A SEASONAL BRAIN-TEASER

I learnt about this nice brain-teaser from Hugh Gilbert.

A married couple, Noel and Holly, invite N other couples over for New Years Eve drinks. It's a rather formal affair, but as everyone in the group is a mathematician they don't want to do more handshaking than is necessary. Each person only shakes hands with the people they haven't met before. As the brandy begins to flow, Noel asks everyone at the party how many hands they shook, and he receives $2N + 1$ different answers. How many people did Holly shake hands with?

Solution to brain-teaser:

Suppose the couples are $\{X_a, X_b\}$ for $X = 1, 2, \ldots, N$. Being a couple implies that X_a and X_b know each other. Being invited guests implies that one or both of Noel and Holly know at least one of $\{X_a, X_b\}$.

Now suppose that the brain-teaser has a unique solution. For example, if $N = 1$ an example of the event is given in the following table.

person	others the person knows already	number of handshakes
1_a	$\{\text{Noel, Holly}, 1_b\}$	0
Holly	$\{\text{Noel}, 1_a\}$	1
1_b	$\{1_a\}$	2

Since Holly makes 1 handshake in this example the uniqueness assumption asserts every example will produce 1 Holly handshake when $N = 1$!

Now I am going to show how to table an event example table for N couples and consistently make one for $N + 1$ couples. To the N-couple table add a new top row

$(N+1)_a$	$\{\text{Noel, Holly}, 1_a, 1_b, 2_a, 2_b, \ldots, N_b, (N+1)_b\}$	0

Since $(N + 1)_a$ knows everyone each row must contain $(N + 1)_a$. So to each row of the N-couple event add in $(N + 1)_a$ as an acquaintance and insert a new bottom row, which must be,

$(N+1)_b$	$\{(N+1)_a\}$	$2N$

This gives an example of an $N + 1$-couple event in which Holly's does $N + 1$ handshakes, which is therefore the general answer.

By way of example, applying the inductive algorithm to the $N = 1$ table and its success gives:

person	others the person knows already	number of handshakes
2_a	$\{\text{Noel, Holly}, 1_a, 1_b, 2_b\}$	0
1_a	$\{\text{Noel, Holly}, 1_b, 2_a\}$	1
Holly	$\{\text{Noel}, 1_a, 2_a\}$	2
1_b	$\{1_a, 2_a\}$	3
2_b	$\{2_a\}$	4

and

person	others the person knows already	number of handshakes
3_a	$\{\text{Noel, Holly}, 1_a, 1_b, 2_a, 2_b, 3_b\}$	0
2_a	$\{\text{Noel, Holly}, 1_a, 1_b, 2_b, 3_a\}$	1
1_a	$\{\text{Noel, Holly}, 1_b, 2_a, 3_a\}$	2
Holly	$\{\text{Noel}, 1_a, 2_a, 3_a\}$	3
1_b	$\{1_a, 2_a, 3_a\}$	4
2_b	$\{2_a, 3_a\}$	5
3_b	$\{3_a\}$	6

Second solution: (Hugh Gilbert and Nina Snaith solved the puzzle this way - by reverse induction)

If there are N invited couples and, say, N_b knows everyone else then N_a knows only N_b. For all other invited guests knows at least N_b plus their partner. There is of course a chance that Holly knows everyone but then Noel knows only one and this soon contradicts the set of answers he is supposed to get. So suppose N_b and N_a don't turn up then we have the $N - 1$ couple situation and by induction Holly shook $N - 1$ of those hands. But in reality she shook those $N - 1$ hands plus the hand of N_a for a total of N handshakes!