The Bernstein Centre of Smooth Representations

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G locally compact, totally disconnected group

usually reductive algebraic group over a local field K e.g. SL_nK

A smooth representation is a $G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$

(continuous) and for $v \in V$ $\mathrm{Stab}_G(v)$ is open.

Abelian category $\mathrm{Rep}^{sm}(G)$.

The centre $Z(\mathcal{A})$ of abelian category $\mathcal{A}$ is the ring of endomorphisms of the identity functor

$A \in \mathrm{Ob}(\mathcal{A})$, $z_A \in \mathrm{End}_{\mathcal{A}}(A)$ such that for all $f : A \longrightarrow B$ we have $fz_A = z_Bf$.

The Bernstein Centre is $Z(\mathrm{Rep}^{sm}(G))$.

Bernstein-Zelevinski determined $Z(\mathrm{Rep}^{sm}(GL_nK))$

Deligne [Le “centre” de Bernstein; Hermann Travaux en Cours (1984)] generalised to all G as above.

Hecke algebras

Assume that G is unimodular - the left/right invariant Haar measures are equal

The Hecke algebra of G, $\mathcal{H}_G$, is the space $C_c^\infty(G)$ of locally constant,compactly supported functions on G with the convolution product

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(gh)\phi_2(h^{-1})dh$$

Suppose that $K_0 \subseteq G$ is a compact, open subgroup. Define an idempotent

$$e_{K_0} = \frac{1}{\mathrm{vol}(K_0)} \cdot \chi_{K_0}$$

where χ_{K_0} is the characteristic function of K_0. If $K_0 \subseteq K_1$ then $e_{K_0} * e_{K_1} = e_{K_1}$.

$\mathcal{H}_G$ is an idempotented algebra - $\mathcal{H}_G = \bigcup e\mathcal{H}_G e$ - e runs through all the poset of idempotents

Let X be a totally disconnected, locally compact topological space. As before $C_c^\infty(X)$ is the space of locally constant, compactly supported functions on X and $\mathcal{D}(X)$ is the space of distributions on X (i.e. linear functionals on $C_c^\infty(X)$).

$\mathcal{H}_G^\wedge \cong \lim_{\leftarrow e_{K_0}} \mathcal{H}_G * e_{K_0}$ an algebra whose centre

$Z(\mathcal{H}_G^\wedge)$ is the space of distributions T on G conjugation-fixed such that $T * e_{K_0}$ has compact support.

Essentially:

Theorem (B-Z and D) $Z(\mathcal{H}_G^\wedge) \cong Z(\mathrm{Rep}^{sm}(G))$

The link between $\mathrm{Rep}^{sm}(G)$ and smooth $\mathcal{H}_G$-modules is related to Bruhat's thesis.

Let σ be a smooth representation of H on a vector space V, the induced representation π

$\pi = \mathrm{Ind}_H^G(\chi^{1/2}\sigma)$

χ is some character depending on H, G

the representation space $\mathcal{H}$ for π is the space of all smooth functions $\psi : G \longrightarrow V$ that are compactly supported modulo H and satisfy, for $x \in G, h \in H$,

$$\psi(xh^{-1}) = \chi(h)^{1/2}\sigma(h)\psi(x).$$

The action of G is given by

$$(\pi(y)\psi)(x) = \psi(y^{-1}x).$$

Let π_1, π_2 be induced representations of G on vector spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively. An intertwining form is a bilinear form $B : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathbb{C}$ such that

$$B(\pi_1(g)v_1, \pi_2(g)v_2) = B(v_1, v_2)$$

for all $g \in G, v_i \in \mathcal{H}_i$.

An intertwining operator $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a linear map such that $T(\pi_1(g)v_1) = \pi_2(g)T(v_1)$.

Bruhat classified B's which is equivalent to classifying T's.

Method: by embedding the B's into a space of invariant distributions on G.

The "classical" account is to be found in §4.2 pp.424-435 of

Daniel Bump: *Automorphic forms and representations*; Cambridge studies in advanced math. **55** (1998).

Let $(h_1, h_2) \in H_1 \times H_2$ act on the right on G by $x \cdot v = h_1^{-1} x h_2$. Fix an orbit $\mathcal{O}$ and choose $y \in \mathcal{O}$ with $\text{stab}_{H_1 \times H_2}(y) = \mathcal{H}(y)$. There is the usual mapping

$$\mathcal{H}(y) \backslash H_1 \times H_2 \longrightarrow \mathcal{O}$$

which is a homeomorphism. The stabiliser is given by $\mathcal{H}(y) = \{(h_1, y^{-1}h_1 y)\} = H_2 \bigcap y^{-1} H_1 y$.

Bruhat refines his embedding into a disjoint union of orbit-by-orbit embeddings (via supports of distributions).

Firstly recall the Double Coset formula

$\mathrm{Res}^G_J \mathrm{Ind}^G_H(\rho) \xrightarrow{\cong} \oplus_{z \in J\backslash G/H} \mathrm{Ind}^J_{J \cap zHz^{-1}}((z^{-1})^*(\rho))$

The DCF plus Frobenius reciprocity splits

$\mathrm{Hom}_G(\pi_1, \pi_2) = \mathrm{Hom}_{H_1}(\chi_1^{1/2}\sigma_1, \mathrm{Res}^G_{H_1}(\pi_2))$

into a sum over $y \in H_1\backslash G/H_2$ of summands

$$\mathrm{Hom}_{H_1}(\chi_1^{1/2}\sigma_1, \mathrm{Ind}^{H_1}_{H_1 \cap yH_2y^{-1}}((y^{-1})^*(\pi_2))).$$

One can derive Bruhat's results via the DCF and the contribution from the $\mathcal{H}(y)$-orbit is that from the double coset $y \in H_1\backslash G/H_2$.

With a dictionary between Bruhat's distributional approach and the DCF approach one can attempt to establish Deligne's result by means of monomial resolutions which I will now recall. Details are in

V.P. Snaith: *Derived Langlands*; research monograph (268 pages) available at http://victor-snaith.staff.shef.ac.uk (June 2016).

Let $\mathcal{C}$ denote the compact open modulo the centre subgroups $H \subseteq G$ containing the centre. Tammo tom Dieck constructed a G-simplicial complex $\underline{E}(G, \mathcal{C})$ such that for any $H \in \mathcal{C}$ $\underline{E}(G, \mathcal{C})^H$ is non-empty and contractible.

For a p-adic Lie group this space is G-homotopy equivalent to the Bruhat-Tits building.

Let G be a locally p-adic Lie group and let V be a smooth representation defined on a $\mathbb{C}$-vector space with central character $\underline{\phi}$.

There is an additive but not abelian category ${}_{\mathbb{C}[G]}\mathbf{mon}$ - the monomial category - whose irreducibles are induced representations $\underline{\mathrm{Ind}}_H^G(\phi)$ where $H \in \mathcal{C}$ and ϕ is a character extending $\underline{\phi}$.

Induced modules have a basis indexed by G/H and the definition of a morphism and of exactness is defined by reference to these bases.

The morphisms in ${}_{\mathbb{C}[G]}\mathbf{mon}$ a linear combinations of

$$((K, \psi), g, (H, \phi)) : \underline{\mathrm{Ind}}_K^G(\psi) \longrightarrow \underline{\mathrm{Ind}}_H^G(\phi)$$

when $(K, \psi) \leq (gHg^{-1}, (g^{-1})^*(\phi))$

Write the basis elements as $g' \otimes_H v$ (as if the groups were finite) then

$$((K, \psi), g, (H, \phi))(g' \otimes_K v) = g'g \otimes_H v$$

The DCF and Frobenius reciprocity hold in ${}_{\mathbb{C}[G]}\mathbf{mon}$ and in the decomposition of

$$\mathrm{Hom}_{{}_{\mathbb{C}[G]}\mathbf{mon}}(\underline{\mathrm{Ind}}_K^G(\psi), \underline{\mathrm{Ind}}_H^G(\phi))$$

This morphism corresponds to the basis of the one-dimensional summand corresponding to KgH.

Therefore it is easy to find in the DCF dictionary for Bruhat's thesis.

A ${}_{\mathbb{C}[G],\underline{\phi}}\mathbf{mon}$-resolution of V is a chain complex

$$M_* : \quad \ldots \stackrel{\partial_{i+1}}{\longrightarrow} M_{i+1} \ldots \stackrel{\partial_1}{\longrightarrow} M_1 \stackrel{\partial_0}{\longrightarrow} M_0$$

with $M_i \in_{k[G],\underline{\phi}} \mathbf{mon}$ such that

$$\ldots \stackrel{\partial_1}{\longrightarrow} M_1^{((H,\phi))} \stackrel{\partial_0}{\longrightarrow} M_0^{((H,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(H,\phi)} \longrightarrow 0$$

is an exact sequence of $\mathbb{C}$-modules for each (H,ϕ).

In particular, when $(H,\phi) = (Z(G),\underline{\phi})$ we see that

$$\ldots \stackrel{\partial_i}{\longrightarrow} M_i \stackrel{\partial_{i-1}}{\longrightarrow} \ldots \stackrel{\partial_1}{\longrightarrow} M_1 \stackrel{\partial_0}{\longrightarrow} M_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is an exact sequence in ${}_{k[G],\underline{\phi}}\mathbf{mod}$.

The bar-monomial resolution is a functorial monomial resolution for $\mathrm{Res}_H^G(V)$ which gives a sheaf of monomial chain complexes on $\underline{E}(G,\mathcal{C})$ whose total complex is a ${}_{\mathbb{C}[G],\underline{\phi}}\mathbf{mon}$-resolution of V.

The morphisms such as $M_1 \xrightarrow{\partial_0} M_0$ are very special cases of the intertwining operators of Bruhat's thesis and can be described via the DCF or via distributions.

This gives the "centre of ${}_{\mathbb{C}[G],\underline{\phi}}\mathbf{mon}$" in terms of distributions that is, morphisms z_{M_1} and z_{M_0} such that $z_{M_0}\partial_0 = \partial_0 z_{M_1}$ for all ∂_0.

The z_{M_i}'s induce z_V in the "centre of ${}_{\mathbb{C}[G],\underline{\phi}}\mathbf{mod}$'.

This should describe the relation between the Bernstein centre and the centre of $\mathcal{H}_G^{\wedge}$ in a new way.

<u>an analogue of Swan's theorem</u>

In [Trans. A.M.Soc. (1962) 264-277] Swan showed that the space of sections of a vector bundle on X is a f.g. projective $C(X)$ and Serre proved the converse.

Let X be a totally disconnected, locally compact topological space. As before $C_c^\infty(X)$ is the space of locally constant, compactly supported functions on X and $\mathcal{D}(X)$ is the space of distributions on X (i.e. linear functionals on $C_c^\infty(X)$).

A basis for the topology on X is given by $U \in \mathcal{T}_c$, the set of compact open subsets of X.

A C^∞-module or a C_c^∞-module $\mathcal{M}$ is defined to be cosmooth if for every $x \in \mathcal{M}$ there exists a compact open U such that $\mathbf{1}_U \cdot x = x$ where $\mathbf{1}_U$ is the characteristic function of U.

Let $\mathcal{M}$ be a cosmooth $C_c^\infty(X)$-module. For $U \in \mathcal{T}_c$ let $\mathcal{M}(U) = \mathbf{1}_U \cdot \mathcal{M}$. The transition map $\rho_{U,V}$ is defined by $\rho_{U,V}(m) = \mathbf{1}_V \cdot m$ for $V \subseteq U$. This is a sheaf.

The stalk at x is $\mathcal{M}_x = M/M(x)$ where $M(x) = \{m \in M \mid \mathbf{1}_U \cdot x = 0\}$ for all $U \in \mathcal{T}_c$ containing x.

We have an analogue of Swan's theorem: the following three categories are equivalent:

(i) the category of sheaves of vector spaces over X,

(ii) the category of sheaves of C^∞-modules and

(iii) the category of cosmooth modules over $C_c^\infty(X)$.

Let $\mathcal{F}$ be a sheaf of C^∞-modules on X. Let $\mathcal{F}_c$ denote the corresponding cosmooth module of compactly supported sections. By an $\mathcal{F}$-distribution on X we mean a linear functional on the vector space $\mathcal{F}_c$. Let $\mathcal{D}(X, \mathcal{F})$ denote the space of $\mathcal{F}$-distributions on X.

Let Z be a locally closed subset of X (i.e. the intersection of an open and a closed subset).

The disjoint union of the stalks of $\mathcal{F}$ is called the étale space of $\mathcal{F}$ and denoted by $\hat{\mathcal{F}}$.

If U is an open subset of Z then let $\mathcal{F}_Z(U)$ be the set of sections $s : U \longrightarrow \hat{\mathcal{F}}$ such that for $x \in U$ and $s(x) \in \mathcal{F}_x$ there exists a section $s' \in \mathcal{F}(V)$ which agrees with s on $V \bigcap U$.

The stalk of $\mathcal{F}_Z$ at x equals that of $\mathcal{F}$ at x so the étale space of $\mathcal{F}_Z$ is just the restriction of the étale space of $\mathcal{F}$. So we have a restriction map $\mathcal{F} \longrightarrow \mathcal{F}_Z$.

If Z is closed in X we have an extension of sections by zero on Z giving a canonical map

$$\mathcal{F}_{X-Z} \longrightarrow \mathcal{F}.$$

There are short exact sequences

$$(\mathcal{F}_{X-Z})_c \longrightarrow \mathcal{F}_c \longrightarrow (\mathcal{F}_Z)_c$$

and its dual

$$\mathcal{D}(Z, \mathcal{F}_Z) \longrightarrow \mathcal{D}(X, \mathcal{F}) \longrightarrow \mathcal{D}(X - Z, \mathcal{F}_{X-Z}).$$

Suppose now that a group G acts on X and let $\mathcal{F}$ be a sheaf on X.

Suppose that G extends to an action on $\mathcal{F}$, which means that we are given for $g \in G$ an isomorphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(gU)$ for each $U \in \mathcal{T}_c$.

Then G acts on $\mathcal{F}_c$ and on $\mathcal{D}(X, \mathcal{F})$.

If Z is a closed subspace mapped to itself by G then G also acts on $(\mathcal{F}_Z)_c$ and $\mathcal{D}(Z, \mathcal{F}_Z)$.

Proposition (Bernstein and Zelevinski)

Let X and Y be totally disconnected, locally compact spaces. Let $p : X \longrightarrow Y$ be a continuous map and let $\mathcal{F}$ be a sheaf on X. Suppose that G acts on X and on $\mathcal{F}$. Assume that $p(gx) = p(x)$ for all $g \in G$, $x \in X$. Let χ be a character on G.

(i) Let $y \in Y$ and let $Z = p^{-1}(y)$. Let $\mathcal{F}_c(\chi)$ (resp. $(\mathcal{F}_Z)_c(\chi)$) denote the submodule of $\mathcal{F}_c$ (resp. $(\mathcal{F}_Z)_c$) generated by the elements of the form $g \cdot f - \chi(g)^{-1} f$ for $f \in \mathcal{F}_c$ (resp. $f \in (\mathcal{F}_Z)_c$) and $g \in G$. Then $M = \mathcal{F}_c / \mathcal{F}_c(\chi)$ is a cosmooth C_Y^∞-module. Let $\mathcal{G}$ denote the corresponding sheaf on Y. If $y \in Y$ the stalk satisfies

$$\mathcal{G}_y \cong (\mathcal{F}_Z)_c / (\mathcal{F}_Z)_c(\chi).$$

(ii) Assume there are no non-zero distributions D in $\mathcal{D}(p^{-1}(y), \mathcal{F}_{p^{-1}(y)})$ for any $y \in Y$ which satisfy $gD = \chi(g) D$ for all $g \in G$. Then there are no non-zero distributions $D' \in \mathcal{D}(X, \mathcal{F})$ which satisfy $gD' = \chi(g) D'$.

Example A GL_2F application of the Proposition

Consider $G = GL_2K$ where K is a p-adic local field in characteristic zero. Then if D is a distribution on G which is invariant under conjugation then D is invariant under transpose.

Let $tr(D)$ be the transpose of D and consider $D - tr(D)$ which is invariant under conjugation. We must show it is zero. Changing notation then we may assume we have conjugation-invariant D such that $tr(D) = -D$. Now let $C_2 \propto G$ denote the semi-direct product of C_2, generated by τ, and G. Then $C_2 \propto G$ acts on D by G fixing it and $\tau(D) = -D$ where $\tau(D) = tr(D) = -D$. Define $\chi : C_2 \propto G \longrightarrow \{\pm 1\}$ to be trivial on G and to send τ to -1. Hence $g(D) = \chi(g)D$.

Denote by G_{reg} the subset of G having distinct eigenvalues with complement G_{sing}. Then G_{sing} is the locus of $tr(g)^2 - 4\det(g) = 0$. Hence it is closed. Therefore we have a short exact sequence

$$0 \longrightarrow \mathcal{D}(G_{sing}) \longrightarrow \mathcal{D}(G) \longrightarrow \mathcal{D}(G_{reg}) \longrightarrow 0.$$

To show that D maps to zero in the right hand group set

$$Y = \{(x, y) \in F \oplus F \mid x^2 \neq 4y\}$$

and set $p : X \longrightarrow Y$ equal to $p(g) = (tr(g), \det(g))$. By Proposition we have to show that there are no distributions on $p^{-1}(y) = Z$ for $y \in Y$ which is a χ eigen-distribution on the (regular) conjugacy class Z.

This is done using some properties of invariant distributions and their relation certain Haar integrals.

Now let C be the scalar matrices. We have an exact sequence

$$0 \longrightarrow \mathcal{D}(C) \longrightarrow \mathcal{D}(G_{sing}) \longrightarrow \mathcal{D}(G_{sing}-C) \longrightarrow 0.$$

The above method shows that D maps to zero in the right-hand group in this sequence, too. This time a parameter space for Y is F^* since every conjugacy class in $G_{sing}-C$ is represented by a matrix of the form

$$g = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

Hence D lies in the left-hand group but anything in there is transposition invariant so cannot be in the -1 χ-eigenspace, which is a contradiction. □