G/H . The stabiliser pair for $1 \otimes_H \mathbb{C}_\lambda$ is (H, λ) and for $g \otimes_H \mathbb{C}_\lambda$ it is $(gHg^{-1}, (g^{-1})^*(\lambda))$.

We have the identity map

$$\underline{\text{Ind}}_{\langle x \rangle}^G(\phi) \longrightarrow \rho$$

which yields isomorphisms

$$\underline{\text{Ind}}_{\langle x \rangle}^G(\phi)^{(\langle \langle x \rangle, \phi \rangle)} \longrightarrow \rho^{(\langle x \rangle, \phi)}$$

and

$$\underline{\text{Ind}}_{\langle x \rangle}^G(\phi)^{(\langle \langle x \rangle, \phi^3 \rangle)} \longrightarrow \rho^{(\langle x \rangle, \phi^3)},$$

these two isomorphisms being conjugate by y .

Similarly there are isomorphisms

$$\underline{\text{Ind}}_{\langle x^2, y \rangle}^G(\tilde{\chi}_1) \longrightarrow \rho$$

and

$$\underline{\text{Ind}}_{\langle x^2, xy \rangle}^G(\hat{\chi}_1) \longrightarrow \rho$$

which induce isomorphisms

$$\underline{\text{Ind}}_{\langle x^2, y \rangle}^G(\tilde{\chi}_1)^{(\langle \langle x^2, y \rangle, \lambda \rangle)} \longrightarrow \rho^{(\langle x^2, y \rangle, \lambda)}$$

and

$$\underline{\text{Ind}}_{\langle x^2, xy \rangle}^G(\hat{\chi}_1)^{(\langle \langle x^2, xy \rangle, \lambda \rangle)} \longrightarrow \rho^{(\langle x^2, xy \rangle, \lambda)}$$

for all λ 's.

Therefore we have a monomial surjection

$$P_0 = \underline{\text{Ind}}_{\langle x \rangle}^G(\phi) \oplus \underline{\text{Ind}}_{\langle x^2, y \rangle}^G(\tilde{\chi}_1) \oplus \underline{\text{Ind}}_{\langle x^2, xy \rangle}^G(\hat{\chi}_1) \longrightarrow \rho$$

in which the six lines on the left are

$$1 \otimes_{\langle x \rangle} 1, \quad y \otimes_{\langle x \rangle} 1, \quad 1 \otimes_{\langle x^2, y \rangle} 1, \quad x \otimes_{\langle x^2, y \rangle} 1, \quad 1 \otimes_{\langle x^2, xy \rangle} 1, \quad x \otimes_{\langle x^2, xy \rangle} 1$$

with stabilising pairs

$$(\langle x \rangle, \phi), (\langle x \rangle, \phi^3), (\langle x^2, y \rangle, \tilde{\chi}_1), (\langle x^2, y \rangle, \tilde{\chi}_1 \tilde{\chi}_2), (\langle x^2, xy \rangle, \hat{\chi}_1), (\langle x^2, xy \rangle, \hat{\chi}_1 \hat{\chi}_2)$$

respectively. Every one of these lines belongs to $P_0^{(\langle \langle x^2 \rangle, \chi \rangle)}$ and $P_0^{(\langle \langle 1 \rangle, 1 \rangle)}$ so $P_0 \longrightarrow \rho$ is a monomial surjection of a D_8 -Line Bundle onto a D_8 -representation.

Next consider the kernel. If

$a_1 \otimes_{\langle x \rangle} 1 + a_2 y \otimes_{\langle x \rangle} 1 + a_3 \otimes_{\langle x^2, y \rangle} 1 + a_4 x \otimes_{\langle x^2, y \rangle} 1 + a_5 \otimes_{\langle x^2, xy \rangle} 1 + a_6 x \otimes_{\langle x^2, xy \rangle} 1$
maps to zero we have

$$\begin{aligned} 0 &= a_1 \otimes_{\langle x \rangle} 1 + a_2 y \otimes_{\langle x \rangle} 1 \\ &+ a_3(1 \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1) + a_4(i \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1) \\ &+ a_5(1 \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1) + a_6(i \otimes_{\langle x \rangle} 1 - y \otimes_{\langle x \rangle} 1) \end{aligned}$$

so that we must have

$$0 = a_1 + a_3 + ia_4 + a_5 + ia_6 \text{ and } 0 = a_2 + a_3 - ia_4 - ia_5 - a_6.$$

Setting $a_3 = 1, a_4 = 0 = a_5 = a_6, a_4 = 1, a_3 = 0 = a_5 = a_6, a_5 = 1, a_3 = 0 = a_4 = a_6$ and $a_6 = 1, a_3 = 0 = a_4 = a_5$ in turn we obtain a basis for the kernel

$$\begin{aligned} e_1 &= -1 \otimes_{\langle x \rangle} 1 - y \otimes_{\langle x \rangle} 1 + 1 \otimes_{\langle x^2, y \rangle} 1 \\ e_2 &= -i \otimes_{\langle x \rangle} 1 + iy \otimes_{\langle x \rangle} 1 + x \otimes_{\langle x^2, y \rangle} 1 \\ e_3 &= -1 \otimes_{\langle x \rangle} 1 + iy \otimes_{\langle x \rangle} 1 + 1 \otimes_{\langle x^2, xy \rangle} 1 \\ e_4 &= -i \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 + x \otimes_{\langle x^2, xy \rangle} 1. \end{aligned}$$

These are mapped respectively by x to

$$\begin{aligned} e_2 &= -i \otimes_{\langle x \rangle} 1 + iy \otimes_{\langle x \rangle} 1 + x \otimes_{\langle x^2, y \rangle} 1 \\ -e_1 &= 1 \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 - 1 \otimes_{\langle x^2, y \rangle} 1 \\ e_4 &= -i \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 + x \otimes_{\langle x^2, xy \rangle} 1 \\ -e_3 &= 1 \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1 - 1 \otimes_{\langle x^2, xy \rangle} 1. \end{aligned}$$

and by y to

$$\begin{aligned} e_1 &= -y \otimes_{\langle x \rangle} 1 - 1 \otimes_{\langle x \rangle} 1 + 1 \otimes_{\langle x^2, y \rangle} 1 \\ -e_2 &= -iy \otimes_{\langle x \rangle} 1 + i \otimes_{\langle x \rangle} 1 - x \otimes_{\langle x^2, y \rangle} 1 \\ -e_4 &= -y \otimes_{\langle x \rangle} 1 + i \otimes_{\langle x \rangle} 1 - x \otimes_{\langle x^2, xy \rangle} 1 \\ -e_3 &= -iy \otimes_{\langle x \rangle} 1 + 1 \otimes_{\langle x \rangle} 1 - 1 \otimes_{\langle x^2, xy \rangle} 1. \end{aligned}$$

Now $e_1 \in P_0^{((x), \phi)} \oplus P_0^{((x), \phi^3)} \oplus P_0^{((x^2, y), \bar{\chi}_1)} \subset P_0^{((x^2, \chi))}$ and similarly $e_3 \in P_0^{((x^2, \chi))}$. Therefore we have a map

$$P_1 = \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(\chi) \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(\chi) \longrightarrow \text{Ker}(\epsilon)$$

given by $(i \otimes_{\langle x^2 \rangle} 1, 0) \mapsto e_1$ and $(0, i \otimes_{\langle x^2 \rangle} 1) \mapsto e_3$.

The kernel of this map has basis

$$\begin{aligned} &((1 - y) \otimes_{\langle x^2 \rangle} 1, 0), ((x - xy) \otimes_{\langle x^2 \rangle} 1, 0), \\ &(0, (1 - xy) \otimes_{\langle x^2 \rangle} 1), (0, (x + y) \otimes_{\langle x^2 \rangle} 1). \end{aligned}$$

One finds that the group action is given by

$$\begin{aligned} y((1 - y) \otimes_{\langle x^2 \rangle} 1, (1 - xy) \otimes_{\langle x^2 \rangle} 1) &= (-(1 - y) \otimes_{\langle x^2 \rangle} 1, ((x + y) \otimes_{\langle x^2 \rangle} 1)) \\ x((1 - y) \otimes_{\langle x^2 \rangle} 1, (1 - xy) \otimes_{\langle x^2 \rangle} 1) &= ((x - xy) \otimes_{\langle x^2 \rangle} 1, (x + y) \otimes_{\langle x^2 \rangle} 1) \\ xy((1 - y) \otimes_{\langle x^2 \rangle} 1, (1 - xy) \otimes_{\langle x^2 \rangle} 1) &= (-(x - xy) \otimes_{\langle x^2 \rangle} 1, -(1 - xy) \otimes_{\langle x^2 \rangle} 1). \end{aligned}$$

Therefore if we set

$$w = ((1 - y) \otimes_{\langle x^2 \rangle} 1, (1 - xy) \otimes_{\langle x^2 \rangle} 1)$$

we obtain an isomorphism

$$P_2 = \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(\chi) \longrightarrow \text{Ker}(P_1 \rightarrow P_0)$$

given by $1 \otimes_{\langle x^2 \rangle} 1 \mapsto w$. The resulting chain complex

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \rho \longrightarrow 0$$

is a $\mathbb{C}[D_8]$ -monomial resolution of ρ .

Consider the related unaugmented $\mathbb{C}[D_8]$ -monomial chain complex

$$\begin{aligned} \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(\chi) &\longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(\chi) \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(\chi) \\ &\longrightarrow \underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi) \oplus \underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8}(\tilde{\chi}_1) \oplus \underline{\text{Ind}}_{\langle x^2, xy \rangle}^{D_8}(\hat{\chi}_1). \end{aligned}$$

Applying the ‘‘Adams operation’’ ψ^2 to this complex gives

$$\begin{aligned} \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1) &\longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1) \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1) \\ &\longrightarrow \underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi^2) \oplus \underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8}(1) \oplus \underline{\text{Ind}}_{\langle x^2, xy \rangle}^{D_8}(1). \end{aligned}$$

Let us identify the $\mathbb{C}[D_8]$ -monomial morphisms in this sequence to determine whether (a) this is a $\mathbb{C}[D_8]$ -monomial chain complex and (b) if so what is the homology.

The left-hand morphism is given by

$$1 \otimes_{\langle x^2 \rangle} 1 \mapsto ((1 - y) \otimes_{\langle x^2 \rangle} 1, (1 - xy) \otimes_{\langle x^2 \rangle} 1)$$

which, in the earlier notation equals

$$\begin{aligned} &[(\langle x^2 \rangle, \chi), 1, (\langle x^2 \rangle, \chi)] - [(\langle x^2 \rangle, \chi), y, (\langle x^2 \rangle, \chi)], \\ &[(\langle x^2 \rangle, \chi), 1, (\langle x^2 \rangle, \chi)] - [(\langle x^2 \rangle, \chi), xy, (\langle x^2 \rangle, \chi)]. \end{aligned}$$

which becomes, under the “Adams operation”,

$$\begin{aligned} & [(\langle x^2 \rangle, 1), 1, (\langle x^2 \rangle, 1)] - [(\langle x^2 \rangle, 1), y, (\langle x^2 \rangle, 1)], \\ & [(\langle x^2 \rangle, 1), 1, (\langle x^2 \rangle, 1)] - [(\langle x^2 \rangle, 1), xy, (\langle x^2 \rangle, 1)] \end{aligned}$$

or, written otherwise, is

$$1 \otimes_{\langle x^2 \rangle} 1 \mapsto ((1 - y) \otimes_{\langle x^2 \rangle} 1, (1 - xy) \otimes_{\langle x^2 \rangle} 1).$$

The right-hand map has the form is given by

$$(1 \otimes_{\langle x^2 \rangle} 1, 0) \mapsto (-i) \cdot e_1 = iy \otimes_{\langle x \rangle} 1 + i \otimes_{\langle x \rangle} 1 - i \otimes_{\langle x^2, y \rangle} 1$$

which (on the left summand) is the map

$$i[(\langle x^2 \rangle, \chi), 1, (\langle x \rangle, \phi)] + i[(\langle x^2 \rangle, \chi), y, (\langle x \rangle, \phi)] - i[(\langle x^2 \rangle, \chi), 1, (\langle x^2, y \rangle, \tilde{\chi}_1)]$$

which becomes, under the “Adams operation”,

$$i[(\langle x^2 \rangle, 1), 1, (\langle x \rangle, \phi^2)] + i[(\langle x^2 \rangle, 1), y, (\langle x \rangle, \phi^2)] - i[(\langle x^2 \rangle, 1), 1, (\langle x^2, y \rangle, 1)]$$

or, written otherwise, is

$$(1 \otimes_{\langle x^2 \rangle} 1, 0) \mapsto iy \otimes_{\langle x \rangle} 1 + i \otimes_{\langle x \rangle} 1 - i \otimes_{\langle x^2, y \rangle} 1.$$

The right-hand map on the right summand has the form

$$(0, 1 \otimes_{\langle x^2 \rangle} 1) \mapsto (-i) \cdot e_3 = i \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 - i \otimes_{\langle x^2, xy \rangle} 1$$

which is the map

$$i[(\langle x^2 \rangle, \chi), 1, (\langle x \rangle, \phi)] + [(\langle x^2 \rangle, \chi), y, (\langle x \rangle, \phi)] - i[(\langle x^2 \rangle, \chi), 1, (\langle x^2, xy \rangle, \hat{\chi}_1)]$$

which becomes, under the “Adams operation”,

$$i[(\langle x^2 \rangle, i), 1, ((\langle x \rangle, \phi^2))] + [(\langle x^2 \rangle, 1), y, (\langle x \rangle, \phi^2)] - i[(\langle x^2 \rangle, 1), 1, (\langle x^2, xy \rangle, 1)]$$

or, written otherwise, is

$$(0, 1 \otimes_{\langle x^2 \rangle} 1) \mapsto i \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 - i \otimes_{\langle x^2, xy \rangle} 1.$$

Question 3.1. Has the ψ^2 -process produced a chain complex in the above example?

The composition has the form

$$\begin{aligned}
1 \otimes_{\langle x^2 \rangle} 1 &\mapsto ((1 - y) \otimes_{\langle x^2 \rangle} 1, 0) + (0, (1 - xy) \otimes_{\langle x^2 \rangle} 1) \\
&\longrightarrow \\
&(1 - y)(iy \otimes_{\langle x \rangle} 1 + i \otimes_{\langle x \rangle} 1 - i \otimes_{\langle x^2, y \rangle} 1) \\
&+ (1 - xy)(i \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 - i \otimes_{\langle x^2, xy \rangle} 1) \\
&= (1 - y)(iy \otimes_{\langle x \rangle} 1 + i \otimes_{\langle x \rangle} 1) \\
&+ (1 - xy)(i \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1) \\
&= iy \otimes_{\langle x \rangle} 1 - i \otimes_{\langle x \rangle} 1 + i \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1 \\
&+ i \otimes_{\langle x \rangle} 1 - ixy \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 - x \otimes_{\langle x \rangle} 1 \\
&= i \otimes_{\langle x \rangle} 1 + iy \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 + 1 \otimes_{\langle x \rangle} 1
\end{aligned}$$

which is non-zero so directly applying the ‘‘Adams operation’’ does not give a chain complex.

Let us examine the effect of combining the ‘‘Adams operation’’ with some scalars on the right-hand map.

Under a ‘‘scaled Adams operation’’ the right-hand map on the left summand becomes

$$ai[(\langle x^2 \rangle, 1), 1, (\langle x \rangle, \phi^2)] + bi[(\langle x^2 \rangle, 1), y, (\langle x \rangle, \phi^2)] - ci[(\langle x^2 \rangle, \chi), 1, (\langle x^2, y \rangle, 1)]$$

and the right-hand map under a ‘‘scaled Adams operation’’ the right-hand map on the right summand becomes

$$di[(\langle x^2 \rangle, 1), 1, (\langle x \rangle, \phi^2)] + e[(\langle x^2 \rangle, 1), y, (\langle x \rangle, \phi^2)] - fi[(\langle x^2 \rangle, 1), 1, (\langle x^2, xy \rangle, 1)]$$

which, in the other notation, respectively become

$$(1 \otimes_{\langle x^2 \rangle} 1, 0) \mapsto aiy \otimes_{\langle x \rangle} 1 + bi \otimes_{\langle x \rangle} 1 - ci \otimes_{\langle x^2, y \rangle} 1$$

and

$$(0, 1 \otimes_{\langle x^2 \rangle} 1) \mapsto di \otimes_{\langle x \rangle} 1 + ey \otimes_{\langle x \rangle} 1 - fi \otimes_{\langle x^2, xy \rangle} 1.$$

Now the composition has the form

The composition has the form

$$\begin{aligned}
1 \otimes_{\langle x^2 \rangle} 1 &\mapsto ((1 - y) \otimes_{\langle x^2 \rangle} 1, 0) + (0, (1 - xy) \otimes_{\langle x^2 \rangle} 1) \\
&\longrightarrow \\
&(1 - y)(aiy \otimes_{\langle x \rangle} 1 + bi \otimes_{\langle x \rangle} 1 - ci \otimes_{\langle x^2, y \rangle} 1) \\
&+ (1 - xy)(di \otimes_{\langle x \rangle} 1 + ey \otimes_{\langle x \rangle} 1 - fi \otimes_{\langle x^2, xy \rangle} 1) \\
&= (1 - y)(aiy \otimes_{\langle x \rangle} 1 + bi \otimes_{\langle x \rangle} 1) \\
&+ (1 - xy)(di \otimes_{\langle x \rangle} 1 + ey \otimes_{\langle x \rangle} 1) \\
&= aiy \otimes_{\langle x \rangle} 1 - ai \otimes_{\langle x \rangle} 1 + bi \otimes_{\langle x \rangle} 1 - biy \otimes_{\langle x \rangle} 1 \\
&+ di \otimes_{\langle x \rangle} 1 - dixy \otimes_{\langle x \rangle} 1 + ey \otimes_{\langle x \rangle} 1 - ex \otimes_{\langle x \rangle} 1 \\
&= aiy \otimes_{\langle x \rangle} 1 - ai \otimes_{\langle x \rangle} 1 + bi \otimes_{\langle x \rangle} 1 - biy \otimes_{\langle x \rangle} 1 \\
&+ di \otimes_{\langle x \rangle} 1 + diy \otimes_{\langle x \rangle} 1 + ey \otimes_{\langle x \rangle} 1 + e \otimes_{\langle x \rangle} 1 \\
&= (-ai + bi + di + e) \otimes_{\langle x \rangle} 1 + (ai - bi + di + e)y \otimes_{\langle x \rangle} 1.
\end{aligned}$$

Hence, for example, $a = b = e = 1$ and $d = i$ gives a chain complex of $\mathbb{Z}[D_8]$ -monomial morphisms.

Now we know that the Euler characteristic of the complex, as a virtual representation, is $\psi^2(\rho)$. This is the same as the Euler characteristic of the homology groups. However, to be completely thorough, I shall compute the homology groups in the monomial sense (i.e. applying the superfix (H, μ) to each Line bundle and examining the homology of the resulting complex).

The ‘‘scaled Adams operation’’ monomial chain complex has the form

$$\begin{aligned}
\underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1) &\longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1) \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1) \\
&\longrightarrow \underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi^2) \oplus \underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8}(1) \oplus \underline{\text{Ind}}_{\langle x^2, xy \rangle}^{D_8}(1).
\end{aligned}$$

The stabiliser pairs of Lines in this complex are $(\langle x^2 \rangle, 1)$, $(\langle x \rangle, \phi^2)$, $(\langle x^2, y \rangle, 1)$ and $(\langle x^2, xy \rangle, 1)$.

The complex of vector spaces

$$\begin{aligned}
\underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle x^2 \rangle, 1))} &\longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle x^2 \rangle, 1))} \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle x^2 \rangle, 1))} \\
&\longrightarrow \underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi^2)^{((\langle x^2 \rangle, 1))} \oplus \underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8}(1)^{((\langle x^2 \rangle, 1))} \oplus \underline{\text{Ind}}_{\langle x^2, xy \rangle}^{D_8}(1)^{((\langle x^2 \rangle, 1))}
\end{aligned}$$

is just the complex

$$\begin{aligned} \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1) &\longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1) \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1) \\ &\longrightarrow \underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi^2) \oplus \underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8}(1) \oplus \underline{\text{Ind}}_{\langle x^2, xy \rangle}^{D_8}(1). \end{aligned}$$

whose degree two homology consists of those elements

$$a \otimes_{\langle x^2 \rangle} 1 + bx \otimes_{\langle x^2 \rangle} 1 + cy \otimes_{\langle x^2 \rangle} 1 + dxy \otimes_{\langle x^2 \rangle} 1$$

such that

$$0 = (1 - y)(a \otimes_{\langle x^2 \rangle} 1 + bx \otimes_{\langle x^2 \rangle} 1 + cy \otimes_{\langle x^2 \rangle} 1 + dxy \otimes_{\langle x^2 \rangle} 1)$$

which implies that $a = c$, $b = d$ and, in addition

$$0 = (1 - xy)(a \otimes_{\langle x^2 \rangle} 1 + bx \otimes_{\langle x^2 \rangle} 1 + cy \otimes_{\langle x^2 \rangle} 1 + dxy \otimes_{\langle x^2 \rangle} 1)$$

which implies $a = d$, $b = c$ so the second homology is

$$\mathbb{C} = \langle (1 + y + x + xy) \otimes_{\langle x^2 \rangle} 1 \rangle.$$

The kernel in degree one consists of elements

$$(a + by + cx + dxy) \otimes_{\langle x^2 \rangle} 1, (\alpha + \beta y + \gamma x + \delta xy) \otimes_{\langle x^2 \rangle} 1)$$

such that

$$\begin{aligned} 0 &= (a + by + cx + dxy)(iy \otimes_{\langle x \rangle} 1 + i \otimes_{\langle x \rangle} 1 - i \otimes_{\langle x^2, y \rangle} 1) \\ &\quad + (\alpha + \beta y + \gamma x + \delta xy)(-1 \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 - i \otimes_{\langle x^2, xy \rangle} 1) \end{aligned}$$

Therefore $0 = a + b = c + d = \alpha + \delta = \beta + \gamma$ and

$$\begin{aligned} 0 &= (a - ay + cx - cxy)(iy \otimes_{\langle x \rangle} 1 + i \otimes_{\langle x \rangle} 1) \\ &\quad + (\alpha + \beta y - \beta x - \alpha xy)(-1 \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1) \\ &= (a - ay + cx - cxy)(iy \otimes_{\langle x \rangle} 1) \\ &\quad + (a - ay + cx - cxy)(i \otimes_{\langle x \rangle} 1) \\ &\quad - (\alpha + \beta y - \beta x - \alpha xy)(1 \otimes_{\langle x \rangle} 1) \\ &\quad + (\alpha + \beta y - \beta x - \alpha xy)(y \otimes_{\langle x \rangle} 1) \\ &= (aiy - ai - icy + ci)(1 \otimes_{\langle x \rangle} 1) \\ &\quad + (ia - iay - ic + icy)(1 \otimes_{\langle x \rangle} 1) \\ &\quad - (\alpha + \beta y + \beta + \alpha y)(1 \otimes_{\langle x \rangle} 1) \\ &\quad + (\alpha y + \beta + \beta y + \alpha)(1 \otimes_{\langle x \rangle} 1) \end{aligned}$$

so that $a = c = -b = -d$ and $\alpha = -\beta = \gamma = -\delta$. Therefore the kernel consists of elements of the form

$$\begin{aligned} & (a(1 - y + x - xy) \otimes_{\langle x^2 \rangle} 1, \alpha(1 - y + x - xy) \otimes_{\langle x^2 \rangle} 1) \\ & = (a(1 + x)(1 - y) \otimes_{\langle x^2 \rangle} 1, \alpha(1 + x)(1 - xy) \otimes_{\langle x^2 \rangle} 1) \end{aligned}$$

One may verify that $\psi^2(\rho) = 1 + \chi_1 - \chi_2 + \chi_1\chi_2$. Therefore

$$H_2 = 1, H_1 = \chi_2, H_0 = \chi_1 + \chi_1\chi_2$$

since the homology Euler characteristic is $\psi^2(\rho)$.

Now consider

$$\begin{aligned} \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle x \rangle, \mu))} & \longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle x \rangle, \mu))} \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle x \rangle, \mu))} \\ & \longrightarrow \end{aligned}$$

$$\underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi^2)^{D_8}(1)^{((\langle x \rangle, \mu))} \oplus \underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8}(1)^{((\langle x \rangle, \mu))} \oplus \underline{\text{Ind}}_{\langle x^2, xy \rangle}^{D_8}(1)^{((\langle x \rangle, \mu))}$$

which is just

$$\underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi^2)^{D_8}(1)^{((\langle x \rangle, \mu))}$$

in degree zero. The lines in here are $1 \otimes_{\langle x \rangle} 1$ and $y \otimes_{\langle x \rangle} 1$ which are acted upon by x as follows

$$1 \otimes_{\langle x \rangle} 1 \mapsto (-1) \otimes_{\langle x \rangle} 1 \text{ and } y \otimes_{\langle x \rangle} 1 \mapsto (-1)y \otimes_{\langle x \rangle} 1.$$

Hence

$$\underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi^2)^{D_8}(1)^{((\langle x \rangle, \mu))}$$

is zero unless $\mu = \phi^2$ in which case it is $2\phi^2$. On the other hand $\psi^2(\rho)^{(\langle x \rangle, \mu)}$ is also zero unless $\mu = \phi^2$ in which case it also equals $2\phi^2$.

Next consider

$$\begin{aligned} \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle y \rangle, \mu))} & \longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle y \rangle, \mu))} \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle y \rangle, \mu))} \\ & \longrightarrow \end{aligned}$$

$$\underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi^2)^{D_8}(1)^{((\langle y \rangle, \mu))} \oplus \underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8}(1)^{((\langle y \rangle, \mu))} \oplus \underline{\text{Ind}}_{\langle x^2, xy \rangle}^{D_8}(1)^{((\langle y \rangle, \mu))}$$

which is just

$$\underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8}(1)^{((\langle y \rangle, \mu))}$$

in degree zero, which is zero unless $\mu(y) = 1$, when it is 2-dimensional. Once again this agrees with the y -eigenspaces of $\psi^2(\rho)$.

Finally we examine

$$\begin{aligned} \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{((\langle x^2, y \rangle, \mu))} & \longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{D_8}(1)^{((\langle x^2, y \rangle, \mu))} \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{D_8}(1)^{D_8}(1)^{((\langle x^2, y \rangle, \mu))} \\ & \longrightarrow \end{aligned}$$

$$\underline{\text{Ind}}_{\langle x \rangle}^{D_8}(\phi^2)^{D_8}(1)^{((\langle x^2, y \rangle, \mu))} \oplus \underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8}(1)^{D_8}(1)^{((\langle x^2, y \rangle, \mu))} \oplus \underline{\text{Ind}}_{\langle x^2, xy \rangle}^{D_8}(1)^{D_8}(1)^{((\langle x^2, y \rangle, \mu))}$$

which is just

$$\underline{\text{Ind}}_{\langle x^2, y \rangle}^{D_8} (1)^{D_8} (1)^{((x^2, y), \mu)}$$

in degree zero. This is zero unless $\mu = 1$ in which case it is 2-dimensional which again coincides with the behaviour of the eigenspaces of $\psi^2(\rho)$.

4. A CALCULATION WITH Q_8

4.1. A $\mathbb{C}[Q_8]$ -monomial resolution

The quaternion group of order eight is given by

$$Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, yxy^{-1} = x^3 \rangle.$$

The subgroups and the one-dimensional characters on them are

$$\begin{aligned} Q_8 &: 1, \chi_1, \chi_2, \chi_1\chi_2 \\ &\quad \chi_1(x) = -1, \chi_1(y) = 1, \chi_2(x) = 1, \chi_2(y) = -1 \\ \langle x \rangle &: 1, \phi_x, \phi_x^2, \phi_x^3 \\ &\quad \phi_x(x) = i \\ \langle y \rangle &: 1, \phi_y, \phi_y^2, \phi_y^3 \\ &\quad \phi_y(y) = i \\ \langle xy \rangle &: 1, \phi_{xy}, \phi_{xy}^2, \phi_{xy}^3 \\ &\quad \phi_{xy}(xy) = i \\ \langle x^2 \rangle &: 1, \chi \\ \langle 1 \rangle &: 1 \end{aligned}$$

Now we shall try to construct a monomial resolution for $\rho = \text{Ind}_{\langle x \rangle}^{Q_8}(\phi_x)$, the unique two-dimensional irreducible. The conjugacy classes in Q_8 are $1, x, x^2, y, xy$. The character values of ρ on these are respectively $2, 0, -2, 0, 0$. This is because

$$x(1 \otimes_{\langle x \rangle} 1) = i \otimes_{\langle x \rangle} 1, x(y^3 \otimes_{\langle x \rangle} 1) = y^3 y x y^3 \otimes_{\langle x \rangle} 1 = -i y^3 \otimes_{\langle x \rangle} 1,$$

$$y(1 \otimes_{\langle x \rangle} 1) = y \otimes_{\langle x \rangle} 1 = y^3 x^2 \otimes_{\langle x \rangle} 1 = -y^3 \otimes_{\langle x \rangle} 1, y(y^3 \otimes_{\langle x \rangle} 1) = 1 \otimes_{\langle x \rangle} 1.$$

Therefore the non-zero $\rho^{(H, \lambda)}$'s are:

$$\begin{aligned} \rho^{(\langle x \rangle, \phi_x)} &= \phi_x, \quad \rho^{(\langle x \rangle, \phi_x^3)} = \phi_x^3 \\ \rho^{(\langle y \rangle, \phi_y)} &= \phi_y, \quad \rho^{(\langle x \rangle, \phi_y^3)} = \phi_y^3 \\ \rho^{(\langle xy \rangle, \phi_{xy})} &= \phi_{xy}, \quad \rho^{(\langle x \rangle, \phi_{xy}^3)} = \phi_{xy}^3 \\ \rho^{(\langle x^2 \rangle, \chi)} &= 2\chi \\ \rho^{(\langle 1 \rangle, 1)} &= 2. \end{aligned}$$

Set $G = Q_8$ pro tem. Let $\underline{\text{Ind}}_H^G(\lambda)$ denote the G -line bundle given by $\text{Ind}_H^G(\lambda)$ with lines given by $g \otimes_H \mathbb{C}(\lambda)$ as g runs through coset representatives of G/H . The stabiliser pair for $1 \otimes_H \mathbb{C}(\lambda)$ is (H, λ) and for $g \otimes_H \mathbb{C}(\lambda)$ it is $(gHg^{-1}, (g^{-1})^*(\lambda))$.

We have the identity map

$$\underline{\text{Ind}}_{\langle x \rangle}^G(\phi_x) \longrightarrow \rho$$

which yields isomorphisms

$$\underline{\text{Ind}}_{\langle x \rangle}^G(\phi_x)^{(\langle x \rangle, \phi_x)} \longrightarrow \rho^{\langle x \rangle, \phi_x}$$

and

$$\underline{\text{Ind}}_{\langle x \rangle}^G(\phi_x)^{(\langle x \rangle, \phi_x^3)} \longrightarrow \rho^{\langle x \rangle, \phi_x^3},$$

these two isomorphisms being conjugate by y .

Similarly there are isomorphisms

$$\underline{\text{Ind}}_{\langle y \rangle}^G(\phi_y) \longrightarrow \rho$$

and

$$\underline{\text{Ind}}_{\langle xy \rangle}^G(\phi_{xy}) \longrightarrow \rho$$

which induce isomorphisms

$$\underline{\text{Ind}}_{\langle y \rangle}^G(\phi_y)^{(\langle y \rangle, \phi_y)} \longrightarrow \rho^{\langle y \rangle, \phi_y},$$

$$\underline{\text{Ind}}_{\langle y \rangle}^G(\phi_y)^{(\langle y \rangle, \phi_y^3)} \longrightarrow \rho^{\langle y \rangle, \phi_y^3},$$

$$\underline{\text{Ind}}_{\langle xy \rangle}^G(\phi_{xy})^{(\langle xy \rangle, \phi_{xy})} \longrightarrow \rho^{\langle xy \rangle, \phi_{xy}},$$

$$\underline{\text{Ind}}_{\langle xy \rangle}^G(\phi_{xy})^{(\langle xy \rangle, \phi_{xy}^3)} \longrightarrow \rho^{\langle xy \rangle, \phi_{xy}^3}.$$

Therefore we have a surjection

$$P_0 = \underline{\text{Ind}}_{\langle x \rangle}^G(\phi_x) \oplus \underline{\text{Ind}}_{\langle y \rangle}^G(\phi_y) \oplus \underline{\text{Ind}}_{\langle xy \rangle}^G(\phi_{xy}) \longrightarrow \rho$$

such that

$$P_0 = P_0^{(\langle x^2 \rangle, \chi)} \longrightarrow \rho^{\langle x^2 \rangle, \chi}$$

is surjective and similarly for $P_0^{(\langle (1,1) \rangle)}$.

The six lines on the left are generated by

$$1 \otimes_{\langle x \rangle} 1, \quad y \otimes_{\langle x \rangle} 1, \quad 1 \otimes_{\langle y \rangle} 1, \quad x \otimes_{\langle y \rangle} 1, \quad 1 \otimes_{\langle xy \rangle} 1, \quad x \otimes_{\langle xy \rangle} 1$$

with stabilising pairs

$$(\langle x \rangle, \phi_x), (\langle x \rangle, \phi_x^3), (\langle y \rangle, \phi_y), (\langle y \rangle, \phi_y^3), (\langle xy \rangle, \phi_{xy}), (\langle xy \rangle, \phi_{xy}^3)$$

respectively. Every one of these lines belongs to $P_0^{(\langle x^2 \rangle, \chi)}$ and $P_0^{(\langle (1,1) \rangle)}$ so $P_0 \longrightarrow \rho$ is a monomial surjection of a Q_8 -Line Bundle onto a Q_8 -representation.

Next consider the kernel. If

$$a_1 \otimes_{\langle x \rangle} 1 + a_2 y \otimes_{\langle x \rangle} 1 + a_3 \otimes_{\langle y \rangle} 1 + a_4 x \otimes_{\langle y \rangle} 1 + a_5 \otimes_{\langle xy \rangle} 1 + a_6 x \otimes_{\langle xy \rangle} 1$$

maps to zero we have

$$\begin{aligned} 0 &= a_1 \otimes_{\langle x \rangle} 1 + a_2 y \otimes_{\langle x \rangle} 1 \\ &+ a_3(1 \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1) + a_4(i \otimes_{\langle x \rangle} 1 - y \otimes_{\langle x \rangle} 1) \\ &+ a_5(1 \otimes_{\langle x \rangle} 1 - y \otimes_{\langle x \rangle} 1) + a_6(i \otimes_{\langle x \rangle} 1 + iy \otimes_{\langle x \rangle} 1) \end{aligned}$$

so that we must have

$$0 = a_1 + a_3 + ia_4 + a_5 + ia_6 \text{ and } 0 = a_2 - ia_3 - a_4 - a_5 + ia_6.$$

Setting $a_3 = 1, a_4 = 0 = a_5 = a_6, a_4 = 1, a_3 = 0 = a_5 = a_6, a_5 = 1, a_3 = 0 = a_4 = a_6$ and $a_6 = 1, a_3 = 0 = a_4 = a_5$ in turn we obtain a basis for the kernel

$$e_1 = -1 \otimes_{\langle x \rangle} 1 + iy \otimes_{\langle x \rangle} 1 + 1 \otimes_{\langle y \rangle} 1$$

$$e_2 = -i \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 + x \otimes_{\langle y \rangle} 1$$

$$e_3 = -1 \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 + 1 \otimes_{\langle xy \rangle} 1$$

$$e_4 = -i \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1 + x \otimes_{\langle xy \rangle} 1.$$

These are mapped respectively by x to

$$e_2 = -i \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 + x \otimes_{\langle xy \rangle} 1$$

$$-e_1 = 1 \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1 - 1 \otimes_{\langle y \rangle} 1$$

$$e_4 = -i \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1 + x \otimes_{\langle xy \rangle} 1$$

$$-e_3 = 1 \otimes_{\langle x \rangle} 1 - y \otimes_{\langle x \rangle} 1 - 1 \otimes_{\langle xy \rangle} 1.$$

and by y to

$$ie_1 = -i \otimes_{\langle x \rangle} 1 - y \otimes_{\langle x \rangle} 1 + i \otimes_{\langle y \rangle} 1$$

$$-ie_2 = -1 \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1 - ix \otimes_{\langle y \rangle} 1$$

$$-ie_4 = -1 \otimes_{\langle x \rangle} 1 - y \otimes_{\langle x \rangle} 1 - ix \otimes_{\langle xy \rangle} 1$$

$$-ie_3 = i \otimes_{\langle x \rangle} 1 - iy \otimes_{\langle x \rangle} 1 - i \otimes_{\langle xy \rangle} 1.$$

Now $e_1 \in P_0^{((\langle x \rangle, \phi_x))} \oplus P_0^{((\langle x \rangle, \phi_x^3))} \oplus P_0^{((\langle y \rangle, \phi_y))} \subset P_0^{((\langle x^2, \chi))}$ and similarly $e_3 \in P_0^{((\langle x^2, \chi))}$. Therefore we have a map

$$P_1 = \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(\chi) \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(\chi) \longrightarrow \text{Ker}(\epsilon)$$

given by $(1 \otimes_{\langle x^2 \rangle} 1, 0) \mapsto e_1$ and $(0, 1 \otimes_{\langle x^2 \rangle} 1) \mapsto e_3$.

The kernel of this map has basis

$$((1 + iy) \otimes_{\langle x^2 \rangle} 1, 0), ((x + ixy) \otimes_{\langle x^2 \rangle} 1, 0),$$

$$(0, (1 + ixy) \otimes_{\langle x^2 \rangle} 1), (0, (x - iy) \otimes_{\langle x^2 \rangle} 1).$$

One finds that the group action is given by

$$\begin{aligned} y((1 + iy) \otimes_{\langle x^2 \rangle} 1, (1 + ixy) \otimes_{\langle x^2 \rangle} 1) &= -(i - y) \otimes_{\langle x^2 \rangle} 1, ((ix + 1) \otimes_{\langle x^2 \rangle} 1) \\ x((1 + iy) \otimes_{\langle x^2 \rangle} 1, (1 + ixy) \otimes_{\langle x^2 \rangle} 1) &= ((x + ixy) \otimes_{\langle x^2 \rangle} 1, (x - iy) \otimes_{\langle x^2 \rangle} 1) \\ xy((1 + iy) \otimes_{\langle x^2 \rangle} 1, (1 + ixy) \otimes_{\langle x^2 \rangle} 1) &= -(ix - xy) \otimes_{\langle x^2 \rangle} 1, (x - i) \otimes_{\langle x^2 \rangle} 1. \end{aligned}$$

Therefore if we set

$$w = ((1 + iy) \otimes_{\langle x^2 \rangle} 1, (1 + ixy) \otimes_{\langle x^2 \rangle} 1)$$

we obtain an isomorphism

$$P_2 = \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(\chi) \longrightarrow \text{Ker}(P_1 \rightarrow P_0)$$

given by $1 \otimes_{\langle x^2 \rangle} 1 \mapsto w$. The resulting chain complex

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \rho \longrightarrow 0$$

is a monomial resolution of ρ .

Consider the related unaugmented $\mathbb{C}[Q_8]$ -monomial chain complex

$$\begin{aligned} \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(\chi) &\longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(\chi) \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(\chi) \\ &\longrightarrow \underline{\text{Ind}}_{\langle x \rangle}^{Q_8}(\phi_x) \oplus \underline{\text{Ind}}_{\langle y \rangle}^{Q_8}(\phi_y) \oplus \underline{\text{Ind}}_{\langle xy \rangle}^{Q_8}(\phi_{xy}) \end{aligned}$$

Applying the ‘‘Adams operation’’ ψ^2 to this chain complex yields

$$\begin{aligned} \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(1) &\longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(1) \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(1) \\ &\longrightarrow \underline{\text{Ind}}_{\langle x \rangle}^{Q_8}(\phi_x^2) \oplus \underline{\text{Ind}}_{\langle y \rangle}^{Q_8}(\phi_y^2) \oplus \underline{\text{Ind}}_{\langle xy \rangle}^{Q_8}(\phi_{xy}^2). \end{aligned}$$

where the left-hand summand of the right-hand map is given by

$$(1 \otimes_{\langle x^2 \rangle} 1, 0) \mapsto -1 \otimes_{\langle x \rangle} 1 + iy \otimes_{\langle x \rangle} 1 + 1 \otimes_{\langle y \rangle} 1$$

otherwise written as

$$-[(\langle x^2 \rangle, 1), 1, (\langle x \rangle, \phi_x^2)] + i[(\langle x^2 \rangle, 1), y, (\langle x \rangle, \phi_x^2)] + [(\langle x^2 \rangle, 1), 1, (\langle y \rangle, \phi_y^2)].$$

The right-hand summand of the right-hand map is given by

$$(0, 1 \otimes_{\langle x^2 \rangle} 1) \mapsto -1 \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1 + 1 \otimes_{\langle xy \rangle} 1$$

which is

$$-[(\langle x^2 \rangle, 1), 1, (\langle x \rangle, \phi_x^2)] + [(\langle x^2 \rangle, 1), y, (\langle x \rangle, \phi_x^2)] + [(\langle x^2 \rangle, 1), 1, (\langle y \rangle, \phi_{xy}^2)].$$

As part of the ‘‘rescaled Adams operation’’, we rescale these maps to

$$(1 \otimes_{\langle x^2 \rangle} 1, 0) \mapsto -a \otimes_{\langle x \rangle} 1 + iby \otimes_{\langle x \rangle} 1 + c \otimes_{\langle y \rangle} 1$$

and

$$(0, 1 \otimes_{\langle x^2 \rangle} 1) \mapsto -d \otimes_{\langle x \rangle} 1 + ey \otimes_{\langle x \rangle} 1 + f \otimes_{\langle xy \rangle} 1.$$

The left hand map is given by

$$1 \otimes_{\langle x^2 \rangle} 1 \mapsto ((1 + iy) \otimes_{\langle x^2 \rangle} 1, (1 + ixy) \otimes_{\langle x^2 \rangle} 1).$$

Therefore the necessary and sufficient condition for the rescale Adams operation to give a chain complex is, in $\underline{\text{Ind}}_{\langle x \rangle}^{Q_8}(\phi_x^2) \oplus \underline{\text{Ind}}_{\langle y \rangle}^{Q_8}(\phi_y^2) \oplus \underline{\text{Ind}}_{\langle xy \rangle}^{Q_8}(\phi_{xy}^2)$,

$$\begin{aligned} 0 &= (1 + iy)(-a \otimes_{\langle x \rangle} 1 + iby \otimes_{\langle x \rangle} 1 + c \otimes_{\langle y \rangle} 1) \\ &\quad + (1 + ixy)(-d \otimes_{\langle x \rangle} 1 + ey \otimes_{\langle x \rangle} 1 + f \otimes_{\langle xy \rangle} 1). \end{aligned}$$

From the middle summand we see that $(1 - i)c = 0$ so that $c = 0$ and from the third summands $(1 - i)f = 0$ so $f = 0$. It remains to fulfill the relation

$$\begin{aligned} 0 &= (1 + iy)(-a \otimes_{\langle x \rangle} 1 + iby \otimes_{\langle x \rangle} 1) \\ &\quad + (1 + ixy)(-d \otimes_{\langle x \rangle} 1 + ey \otimes_{\langle x \rangle} 1) \\ &= (-a - b - d - ie) \otimes_{\langle x \rangle} 1 \\ &\quad + (-ia + ib + id + e)y \otimes_{\langle x \rangle} 1 \end{aligned}$$

which means that $b + d = 0 = a + ie$ so we may set

$$a = -i, b = e = 1, d = -1.$$

To recapitulate, our “scaled Adams operation” complex has morphisms,

$$1 \otimes_{\langle x^2 \rangle} 1 \mapsto ((1 + iy) \otimes_{\langle x^2 \rangle} 1, (1 + ixy) \otimes_{\langle x^2 \rangle} 1)$$

and

$$(1 \otimes_{\langle x^2 \rangle} 1, 0) \mapsto i \otimes_{\langle x \rangle} 1 + iy \otimes_{\langle x \rangle} 1$$

and

$$(0, 1 \otimes_{\langle x^2 \rangle} 1) \mapsto 1 \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1$$

Now we turn to homology of the complex. The element

$$(a + bx + cy + dxy) \otimes_{\langle x^2 \rangle} 1$$

goes to zero in the left-hand summand of

$$\underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(1) \longrightarrow \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(1) \oplus \underline{\text{Ind}}_{\langle x^2 \rangle}^{Q_8}(1)$$

if and only if the following relation holds

$$\begin{aligned} 0 &= (a + bx + cy + dxy)(1 + iy) \otimes_{\langle x^2 \rangle} 1 \\ &= (a + bx + cy + dxy + iay + ibxy + ic + idx) \otimes_{\langle x^2 \rangle} 1 \end{aligned}$$

so $a = -ic$, $c = -ia$ so $a = -i(-i)a = -a$ so that $a = c = 0$ and also $b = -id$, $d = -ib$ so $d = b = 0$ and $H_2 = 0$.

Now consider the kernel of the (rescaled) right-hand map in the monomial complex, which consists of

$$((a + bx + cy + dxy) \otimes_{\langle x^2 \rangle} 1, (\alpha + \beta x + \gamma y + \delta xy) \otimes_{\langle x^2 \rangle} 1)$$

such that

$$0 = (a + bx + cy + dxy)e_1 + (\alpha + \beta x + \gamma y + \delta xy)e_3$$

in

$$\underline{\text{Ind}}_{\langle x \rangle}^{Q_8}(\phi_x^2) \oplus \underline{\text{Ind}}_{\langle y \rangle}^{Q_8}(\phi_y^2) \oplus \underline{\text{Ind}}_{\langle xy \rangle}^{Q_8}(\phi_{xy}^2).$$

Written out in full this means (after rescaling) that the following element is zero

$$\begin{aligned} & (a + bx + cy + dxy)(-1 \otimes_{\langle x \rangle} 1 + iy \otimes_{\langle x \rangle} 1) \\ & + (\alpha + \beta x + \gamma y + \delta xy)(-1 \otimes_{\langle x \rangle} 1 + y \otimes_{\langle x \rangle} 1) \\ & = (a + bx + cy + dxy)(-1 \otimes_{\langle x \rangle} 1) \\ & + (a + bx + cy + dxy)(iy \otimes_{\langle x \rangle} 1) \\ & + (\alpha + \beta x + \gamma y + \delta xy)(-1 \otimes_{\langle x \rangle} 1) \\ & + (\alpha + \beta x + \gamma y + \delta xy)(y \otimes_{\langle x \rangle} 1) \\ & = (-a + b - cy + dy)(1 \otimes_{\langle x \rangle} 1) \\ & + (ia y - ib y + ic - id)(1 \otimes_{\langle x \rangle} 1) \\ & + (-\alpha + \beta - \gamma y + \delta y)(1 \otimes_{\langle x \rangle} 1) \\ & + (\alpha y - \beta y + \gamma - \delta)(1 \otimes_{\langle x \rangle} 1) \end{aligned}$$

Therefore

$$-a + b + ic - id - \alpha + \beta + \gamma - \delta = 0$$

and

$$ia - ib - c + d - \gamma + \delta + \alpha - \beta = 0.$$

This would enable me to calculate H_1 and then H_0 would follow from the Euler characteristic, which equals $\psi^2(\rho)$.

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