A history of the Arf-Kervaire invariant problem

Victor Snaith

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The outcome: "mostly framed manifolds of Arf-Kervaire invariant one do not exist".

Here is what Lewis Carroll has to say about non-existent things!

"I know what you're thinking about," said Tweedledum; "but it isn't so, nohow." "Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't; it ain't. That's logic."

Through the Looking Glass by Lewis Carroll (aka Charles Lutwidge Dodgson)

Exotic Spheres

$$S^{m+n+1} = \partial(D^{m+1} \times D^{n+1})$$

$$= D^{m+1} \times S^n \bigcup S^m \times D^{n+1}$$

Given a self-diffeomorphism f of $S^m \times S^n$ we can form

$$M_f = D^{m+1} \times S^n \bigcup_f S^m \times D^{n+1}$$

which can be "rounded off" to be a smooth orientable differentiable manifold with the same homology as S^{m+n+1}.

Given differentiable maps

$$f_1 : S^m \longrightarrow SO(n+1) \; f_2 : S^n \longrightarrow SO(m+1)$$

$$f(x,y) = (f_2(f_1(x)(y))^{-1}(x), f_1(x)(y))$$

yields $M_f = M(f_1, f_2)$ - construction due to John Milnor (1956-1959).

Question $M(f_1, f_2)$ is homeo to S^{m+n+1}. Is it diffeomorphic to S^{m+n+1}?

Answer Mostly no - although for S^3 yes - for example S^{31} has more that 16×10^6 differentiable structures!

Milnor's method: If $m+n+1 = 4k-1$ there exists a $4k$ dimensional smooth manifold W with $\partial W = M(f_1, f_2)$. Applying Hirzebruch's Signature Theorem to W gives an invariant

$$\lambda(M(f_1, f_2) \in \mathbb{Q}/\mathbb{Z}$$

dependng only on the differentiable manifold $M(f_1, f_2)$.

The invariant works for any exotic $4k-1$-sphere.

Reversing orientation reverses the sign of $\lambda(M)$.

The invariant of a connected sum of two exotic spheres is the sum of the invariants.

There is a formula for $\lambda(M)$ from which one can estimate the size of the subgroup of $\lambda(M)$-values in $\mathbb{Q}/\mathbb{Z}$.

Stable homotopy groups of spheres

$\pi_r(X) =$ (based) homotopy classes of continuous maps $h : S^r \longrightarrow X$ such that $h(\text{North pole}) =$ base $-$ point.

$$\pi_r^S(S^0) = \lim_{\substack{\rightarrow \\ n}} (\ldots \pi_{r+n}(S^n) \longrightarrow \pi_{r+n+1}(S^{n+1}) \ldots).$$

There are 2 famous constructions which land in the abelian group $\pi_r^S(S^0)$.

The J-homomorphism

$$J : \pi_r(SO) \longrightarrow \pi_r^S(S^0)$$

sends $h : S^r \longrightarrow SO(n)$ to the adjoint of

$h : S^r \longrightarrow \text{Map}(S^{n-1}, S^{n-1})$ which is

$$J(h) : S^{r+n-1} \longrightarrow S^{n-1}.$$

By work (in the 1960's and 1970's) on J.F. Adams, M.F. Atiyah, M. Mahowald, D. Sullivan, D.G. Quillen et al the image of J is known.

The Pontrjagin-Thom construction

Framed manifolds and stable homotopy groups

Let M^n be a compact C$^\infty$ manifold without boundary and let $i : M^n \longrightarrow \mathbb{R}^{n+r}$ be an embedding.

The tangent bundle of $\mathbb{R}^{n+r}$, denoted by $\tau(\mathbb{R}^{n+r})$, may be identified with $\mathbb{R}^{n+r} \times \mathbb{R}^{n+r}$.

The normal bundle of i, denoted by $\nu(M, i)$, is the vector bundle whose fibre at $z \in M$ is the subspace of vectors $(i(z), x) \in \mathbb{R}^{n+r} \times \mathbb{R}^{n+r}$ such that x is orthogonal to $i_* \tau(M)_z$ where i_* is the induced embedding of tangent bundles - i.e. $\tau(M)$ into $\tau(\mathbb{R}^{n+r})$.

Let ξ be a vector bundle over a compact manifold M endowed with a Riemannian metric on the fibres. Then the Thom space is defined to be the quotient of the unit disc bundle $D(\xi)$ of ξ with the unit sphere bundle $S(\xi)$ collapsed to a point.

Hence

$$T(\xi) = \frac{D(\xi)}{S(\xi)}$$

is a compact topological space with a base-point given by the image of $S(\xi)$.

If M admits an embedding with a trivial normal bundle we say that M has a stably trivial normal bundle.

Write M_+ for the disjoint union of M and a disjoint base-point. Then there is a canonical homeomorphism

$$T(M \times \mathbb{R}^r) \cong \Sigma^r(M_+)$$

between the Thom space of the trivial r-dimensional vector bundle and the r-fold suspension of M_+, $(S^r \times (M_+))/(S^r \vee (M_+)) = S^r \wedge (M_+)$.

The Pontrjagin-Thom construction

Suppose that M^n is a manifold together with a choice of trivialisation of normal bundle $\nu(M, i)$.

This choice gives a choice of homeomorphism

$$T(\nu(M, i)) \cong \Sigma^r(M_+).$$

Such a homeomorhism is called a framing of (M, i).

Now consider the embedding $i : M^n \longrightarrow \mathbb{R}^{n+r}$ and identify the $(n+r)$-dimensional sphere S^{n+r} with the one-point compactification $\mathbb{R}^{n+r} \bigcup \{\infty\}$. The Pontrjagin-Thom construction is the map

$$S^{n+r} \longrightarrow T(\nu(M, i))$$

given by collapsing the complement of the interior of the unit disc bundle $D(\nu(M, i))$ to the point corresponding to $S(\nu(M, i))$ and by mapping each point of $D(\nu(M, i))$ to itself.

Identifying the r-dimensional sphere with the r-fold suspension $\Sigma^r S^0$ of the zero-dimensional sphere (i.e. two points, one the basepoint) the map which collapses M to the non-basepoint yields a basepoint preserving map $\Sigma^r(M_+) \longrightarrow S^r$.

Therefore, starting from a framed manifold M^n, the Pontrjagin-Thom construction yields a based map

$$S^{n+r} \longrightarrow T(\nu(M,i)) \cong \Sigma^r(M_+) \longrightarrow S^r,$$

whose homotopy class defines an element of $\pi_{n+r}(S^r)$.

To sum up:

$$PT : \{\mathsf{framed\ n-manifolds}\} \longrightarrow \pi_{\mathsf{n}}^{\mathsf{S}}(\mathsf{S}^0).$$

Lev Pontrjagin (1947) introduced this construction in order to use framed manifolds to study stable homotopy groups. Later René Thom (1954) developed a generalisation for the opposite reason, to calculate equivalence classes of manifolds by reducing to stable homotopy group calculations.

The Arf invariant of a quadratic form

Let V be a finite dimensional vector space over the field $\mathbb{F}_2$ of two elements. A quadratic form is a function $q : V \longrightarrow \mathbb{F}_2$ such that $q(0) = 0$ and

$$q(x+y) - q(x) - q(y) = (x, y)$$

is $\mathbb{F}_2$-bilinear (and, of course, symmetric). Notice that $(x, x) = 0$ so that $(-,-)$ is a symplectic bilinear form.

Hence $\dim(V) = 2n$ and to say that q is non-singular means that there is an $\mathbb{F}_2$-basis of V, $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ say, such that $(a_i, b_j) = 0$ if $i \neq j$, $(a_i, b_i) = 1$ and $(a_i, a_j) = 0 = (b_i, b_j)$ for all i and j.

In this case the Arf invariant of q is defined to be

$$c(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbb{F}_2.$$

Browder used an equivalent definition of the Arf invariant as the following "democratic invariant". The elements of V "vote" for either 0 or 1 by the function q. The winner of the election (which is never a tie) is c(q). Here is a table illustrating this for three possibilities q, q', q'' when V is two-dimensional with basis $\{e_1, e_2\}$. Having equal Arf invariants q is isomorphic to q . Thus the vote is three to one in each case.

x	0	e_1	e_2	$e_1 + e_2$	value of c
values of q	0	0	0	1	0
values of q'	0	1	1	1	1
values of q''	0	1	0	0	0

Theorem (C. Arf 1941) The invariant $c(q)$ is independent of the choice of basis and two quadratic forms on V are equivalent if and only if their Arf invariants coincide.

The Arf-Kervaire invariant of a framed manifold

Using the Arf invariant, Michel Kervaire (1960) defined an $\mathbb{F}_2$-valued invariant for compact, $(2l-2)$-connnected framed $(4l-2)$-manifolds which are smooth in the complement of a point. He applied it to exhibit a manifold which does not admit any differentiable structure!

Bill Browder (1969) extended this definition to any framed, closed $(4l-2)$-manifold.

Given a framed manifold M^{2k} and

$$a \in H^k(M; \mathbb{Z}/2) \cong [M_+, K(\mathbb{Z}/2, k)]$$

we compose with the Pontrjagin-Thom map

$$S^{2k+N} \longrightarrow T(\nu(M, i)) \cong \Sigma^N(M_+)$$

to obtain an element of

$$q_{M,t}(a) \in \pi_{2k+N}(\Sigma^N K(\mathbb{Z}/2, k)) \cong \mathbb{F}_2.$$

This is a non-singular quadratic form $q_{M,t}$ on $H^k(M;\mathbb{Z}/2)$, depending on the framing t and the Arf-Kervaire invariant of (M,t) is

$$c(q_{M,t}) \in \mathbb{F}_2.$$

Theorem (Browder 1969) The Arf invariant of a framed manifold M^{4l-2} is trivial unless $l = 2^s$ for some s.

Via the Pontrjagin-Thom construction the Arf-Kervaire invariant may be considered as a homomorphism

$$\mathrm{Arf}_n : \pi^S_{2^n-2}(S^0) \longrightarrow \mathbb{Z}/2$$

for $n \geq 2$.

The Arf-Kervaire invariant problem
Is Arf_n non-zero?

Let Θ_k denote the group of diffeomorphism classes of smooth manifolds Σ^k which are homotopy equivalent to S^k with group operation induced by connected sum. When $k \geq 5$ Smale's proof of the Poincaré conjecture (1962) implies Σ^k is homeomorphic to S^k.

An exotic sphere embeds into Euclidean space with a framing on its normal bundle and, by the Pontrjagin-Thom construction, defines an element of $\pi_k(\Sigma^\infty S^0)$.

Two framings in the normal bundle of Σ^k differ by a map into SO so that the above construction yields a homomorphism ($k \geq 5$)

$$\tau_k : \Theta_k \longrightarrow \pi_k(\Sigma^\infty S^0)/\mathrm{Im}(J).$$

where J is the J-homomorphism introduced earlier.

The Arf-Kervaire invariant influences the behaviour of τ_k in the following manner:

Theorem If $\mathrm{Arf}_{4l+2} = 0$ then τ_{4l+2} is surjective and $\mathrm{Ker}(\tau_{4l+1}) \cong \mathbb{Z}/2$.

Now we skip ahead to the "stop press":

Theorem (Mike Hill, Mike Hopkins and Doug Ravenel - announced April 2009) The homomorphism $\mathrm{Arf}_n = 0$ for $n \geq 8$.

Going into the details is the domain of the specialists. On the other hand $\mathrm{Arf}_n \neq 0$ for $n = 2, 3, 4, 5, 6$ which leaves only the case $n = 7$ to resolve (Fall 2010: Dung Yung Yan claims that $\mathrm{Arf}_7 = 0$, too! The case $n = 6$ is a long and brutal calculation (Barratt-Jones-Mahowald 1987; Kochman 1990).

We can sketch the cases $n = 2, 3, 4, 5$ without too much technicality by rephrasing the problem.

If X is a base-pointed space then

$$QX = \lim_{\overrightarrow{n}} \mathrm{Map}_0(\mathrm{S}^{\mathrm{n}}, \Sigma^{\mathrm{n}}\mathrm{X})$$

satisfies

$$\pi_r^S(X) \cong \pi_r(QX) \text{ for all } r \geq 0.$$

For each integer k the maps of degree k gives a component $Q_k S^0$ of QS^0, all homotopy equivalent. Therefore for $r \geq 1$

$$\pi_r^S(S^0) \cong \pi_r(QS^0) \cong \pi_r(Q_0 S^0) \cong \pi_r(Q_1 S^0).$$

Surgery theory yields a mod 2 cohomology class for $n \geq 2$

$$\underline{Arf}_n \in H^{2^n-2}(Q_1S^0),\mathbb{Z}/2)$$

such that

$$\mathrm{Arf_n}(\mathrm{f}: \mathrm{S}^{2^n-2} \to \mathrm{Q_1S^0})$$

$$= f^*(\underline{Arf}_n) \in H^{2^n-2}(S^{2^n-2},\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Dually this result may be stated in mod 2 homology: If $f : S^{2^n-2} \to Q_0S^0$ has $\mathrm{Arf_n(f)} \neq 0$ then

$$0 \neq f_*(H_{2^n-2}(S^{2^n-2},\mathbb{Z}/2)) \subseteq H_{2^n-2}(Q_0S^0,\mathbb{Z}/2)$$

in the subgroup of primitive elements

$$PH_{2^n-2}(Q_0S^0,\mathbb{Z}/2)$$

$$= \{x \mid \mathrm{diag}_*(x) = x \times 1 + 1 \otimes x\}.$$

In fact, $f_*(H_{2^n-2}(S^{2^n-2}, \mathbb{Z}/2))$ must lie in

$$PH_{2^n-2}(Q_0S^0, \mathbb{Z}/2)_{\mathcal{A}},$$

the primitives which are annihilated by the duals of the mod Steenrod cohomology operations.

Theorem (Snaith and Tornehave 1981) The $\mathbb{F}_2$-vector space

$$PH_{2^n-2}(Q_0S^0, \mathbb{Z}/2)_{\mathcal{A}}$$

is one dimensional. Hence $\mathrm{Arf}_n(f) \neq 0$ if and only if $0 \neq f_*(H_{2^n-2}(S^{2^n-2}, \mathbb{Z}/2))$.

This result actually gives a formula for the non-zero element of $f_*(H_{2^n-2}(S^{2^n-2}, \mathbb{Z}/2))$.

The proof of the Adams conjecture (Quillen, Sullivan, Becker-Gottlieb c.1970-75), which almost completely determines the image of the J-homomorphism, relates the classifying spaces of surgery theory to BO, the classifying space of the infinite orthogonal group.

This enables us to translate the Arf-Kervaire problem in terms of $\pi^S_{2^n-2}(BO)$.

The key diagram is ($SG = Q_1 S^0$) in which the left-hand vertical is a split surjection (Priddy 1978).

$$\begin{array}{ccccc}
QBD_8 & \longrightarrow & Q(BO(2)) & \longrightarrow & Q(BO) \\
\downarrow{\scriptstyle QA'} & & \downarrow{\scriptstyle QA} & & \downarrow{\scriptstyle QA} \\
Q(SG) & \longrightarrow & Q(G/O) & \xrightarrow{1} & Q(G/O) \\
\downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & & \\
SG & \xrightarrow{\pi} & G/O & &
\end{array}$$

This formula together with the relation leads to the following construction.

If M^{2^n-2} is a connected, framed manifold the Pontrjagin-Thom construction gives us an element $\theta \in \pi^S_{2^n-2}(M_+)$. If E is an k-dimensional vector bundle over M ($k \geq 2$) classified by

$$h(E) : M \longrightarrow BO$$

we can form

$$(h(E)_+)_*(\theta) \in \pi^S_{2^n-2}(BO_+)$$

and from this an element

$$\Theta(M, E) \in \pi^S_{2^n-2}(S^0).$$

Theorem (Snaith and Tornehave 1981) If $w_2(E) \in H^2(M, \mathbb{Z}/2)$ is the 2nd Stiefel-Whitney class of E and $[M] \in H^{2^n-2}(M, \mathbb{Z}/2)$ is the fundamental class of M then

$$\mathrm{Arf}_n(\Theta(M, E)) = \langle w_2(E)^{2^{n-1}-1}, [M] \rangle \in \mathbb{Z}/2.$$

THIS FORMULA DOES NOT REQUIRE KNOWLEDGE OF THE FRAMING - EXCEPT ITS EXISTENCE!

Examples

(a) $M = \mathbb{R}P^1 \times \mathbb{R}P^1 \subseteq BO(1) \times BO(1) \to BO$

(b) $M = \mathbb{R}P^3 \times \mathbb{R}P^3 \subseteq BO(1) \times BO(1) \to BO$

(c) $M = \mathbb{R}P^7 \times \mathbb{R}P^7 \subseteq BO(1) \times BO(1) \to BO$

(d) Let C denote the Riemann surface obtained by putting a thin tube around each edge of a cube. The natural action D_8, of the dihedral group of order 8, on the cube induces a free D_8-action on C. Since D_8 is the 2-Sylow subgroup of the permutation on 4 object D_8 also acts on

$$\mathbb{R}P^7 \times \mathbb{R}P^7 \times \mathbb{R}P^7 \times \mathbb{R}P^7.$$

Form

$$M = C \times_{D_8} (\mathbb{R}P^7 \times \mathbb{R}P^7 \times \mathbb{R}P^7 \times \mathbb{R}P^7).$$

A simple calculation in K-theory shows that M is framed (first noticed by John Jones 1975) and by construction M is a subcomplex of $BO(4)$.

Examples(a)-(d) show the existence non-trivial Arf-Kervaire invariants in dimensions $2, 6, 14$ and 30.