

Computing Borel's Regulator

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Abstract

We present an infinite series formula based on the Karoubi-Hamida integral, for the universal Borel class evaluated on $H_{2n+1}(GL(\mathbb{C}))$. For a cyclotomic field F we define a canonical set of elements in $K_3(F)$ and present a novel approach (based on a free differential calculus) to constructing them. Applying our computer algorithm to this set yields a value $V_1(F)$, which coincides with the Borel regulator $R_1(F)$ when our set is a basis modulo torsion. For $F = \mathbb{Q}(e^{2\pi i/3})$ we compute $V_1(F)$.

1 Introduction

Let F be an algebraic number field and \mathcal{O}_F its ring of integers. For $n \geq 1$, the Borel regulator $R_n(F)$ is a real valued numerical invariant of F . It measures the covolume of the algebraic K -theory groups $K_{2n+1}(F)$ modulo torsion, embedded as a lattice in \mathbb{R}^{d_n} (where the integer d_n is to be defined later). Although defined in terms of the odd-dimensional K -theory groups, knowledge of the Borel regulator has implications for the even dimensional K -theory groups. In particular, the Lichtenbaum conjecture (proven in many cases, such as abelian extensions of the rationals [6, 16, 19, 23]) gives the order of $K_{2n}(\mathcal{O}_F)$ up to a power of 2 in terms of the Dedekind zeta function of F , the order of $\text{Tor}(K_{2n+1}(F))$ and $R_n(F)$.

However explicitly computing the Borel regulator is a very difficult problem, even in the case of cyclotomic number fields. The standard approach to the Borel regulator is via comparison with the Beilinson regulator [5], which in turn is expressed via polylogarithms; moreover, Zagier's conjectures [26], which generalize classical results of Bloch [3, 13], allow one to map the higher Bloch group $\mathcal{B}_{n+1}(F)$ modulo torsion to a lattice in \mathbb{R}^{d_n} . However the identification of $\mathcal{B}_{n+1}(F)$ modulo torsion with a full sublattice of $K_{2n+1}(F)$ is delicate [7].

We present a new approach to computing $R_1(F)$ when F is a cyclotomic field. We first describe a set of elements $z_u \in K_3(F)$ corresponding to primitive roots of unity $u \in F$ (Section 2). We wish to compute the covolume $V_1(F)$ of the lattice generated by the images of these elements in the real vector space \mathbb{R}^{d_1} . If the elements $z_u \in K_3(F)$ form a basis modulo torsion, then we have $R_1(F) = V_1(F)$ and we have computed the desired Borel regulator. Otherwise $V_1(F)$ is an integer multiple of $R_1(F)$.

Determining for which cyclotomic fields the elements z_u form a basis modulo torsion is beyond the scope of this article. However by defining and showing how to compute $V_1(F)$ we hope to have laid foundations for the computation of $R_1(F)$ by these means.

Computing $V_1(F)$ breaks down into two stages. Firstly we construct the images of the z_u in $H_3(E(\mathbb{C}))$ as explicit chains (Section 3). This involves constructing a free differential calculus, motivated by the one invented by Fox as a tool in knot theory [11].

Then (Section 4) we apply the universal Borel class to these chains by expanding the Karoubi-Hamida integral [14] as an infinite power series (Theorem 4.11). In fact

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this method works for evaluating the universal Borel class on any number field and any $n \geq 1$ (Theorem 4.4). Finally we compute $V_1(F)$ for $F = \mathbb{Q}(e^{2\pi i/3})$ (Section 5).

Part of the contents of this article are contained in the 1st author's PhD thesis [8]. Also the 1st and 4th authors published analogous results for the p -adic regulator [9].

2 Defining V_1 of a cyclotomic field

Let F be the cyclotomic number field $\mathbb{Q}(\omega)$, where ω is a q^{th} root of unity ($q \geq 3$). In this section we will define a canonical set of elements in $K_3(F)$ (defined up to torsion) (§2.2). The images of this set under the Borel regulator map generate a lattice and we define $V_1(F)$ to be its covolume (§2.3). We first recall the definition of the Borel regulator of a number field (§2.1).

2.1 The Borel regulator

Let F be a number field, \mathcal{O}_F its ring of integers and q_1, q_2 the number of real embeddings, respectively conjugate pairs of complex embeddings, $F \hookrightarrow \mathbb{C}$. The *Borel regulator maps* are homomorphisms

$$K_{2n+1}(\mathcal{O}_F) \cong K_{2n+1}(F) \longrightarrow \mathbb{R}^{d_n} \quad (n \geq 1) \quad (1)$$

from the odd algebraic K-theory groups of F (or \mathcal{O}_F) to \mathbb{R}^{d_n} , where d_n is $q_1 + q_2$ if n is even and q_2 if n is odd. They can be defined in the following way [12].

The Hurewicz homomorphism induces the following homomorphism from K-theory into the homology (with integral coefficients) of the discrete group $GL(\mathbb{C})$:

$$h_{2n+1}: K_{2n+1}(\mathbb{C}) = \pi_{2n+1}(BGL(\mathbb{C})^+) \longrightarrow H_{2n+1}(BGL(\mathbb{C})^+) \cong H_{2n+1}(GL(\mathbb{C})).$$

Remark 1. Suslin's stability result [17, Corollary 2.5.3], gives that if $N \geq 2n + 1$ then

$$H_n(GL(\mathbb{C})) \cong H_n(GL_N(\mathbb{C})).$$

Let $\mathbb{R}(n) = (2\pi i)^n \mathbb{R}$ for $n \geq 1$. There exists a *universal Borel class*

$$b_n \in H_c^{2n+1}(GL(\mathbb{C}); \mathbb{R}(n))$$

in the continuous cohomology of $GL(\mathbb{C}) = \text{colim}_r GL_r(\mathbb{C})$ [5]. Application of b_n induces a map $H_{2n+1}(GL(\mathbb{C})) \rightarrow \mathbb{R}(n)$. The *universal Borel regulator map*

$$r_n: K_{2n+1}(\mathbb{C}) \longrightarrow \mathbb{R}(n)$$

is defined to be the composition $r_n = b_n \circ h_{2n+1}$.

For the definition on an arbitrary number field F , we compose with the maps induced on K-theory by the different embeddings $F \hookrightarrow \mathbb{C}$:

$$K_{2n+1}(F) \longrightarrow \bigoplus_{\text{Hom}(F, \mathbb{C})} K_{2n+1}(\mathbb{C}) \longrightarrow X_F \otimes \mathbb{R}(n),$$

where $X_F = \mathbb{Z}[\text{Hom}(F, \mathbb{C})]$. The image of this map is invariant under complex conjugation acting on both $\text{Hom}(F, \mathbb{C})$ and $\mathbb{R}(n)$. Hence we have a map

$$K_{2n+1}(F) \longrightarrow (X_F \otimes \mathbb{R}(n))^c$$

where $(\)^c$ denotes the subgroup of invariants under complex conjugation. If n is odd, we take a basis of $(X_F \otimes \mathbb{R}(n))^c$ consisting of $\psi \otimes i - \overline{\psi} \otimes i$ for each conjugate pair of complex embeddings $\psi, \overline{\psi}: F \rightarrow \mathbb{C}$. If n is even we take a basis of $(X_F \otimes \mathbb{R}(n))^c$ consisting of $\psi \otimes 1 + \overline{\psi} \otimes 1$ for each conjugate pair of complex embeddings $\psi, \overline{\psi}: F \rightarrow \mathbb{C}$, together with $\psi \otimes 1$ for each real embedding $\psi: F \rightarrow \mathbb{R}$. Either way, this yields a natural identification of $(X_F \otimes \mathbb{R}(n))^c$ with \mathbb{R}^{d_n} .

Borel proved that for $n \geq 1$ the Borel regulator map (1) is an embedding of $K_{2n+1}(F)$ modulo torsion in \mathbb{R}^{d_n} and its image is a full lattice in \mathbb{R}^{d_n} . The covolume of this lattice is called the *Borel regulator* for F , written $R_n(F)$.

2.2 A canonical set of elements in $K_3(F)$

Let R be a ring and let $E(R)$ denote the elementary matrices of R . We will make use of the following result:

Lemma 2.1. *Let R be a ring. Then the Hurewicz map induces a surjective map:*

$$h_3: K_3(R) = \pi_3(BE(R)^+) \longrightarrow H_3(BE(R)^+) \cong H_3(BE(R)) \cong H_3(E(R)).$$

Proof. By construction, the space $BE(R)^+$ is simply connected, so the surjectivity of h_3 follows immediately from the Hurewicz Theorem. \square

Now let $F = \mathbb{Q}(\omega)$, where ω is a q^{th} root of unity ($q \geq 3$). Let $E(F)$ denote the group of elementary matrices over F . The kernel of $h_3: K_3(F) \rightarrow H_3(E(F))$ is contained in the kernel of the Borel regulator map which contains only torsion elements. Thus given an element in $H_3(E(F))$, there exists a preimage in $K_3(F)$ (Lemma 2.1), and it is unique up to torsion.

We may therefore specify elements of $K_3(F)$ up to torsion, by giving cycles in the inhomogeneous bar resolution of $E(F)$ tensored with \mathbb{Z} , which we write as

$$\cdots \xrightarrow{d} B_n \xrightarrow{d} B_{n-1} \xrightarrow{d} \cdots$$

For later reference we recall that the boundary map $d: B_3 \rightarrow B_2$ is given by

$$d([g_1|g_2|g_3]) = [g_2|g_3] - [g_1g_2|g_3] + [g_1|g_2g_3] - [g_1|g_2]$$

and the boundary map $d: B_2 \rightarrow B_1$ is given by

$$d([g_1|g_2]) = [g_2] - [g_1g_2] + [g_1].$$

Let $A = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials over \mathbb{Z} . For any primitive q^{th} root of unity $u \in F$, we have a ring homomorphism $\alpha_u: A \rightarrow F$, given by $x \mapsto x|_{t=u}$, the element of F obtained by evaluating t at u .

For a unit $\lambda \in A$, let g_{ij}^λ denote the matrix which differs from the identity in the $(i, i)^{\text{th}}$ and $(j, j)^{\text{th}}$ entries only, which are λ, λ^{-1} respectively. Let $a, b \in E(A)$ be

$$a = g_{12}^t = \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = g_{13}^{-t} = \begin{pmatrix} -t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -t^{-1} \end{pmatrix}.$$

We write the inhomogeneous bar resolution of $E(A)$ as

$$\cdots \xrightarrow{d'} B'_n \xrightarrow{d'} B'_{n-1} \xrightarrow{d'} \cdots$$

As a, b commute, we have $d'([a|b] - [b|a]) = 0$. Thus $[a|b] - [b|a]$ is a cycle and represents an element of $H_2(E(A)) \cong K_2(A)$. In fact ([20] pp. 71–75) this element is trivial, so the cycle is actually the boundary of some $Z_1 \in B'_3$. That is

$$d'(Z_1) = [a|b] - [b|a].$$

Let $h'_3: K_3(A) \rightarrow H_3(E(A))$ denote the surjective homomorphism of Lemma 2.1.

Lemma 2.2. *Given any other Z'_1 such that $d'(Z'_1) = [a|b] - [b|a]$, the difference $Z'_1 - Z_1$ is a cycle and represents a class $h'_3(y) \in H_3(E(A))$, for some element $y \in K_3(A)$.*

Proof. Clearly $d'(Z'_1 - Z_1) = 0$, so $Z'_1 - Z_1$ represents a homology class. As h'_3 is surjective, there exists $y \in K_3(A)$ such that $h'_3(y)$ is this homology class. \square

Thus Z_1 is a well defined chain modulo the image of $K_3(A)$. As its definition does not depend on the cyclotomic field F , we may define:

Definition 2.3. The *universal chain* (defined up to the image of $K_3(A)$) is the chain $Z_1 \in B'_3$ satisfying $d'(Z_1) = [a|b] - [b|a]$.

Definition 2.4. Let $q \geq 3$. We define $Z_2(q) \in B'_3$ as

$$Z_2(q) = \sum_{r=0}^{q-1} ([a^r|a|b] - [a^r|b|a] + [b|a^r|a]).$$

Then (writing I for the identity matrix) we have

$$\begin{aligned} d'Z_2(q) &= \sum_{r=0}^{q-1} (([a|b] - [b|a]) - ([a^{r+1}|b] - [a^r|b]) + ([b|a^{r+1}] - [b|a^r])) \\ &= q([a|b] - [b|a]) - ([a^q|b] - [I|b]) + ([b|a^q] - [b|I]). \end{aligned}$$

Note that $Z_2(q)$ is given explicitly above, whereas for Z_1 we have only proven existence and uniqueness up to the image of $K_3(A)$ under h'_3 .

Let α_u denote the induced chain map:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d'} & B'_n & \xrightarrow{d'} & B'_{n-1} & \xrightarrow{d'} & \cdots \\ & & \downarrow \alpha_u & & \downarrow \alpha_u & & \\ \cdots & \xrightarrow{d} & B_n & \xrightarrow{d} & B_{n-1} & \xrightarrow{d} & \cdots \end{array}$$

Note we use α_u to denote the induced maps on matrices, chains and elements of K-theory. We have

$$d\alpha_u(Z_2(q)) = \alpha_u d'Z_2(q) = q\alpha_u([a|b] - [b|a])$$

as $\alpha_u(a)^q = I$.

Let $Z = qZ_1 - Z_2(q)$. We then have $\alpha_u(Z) \in B_3$ is a cycle representing a homology class $\mathcal{Z}_u = [Z|_{t=u}] \in H_3(E(F))$. Its preimage under h_3 , denoted $z_u \in K_3(F)$, is then determined up to torsion.

Theorem 2.5. *For each primitive q^{th} root of unity in the cyclotomic field $F = \mathbb{Q}(\omega)$, there exists $z_u \in K_3(F)$, unique up to torsion, satisfying $h_3(z_u) = [(qZ_1 - Z_2(q))|_{t=u}]$, where Z_1 is a representative of the universal chain (Definition 2.3) and $Z_2(q)$ is as given in Definition 2.4.*

Proof. Given a different choice of representative of the universal chain, Z'_1 , we would have a cycle $\mathcal{Z}'_u = [\alpha_u(qZ'_1 - Z_2(q))]$. Then

$$\mathcal{Z}'_u - \mathcal{Z}_u = \alpha_u(h'_3(y)) = h_3(\alpha_u(y))$$

for some $y \in K_3(A)$. We may take $z'_u = \alpha_u(y) + z_u$ to be a preimage of \mathcal{Z}'_u under h_3 .

However $K_3(A) \cong K_3(\mathbb{Z}) \oplus K_2(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/48$ ([22], Theorem 5.3.30) finite. Thus $\alpha_u(y) \in K_3$ is an element of torsion as required. \square

Thus modulo torsion we have a well defined canonical set of elements $z_u \in K_3(F)$, corresponding to the primitive q^{th} roots of unity $u \in F$.

2.3 The covolume $V_1(F)$

Let $l = \frac{\varphi(q)}{2}$ (where φ is the Euler totient function) and let $\pm v_1, \dots, \pm v_l$ be the units of $\mathbb{Z}/q\mathbb{Z}$. Also let $\xi = e^{2\pi i/q}$. We have l conjugate pairs of embeddings $\psi_j, \overline{\psi_j}: F = \mathbb{Q}(\omega) \hookrightarrow \mathbb{C}$, given by $\omega \mapsto \xi^{v_j}$ and $\omega \mapsto \xi^{-v_j}$ for $j = 1, \dots, l$. We write ψ_j to denote the induced map on chains and elements of K-theory. The rank of $K_3(F)$ is $d_1 = q_2 = l$.

We also have l conjugate pairs of primitive q^{th} roots of unity, $u_k, \overline{u_k} = \omega^{\pm v_k} \in F$ for $k = 1, \dots, l$. Correspondingly we have elements $z_{u_k}, z_{\overline{u_k}} \in K_3(F)$ for $k = 1, \dots, l$.

For m a unit in $\mathbb{Z}/q\mathbb{Z}$, let $\beta_m: A \rightarrow \mathbb{C}$ be the ring homomorphism sending $t \mapsto \xi^m$. Again we write β_m to denote maps on chains and elements of K-theory. Note that:

$$\psi_j \circ \alpha_{u_k} = \beta_{v_j v_k}$$

Let $(L_{jk})_{j,k}$ be the matrix of co-ordinates of the images of the z_{u_k} under the Borel regulator map, $K_3(F) \rightarrow \mathbb{R}^{d_1} = (X_F \otimes \mathbb{R}(n))^c$, with respect to the basis

$$\{\psi_j \otimes i - \overline{\psi_j} \otimes i \in \mathbb{R}^{d_1} \mid j = 1, \dots, l\}.$$

Let $Z = qZ_1 - Z_2(q)$ as in §2.2.

Lemma 2.6. *We have $L_{jk} = b_1([Z]_{t=\xi^{v_j v_k}})/i$.*

Proof. The coefficient L_{jk} is $r_1(\psi_j(z_{u_k}))/i$ (see §2.1). We have

$$\psi_j(\mathcal{Z}_{u_k}) = [\psi_j \alpha_{u_k}(Z)] = [\beta_{v_j v_k}(Z)] \in H_3(E(\mathbb{C})).$$

Thus

$$\begin{aligned} r_1(\psi_j(z_{u_k})) &= b_1(h_3(\psi_j(z_{u_k}))) = b_1(\psi_j(h_3(z_{u_k}))) = b_1(\psi_j(\mathcal{Z}_{u_k})) \\ &= b_1[\beta_{v_j v_k}(Z)] = b_1([Z]_{t=\xi^{v_j v_k}}). \end{aligned}$$

Hence for $1 \leq j, k \leq l$ we have $L_{jk} = r_1(\psi_j(z_{u_k}))/i = b_1([Z]_{t=\xi^{v_j v_k}})/i$. \square

Lemma 2.7. *Applying the Borel regulator map to $z_{u_k}, \overline{z_{u_k}} \in K_3(F)$ gives elements of \mathbb{R}^{d_1} which differ by a sign. Thus up to torsion $z_{u_k}, \overline{z_{u_k}} \in K_3(F)$ differ by a sign.*

Proof. The invariance under complex conjugation of the image of z_{u_k} in $(X_F \otimes \mathbb{R}(n))^c$, implies that $r_1(\overline{\psi_j}(z_{u_k})) = -r_1(\psi_j(z_{u_k}))$ (see §2.1). Thus

$$r_1(\psi_j(\overline{z_{u_k}})) = b_1[\beta_{-v_j v_k}(Z)] = r_1(\overline{\psi_j}(z_{u_k})) = -r_1(\psi_j(z_{u_k})). \quad \square$$

From Lemma 2.7 we have that the following are well defined, independently of the choice of signs on the v_k :

Definition 2.8. We define $\text{ind}(F) \in \mathbb{Z}$ to be the index of the lattice in $K_3(F)$ modulo torsion, generated by the $z_{u_k}, k = 1, \dots, l$.

Definition 2.9. We define $V_1(F)$ to be the covolume of the lattice in \mathbb{R}^{d_1} generated by the images of the $z_{u_k}, k = 1, \dots, l$, under the Borel Regulator map.

Remark 2. It is not a priori clear that the z_u are non-zero elements of $K_3(F)$, let alone linearly independent. However if one computes a non-zero value of $V_1(F)$ (as we demonstrate for $q = 3$ in Section 5) it follows that the z_u are linearly independent and generate a finite index subgroup of $K_3(F)$; in particular, $\text{ind}(F) \neq 0$.

Theorem 2.10. *If $V_1(F) \neq 0$ then $\text{ind}(F) \neq 0$ and the Borel regulator satisfies $R_1(F) = V_1(F)/\text{ind}(F)$.*

In particular, whenever $\text{ind}(F) = 1$ (that is, whenever the z_{u_k} generate $K_3(F)$ modulo torsion), we are able to compute the Borel regulator $R_1(F)$. Thus if criteria were found to determine for which cyclotomic fields $\text{ind}(F) = 1$, then this approach would allow one to compute the Borel regulator for those fields.

Theorem 2.11. *For a cyclotomic field $F = \mathbb{Q}(\omega)$, we may compute $V_1(F)$ in terms of the universal Borel class b_1 :*

$$V_1(F) = \left| \det \left(b_1 \left([(qZ_1 - Z_2(q))|_{t=\xi^{v_j v_k}}] \right) \right)_{j,k} \right|$$

where Z_1 is the universal chain (Definition 2.3) and $Z_2(q)$ as in Definition 2.4.

Proof. We know that $V_1(F)$ is given by the absolute value of the determinant of the matrix $(L_{jk})_{j,k}$ of co-ordinates of the images of the z_{u_k} . We need only note that factors of i do not effect the absolute value of the determinant. \square

Thus to compute $V_1(F)$ we must evaluate the $b_1([qZ_1 - Z_2(q)]_{t=\xi^{v_j v_k}})$. This comprises two independent stages which are dealt with in Sections 3 and 4.

Firstly, we need to explicitly construct the chain $qZ_1 - Z_2(q)$. As $Z_2(q)$ is given explicitly (Definition 2.4) it remains to find a representative of the universal chain Z_1 . Motivated by ideas in knot theory, in Section 3 we devise techniques for constructing elements in the bar resolution of a group, with specified boundary. Using this, we eventually obtain an expression for Z_1 which fills 863 pages and is available as a PDF or L^AT_EX file¹. One may compute its boundary to independently verify that it is indeed the universal chain.

¹<http://www.personal.soton.ac.uk/rsg1y09/universalchain.pdf>
<http://www.personal.soton.ac.uk/rsg1y09/universalchain.tex>

Secondly, in Section 4 we show how b_n can be computed for arbitrary $n \geq 1$ and any number field by making the Karoubi-Hamida integral [14] explicit. In particular we describe a computer algorithm for computing b_1 (Theorem 4.11). This computer algorithm is available at request from the authors in Maple format.

In Section 5 we give the resulting value of $V_1(F)$ in the case $q = 3$.

Remark 3. To compute $R_n(F)$ for $n > 1$ we would need to construct a basis modulo torsion of $K_{2n+1}(F)$. A potential idea for generalizing our methods in the $n = 1$ case is to let $S = F[x_0, \dots, x_r]/(x_0x_1 \dots x_r(1 - \sum_{i=0}^r x_i))$, the coordinate ring of an algebraic sphere over F . Then if we could construct elements in $K_3(S)$ we could apply the natural homomorphism $K_3(S) \rightarrow KV_3(S)$ to obtain elements in Karoubi-Villamayor K-theory ([24] §3). There is an isomorphism [10]: $KV_3(S) \cong K_3(F) \oplus K_{3+r}(F)$. Projecting onto the second summand would yield elements of $K_{3+r}(F)$.

It is pure speculation that one could devise a procedure along these lines to find a basis (modulo torsion) for $K_{3+r}(F)$. However it does raise the possibility that our low dimensional group homology methods (§3.2) could be extended to aim for R_n , $n > 1$.

3 Boundary relations in the bar resolution

We describe a method for producing boundary relations in the inhomogeneous bar resolutions of groups (§3.2). We use this to construct the universal chain Z_1 (§3.5).

The method will exploit the relationship between the low dimensional homology of a group G and identities in the free group of words in its elements. We introduce a free derivative (3.1) on such words, and exploit the fact that this derivative is independent of the choice of word used to represent an element of the free group.

3.1 A free Fox type derivative

Let G be a discrete group and write $B_n G$ for the degree n part of the inhomogeneous bar resolution [25]. Therefore $B_n G$ is the free left $\mathbb{Z}[G]$ -module with basis consisting of n -tuples $[g_1|g_2|\dots|g_n]$ with each $g_i \in G$. The boundary map is given by

$$\begin{aligned} d([g_1|g_2|\dots|g_n]) &= g_1[g_2|\dots|g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1|g_2|\dots|g_i g_{i+1}|\dots|g_n] \\ &\quad + (-1)^n [g_1|g_2|\dots|g_{n-1}]. \end{aligned}$$

Thus in particular the boundary map $B_3 G \rightarrow B_2 G$ is

$$d([g_1|g_2|g_3]) = g_1[g_2|g_3] - [g_1g_2|g_3] + [g_1|g_2g_3] - [g_1|g_2].$$

The study of knot theory motivated the idea of a free differential [11]: a map from a set of words to an abelian object, satisfying a variant of Leibniz's rule. We now construct the relevant such differential. Let F_G denote the free group on symbols s_x with $x \in G$ and let $\phi: F_G \rightarrow G$ be the group homomorphism mapping $\phi(s_x) \mapsto x$.

Definition 3.1. The free derivative

$$\partial: F_G \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} B_2 G$$

is characterized by the properties:

- (i) $\partial(e) = 0$ (e is the identity element of G) and
- (ii) $\partial(us_x) = \partial(u) + 1 \otimes_{\mathbb{Z}[G]} [\phi(u)|x]$ for $u \in F_G$.

Lemma 3.2. For $1 \leq i \leq r$ suppose $x_i \in G$ and that $\epsilon_i = \pm 1$. Set $z_i = x_i^{-1}$ if $\epsilon_i = -1$ and $z_i = 1$ otherwise. Then (i) and (ii) give that

$$\partial(s_{x_1}^{\epsilon_1} s_{x_2}^{\epsilon_2} \dots s_{x_r}^{\epsilon_r}) = \sum_{i=1}^r \epsilon_i \otimes_{\mathbb{Z}[G]} [x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_{i-1}^{\epsilon_{i-1}} z_i | x_i]. \quad (2)$$

Thus ∂ is well-defined on F_G .

Proof. Firstly, note that (ii) implies

$$\partial(u) = \partial(us_x^{-1} s_x) = \partial(us_x^{-1}) + 1 \otimes_{\mathbb{Z}[G]} [\phi(u)x^{-1}|x]$$

which means that

$$\partial(us_x^{-1}) = \partial(u) - 1 \otimes_{\mathbb{Z}[G]} [\phi(u)x^{-1}|x].$$

Induction on r gives that $\partial(s_{x_1}^{\epsilon_1} s_{x_2}^{\epsilon_2} \dots s_{x_r}^{\epsilon_r})$ must be given by (2). It remains to verify that (2) yields a well defined map $\partial: F_G \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} B_2 G$, satisfying (i) and (ii).

Two words represent the same element of F_G precisely when they differ by a series of insertions and deletions of strings $s_y s_y^{-1}$ and $s_y^{-1} s_y$. Direct calculation shows that (2) is independent of such insertions or deletions. Finally, direct calculation verifies that (2) satisfies (i) and (ii). \square

Definition 3.3. Given a word w written in the form:

$$w = u_1(s_{x_1} s_{y_1} s_{x_1 y_1}^{-1})^{n_1} u_1^{-1} u_2(s_{x_2} s_{y_2} s_{x_2 y_2}^{-1})^{n_2} u_2^{-1} \dots u_k(s_{x_k} s_{y_k} s_{x_k y_k}^{-1})^{n_k} u_k^{-1}$$

with $x_i, y_i \in G$, $u_i \in F_G$ and $n_i \in \mathbb{Z}$, we define $W(w) \in \mathbb{Z} \otimes_{\mathbb{Z}[G]} B_3(G)$ by the formula

$$W(w) = \sum_{i=1}^k n_i \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i|y_i].$$

Note W is not a well defined function on F_G (unlike the free derivative which is).

Lemma 3.4. *The boundary of such a chain is given by the formula*

$$(1 \otimes_{\mathbb{Z}[G]} d)(W(w)) = \left(\sum_{i=1}^k n_i \otimes_{\mathbb{Z}[G]} [x_i|y_i] \right) - \partial(w).$$

Proof. First we note that

$$\phi(u_i(s_{x_i} s_{y_i} s_{x_i y_i}^{-1})^{n_i} u_i^{-1}) = \phi(u_i) \phi(s_{x_i} s_{y_i} s_{x_i y_i}^{-1})^{n_i} \phi(u_i)^{-1} = \phi(u_i) \phi(u_i)^{-1} = e$$

and, by the formula of Proposition 3.2, it is easy to see that

$$\partial(w) = \sum_{i=1}^k n_i \partial(u_i(s_{x_i} s_{y_i} s_{x_i y_i}^{-1}) u_i^{-1}).$$

Let u_i be written out as $\prod_{j=1}^{l_i} z_j^{m_j}$, where m_j is either 1 or -1 . Then by (2)

$$\begin{aligned} & \partial(u_i(s_{x_i} s_{y_i} s_{x_i y_i}^{-1}) u_i^{-1}) \\ &= \sum_{j|m_j=1} 1 \otimes_{\mathbb{Z}[G]} \left[\prod_{p=1}^{j-1} z_p^{m_p} |z_j \right] - \sum_{j|m_j=-1} 1 \otimes_{\mathbb{Z}[G]} \left[\prod_{p=1}^j z_p^{m_p} |z_j \right] \\ &+ 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i] + 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)x_i|y_i] - 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i y_i] \\ &- \sum_{j|m_j=1} 1 \otimes_{\mathbb{Z}[G]} \left[\prod_{p=1}^{j-1} z_p^{m_p} |z_j \right] + \sum_{j|m_j=-1} 1 \otimes_{\mathbb{Z}[G]} \left[\prod_{p=1}^j z_p^{m_p} |z_j \right] \\ &= 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i] + 1 \otimes_{\mathbb{Z}[G]} (\phi(u_i)x_i|y_i) - 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i y_i]. \end{aligned}$$

Thus we have

$$\partial(w) = \sum_{i=1}^k n_i (1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i] + 1 \otimes_{\mathbb{Z}[G]} [(\phi(u_i)x_i|y_i) - 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i y_i])$$

and

$$\begin{aligned} (1 \otimes_{\mathbb{Z}[G]} d)(W(w)) &= \sum_{i=1}^k n_i (1 \otimes_{\mathbb{Z}[G]} [x_i|y_i] - 1 \otimes_{\mathbb{Z}[G]} [(\phi(u_i)x_i|y_i) \\ &+ 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i y_i] - 1 \otimes_{\mathbb{Z}[G]} [\phi(u_i)|x_i]) \\ &= \left(\sum_{i=1}^k n_i (1 \otimes_{\mathbb{Z}[G]} [x_i|y_i]) \right) - \partial(w) \end{aligned}$$

as required. \square

3.2 Constructing boundary relations

We now describe a method, based on Lemma 3.4 for constructing boundary relations: identities of the form $(1 \otimes_{\mathbb{Z}[G]} d)\alpha = \beta$, for $\alpha \in \mathbb{Z} \otimes_{\mathbb{Z}[G]} B_3(G)$ and $\beta \in \mathbb{Z} \otimes_{\mathbb{Z}[G]} B_2(G)$.

Lemma 3.5. *Given $u \in F_G$ we may write*

$$u = (s_{x_1} s_{y_1} s_{x_1 y_1}^{-1})^{n_1} (s_{x_2} s_{y_2} s_{x_2 y_2}^{-1})^{n_2} \dots (s_{x_k} s_{y_k} s_{x_k y_k}^{-1})^{n_k} s_{\phi(u)}$$

where $x_i, y_i \in G$.

Proof. For contradiction let l be the smallest integer such that there is some $u \in F_G$ of length l contradicting the lemma. As $e = (s_e s_e s_e^{-1})^{-1} s_e$, we know $l > 0$. Then either $u = u' s_a$ or $u = u' s_a^{-1}$ with u' of length $l - 1$ and $a \in G$. Thus either

$$u = P s_{(\phi(u')) s_a} = P (s_{(\phi(u')) s_a} s_{\phi(u)}^{-1}) s_{\phi(u)}$$

or

$$u = P s_{(\phi(u')) s_a^{-1}} = P (s_{(\phi(u')) s_a^{-1}} s_{\phi(u)}^{-1}) s_{\phi(u)} = P (s_{\phi(u)} s_a s_{(\phi(u'))}^{-1})^{-1} s_{\phi(u)},$$

where P is a product of $(s_{x_i} s_{y_i} s_{x_i y_i}^{-1})^{n_i}$. \square

Definition 3.6. A *relator* is an element of $\text{Ker}(\phi: F_G \rightarrow G)$.

Lemma 3.7. *Given a relator $R \in F_G$ we may write*

$$R = (s_{x_1} s_{y_1} s_{x_1 y_1}^{-1})^{n_1} (s_{x_2} s_{y_2} s_{x_2 y_2}^{-1})^{n_2} \dots (s_{x_k} s_{y_k} s_{x_k y_k}^{-1})^{n_k}.$$

Proof. We have

$$\begin{aligned} R &= (s_{x_1} s_{y_1} s_{x_1 y_1}^{-1})^{n_1} (s_{x_2} s_{y_2} s_{x_2 y_2}^{-1})^{n_2} \dots (s_{x_k} s_{y_k} s_{x_k y_k}^{-1})^{n_k} s_e \\ &= (s_{x_1} s_{y_1} s_{x_1 y_1}^{-1})^{n_1} (s_{x_2} s_{y_2} s_{x_2 y_2}^{-1})^{n_2} \dots (s_{x_k} s_{y_k} s_{x_k y_k}^{-1})^{n_k} (s_e s_e s_e^{-1}). \end{aligned} \quad \square$$

Now let $C_1, \dots, C_{k+1}, C'_1, \dots, C'_{l+1}$ be products of commutators of the form $[R, u]^{\pm 1} = (RuR^{-1}u^{-1})^{\pm 1}$, where R is a relator and $u \in F_G$. Given an identity in F_G of the form

$$\begin{aligned} C_1 v_1 (s_{x_1} s_{y_1} s_{x_1 y_1}^{-1})^{n_1} v_1^{-1} \dots C_k v_k (s_{x_k} s_{y_k} s_{x_k y_k}^{-1})^{n_k} v_k^{-1} C_{k+1} \\ = C'_1 v'_1 (s_{x'_1} s_{y'_1} s_{x'_1 y'_1}^{-1})^{n'_1} v'_1^{-1} \dots C'_{l+1} v'_{l+1} (s_{x'_{l+1}} s_{y'_{l+1}} s_{x'_{l+1} y'_{l+1}}^{-1})^{n'_{l+1}} v'_{l+1}{}^{-1} C'_{l+1} \end{aligned} \quad (3)$$

we may use Lemma 3.7 to express each relator R as a product of $(s_a s_b s_{ab}^{-1})^m$ and each $uR^{-1}u^{-1}$ as a product of $u(s_a s_b s_{ab}^{-1})^m u^{-1}$. Thus we may express the left and right hand sides of (3) as words w_1, w_2 respectively, to which we may apply W . We get

$$(1 \otimes_{\mathbb{Z}[G]} d)(W(w_1)) = \left(\sum_{i=1}^k n_i \otimes_{\mathbb{Z}[G]} [x_i | y_i] \right) - \partial(w_1)$$

and

$$(1 \otimes_{\mathbb{Z}[G]} d)(W(w_2)) = \left(\sum_{i=1}^l n'_i \otimes_{\mathbb{Z}[G]} [x'_i | y'_i] \right) - \partial(w_2)$$

as the remaining terms coming from each relator R are canceled by the corresponding terms from each $uR^{-1}u^{-1}$.

From (3) we have that $w_1 = w_2$ as elements of F_G , so $\partial(w_1) - \partial(w_2) = 0$. Thus:

Theorem 3.8. *We have a boundary relation:*

$$(1 \otimes_{\mathbb{Z}[G]} d)(W(w_1) - W(w_2)) = \left(\sum_{i=1}^k n_i \otimes_{\mathbb{Z}[G]} [x_i | y_i] \right) - \left(\sum_{i=1}^l n'_i \otimes_{\mathbb{Z}[G]} [x'_i | y'_i] \right).$$

3.3 Examples

Let G be a group and let $x, y \in G$ commute. Then $w = [s_x, s_y]$ is a relator and we have $w = (s_x s_y s_{xy}^{-1})(s_y s_x s_{yx}^{-1})^{-1}$. Then $W(w) = 1 \otimes_{\mathbb{Z}[G]} [e|x|y] - 1 \otimes_{\mathbb{Z}[G]} [e|y|x]$ and

$$(1 \otimes_{\mathbb{Z}[G]} d)W(w) = 1 \otimes_{\mathbb{Z}[G]} [x|y] - 1 \otimes_{\mathbb{Z}[G]} [y|x] - \partial(w).$$

We use the notation $\{x, y\}$ to denote $1 \otimes_{\mathbb{Z}[G]} [x|y] - 1 \otimes_{\mathbb{Z}[G]} [y|x]$.

Example 1. As our first example we consider the identity

$$(s_e s_y s_y^{-1})(s_y s_e s_y^{-1})^{-1} = [s_e, s_y].$$

Letting w_1, w_2 denote the left and right sides of this identity as before, we get

$$W(w_1) - W(w_2) = 1 \otimes_{\mathbb{Z}[G]} [e|e|y] - 1 \otimes_{\mathbb{Z}[G]} [e|y|e] - 1 \otimes_{\mathbb{Z}[G]} [e|e|e] + 1 \otimes_{\mathbb{Z}[G]} [y|e|e].$$

Thus by Theorem 3.8

$$(1 \otimes_{\mathbb{Z}[G]} d)(W(w_1) - W(w_2)) = 1 \otimes_{\mathbb{Z}[G]} [e|y] - 1 \otimes_{\mathbb{Z}[G]} [y|e] = \{e, y\}.$$

In fact $(1 \otimes_{\mathbb{Z}[G]} d)(1 \otimes_{\mathbb{Z}[G]} [e|e|y|e] + 1 \otimes_{\mathbb{Z}[G]} [e|e|e|e]) = 1 \otimes_{\mathbb{Z}[G]} [e|y|e] + 1 \otimes_{\mathbb{Z}[G]} [e|e|e]$. So adding this boundary of a 4-chain to $W(w_1) - W(w_2)$ leaves the boundary relation

$$(1 \otimes_{\mathbb{Z}[G]} d)(1 \otimes_{\mathbb{Z}[G]} [e|e|y] + 1 \otimes_{\mathbb{Z}[G]} [y|e|e]) = \{e, y\}.$$

Example 2. Let $x, y, c \in G$ satisfy $[x, c] = [y, c] = e$. Consider the identity

$$s_x [s_y, s_c] s_x^{-1} [s_x, s_c] = [s_x s_y, s_c] = (s_x s_y s_{xy}^{-1}) [s_{xy}, s_c] s_c (s_x s_y s_{xy}^{-1})^{-1} s_c^{-1}.$$

Let w_1, w_2 denote the left and right hand sides of this identity. Then

$$W(w_1) = 1 \otimes_{\mathbb{Z}[G]} [x|y|c] - 1 \otimes_{\mathbb{Z}[G]} [x|c|y] + 1 \otimes_{\mathbb{Z}[G]} [e|x|c] - 1 \otimes_{\mathbb{Z}[G]} [e|c|x]$$

and

$$W(w_2) = 1 \otimes_{\mathbb{Z}[G]} [e|xy|c] - 1 \otimes_{\mathbb{Z}[G]} [e|c|xy] + 1 \otimes_{\mathbb{Z}[G]} [e|x|y] - 1 \otimes_{\mathbb{Z}[G]} [c|x|y].$$

Thus by Theorem 3.8 we have the boundary relation

$$(1 \otimes_{\mathbb{Z}[G]} d)(W(w_1) - W(w_2)) = \{x, c\} + \{y, c\} - \{xy, c\}.$$

Adding the boundary of the 4-chain

$$1 \otimes_{\mathbb{Z}[G]} [e|x|y|c] + 1 \otimes_{\mathbb{Z}[G]} [e|c|x|y] - 1 \otimes_{\mathbb{Z}[G]} [e|x|c|y]$$

to $W(w_1) - W(w_2)$ we get a 3-chain with the same boundary:

$$\begin{aligned} (1 \otimes_{\mathbb{Z}[G]} d)(1 \otimes_{\mathbb{Z}[G]} [x|y|c] - 1 \otimes_{\mathbb{Z}[G]} [x|c|y]) + 1 \otimes_{\mathbb{Z}[G]} [c|x|y] \\ = \{x, c\} + \{y, c\} - \{xy, c\}. \end{aligned}$$

Example 3. Let $x, y, c \in G$ satisfy $cx c^{-1} = y, cy c^{-1} = x, xy = yx$. Then we have

$$\begin{aligned} s_c (s_x s_y s_{xy}^{-1})(s_y s_x s_{yx}^{-1})^{-1} s_c^{-1} \\ = [(s_c s_x s_{cx}^{-1})(s_y s_c s_{yc}^{-1})^{-1}, s_y s_c s_y s_c^{-1} s_y^{-1}](s_y s_x s_{yx}^{-1})(s_x s_y s_{xy}^{-1})^{-1} \\ [s_x s_y s_{xy}^{-1}, (s_c s_y s_{cy}^{-1})(s_x s_c s_{xc}^{-1})^{-1}]. \end{aligned}$$

Again let w_1, w_2 denote the left and right hand sides of this identity. Theorem 3.8 gives

$$(1 \otimes_{\mathbb{Z}[G]} d)(W(w_1) - W(w_2)) = 2\{x, y\}.$$

Applying W to w_1, w_2 gives

$$\begin{aligned} W(w_1) &= 1 \otimes_{\mathbb{Z}[G]} [c|x|y] - 1 \otimes_{\mathbb{Z}[G]} [c|y|x] \\ W(w_2) &= 1 \otimes_{\mathbb{Z}[G]} [e|c|x] - 1 \otimes_{\mathbb{Z}[G]} [e|y|c] \\ &\quad - 1 \otimes_{\mathbb{Z}[G]} [x|c|x] + 1 \otimes_{\mathbb{Z}[G]} [x|y|c] \\ &\quad + 1 \otimes_{\mathbb{Z}[G]} [e|y|x] - 1 \otimes_{\mathbb{Z}[G]} [e|x|y] \\ &\quad + 1 \otimes_{\mathbb{Z}[G]} [y|c|y] - 1 \otimes_{\mathbb{Z}[G]} [y|x|c] \\ &\quad - 1 \otimes_{\mathbb{Z}[G]} [e|c|y] + 1 \otimes_{\mathbb{Z}[G]} [e|x|c]. \end{aligned}$$

We have a 4-chain

$$\begin{aligned} T &= 1 \otimes_{\mathbb{Z}[G]} [e|y|x|c] + 1 \otimes_{\mathbb{Z}[G]} [e|x|c|x] + 1 \otimes_{\mathbb{Z}[G]} [e|c|x|y] \\ &\quad - 1 \otimes_{\mathbb{Z}[G]} [e|x|y|c] - 1 \otimes_{\mathbb{Z}[G]} [e|y|c|y] - 1 \otimes_{\mathbb{Z}[G]} [e|c|y|x]. \end{aligned}$$

Let

$$\begin{aligned} P &= W(w_1) - W(w_2) + (1 \otimes_{\mathbb{Z}[G]} d)T \\ &= 1 \otimes_{\mathbb{Z}[G]} [c|x|y] - 1 \otimes_{\mathbb{Z}[G]} [x|y|c] - 1 \otimes_{\mathbb{Z}[G]} [c|y|x] \\ &\quad + 1 \otimes_{\mathbb{Z}[G]} [y|x|c] + 1 \otimes_{\mathbb{Z}[G]} [x|c|x] - 1 \otimes_{\mathbb{Z}[G]} [y|c|y]. \end{aligned}$$

Thus we have a boundary relation $(1 \otimes_{\mathbb{Z}[G]} d)(P) = 2\{x, y\}$.

3.4 The strategy for the universal chain

Recall from §2.2 that $A = \mathbb{Z}[t, t^{-1}]$ and $a = g_{12}^t$, $b = g_{13}^{-t}$. Our aim in this section is to describe the idea behind the construction of the chain Z_1 , which satisfies $d'(Z_1) = [a|b] - [b|a]$. Then in §3.5 we describe the actual process of constructing it.

In order to construct this boundary relation we employ the method given in §3.2 and used in Examples 1, 2, 3 of §3.3. In this case our group is $E(A)$ and we seek an identity in the letters of the free group generated by elements of $E(A)$:

$$C_1(s_a s_b s_{ab}^{-1})(s_b s_a s_{ab}^{-1})^{-1} C_2 = C_3 \quad (4)$$

where, as before C_1, C_2, C_3 are products of commutators $[R, u]^{\pm 1}$, with R a relator.

In this subsection we will describe the idea behind the construction of (4). Then in §3.5 we will go through the stages in its construction.

For $i \neq j$ and $\mu \in A$, let $E_{ij}^\mu \in E(A)$ differ from the identity in $E(A)$ in the (i, j) th entry only, which is μ . Let $F_E \subset F_{E(A)}$ be the subgroup generated by the $s_{E_{ij}^\mu}$.

The Steinberg group $St(A)$ is defined to be the group generated by letters X_{ij}^μ , subject to the Steinberg relations $S_{ij}^{\mu, \nu} = T_{ijk}^{\mu, \nu} = U_{ijkl}^{\mu, \nu} = e$, where

$$\begin{aligned} S_{ij}^{\mu, \nu} &= (X_{ij}^{\mu+\nu})^{-1} X_{ij}^\mu X_{ij}^\nu, & i &\neq j \\ T_{ijk}^{\mu, \nu} &= X_{ij}^\mu X_{jk}^\nu (X_{ij}^\mu)^{-1} (X_{jk}^\nu)^{-1} (X_{ik}^{\mu\nu})^{-1}, & i, j, k &\text{ distinct} \\ U_{ijkl}^{\mu, \nu} &= X_{ij}^\mu X_{kl}^\nu (X_{ij}^\mu)^{-1} (X_{kl}^\nu)^{-1}, & i \neq l, j \neq k, i &\neq j, k \neq l. \end{aligned}$$

The homomorphism $\psi: St(A) \rightarrow E(A)$ mapping $X_{ij}^\mu \mapsto E_{ij}^\mu$ is clearly surjective and its kernel $K_2(A)$ is central in $St(A)$ ([20], p.40, Theorem 5.1). Thus given $x, y \in E(A)$ and preimages $x', y' \in St(A)$, the commutator $[x', y']$ is independent of the choice of preimages, and may be denoted $\{x, y\}$ (this differs from our earlier notation).

The first step in our construction is to express $\{a, b\}$ explicitly as a product of the X_{ij}^μ . This is done by writing a and b as products of the E_{ij}^μ , then replacing each E_{ij}^μ with X_{ij}^μ , to get $a', b' \in St(A)$. Then $\{a, b\}$ is represented by the word $[a', b']$.

In [20] pp. 71–75 it is shown that $\{a, b\} = e$ in $St(A)$. This means that we have an identity in the free group generated by the letters X_{ij}^μ :

$$[a', b'] = (w_1 R_1^{\pm 1} w_1^{-1})(w_2 R_2^{\pm 1} w_2^{-1}) \cdots (w_m R_m^{\pm 1} w_m^{-1}) \quad (5)$$

where the R_i are of the form $S_{ij}^{\mu, \nu}$, $T_{ijk}^{\mu, \nu}$ or $U_{ijkl}^{\mu, \nu}$.

Let $R_E \subset F_E$ be the subgroup $\ker(\phi|_{F_E})$, and let $\hat{\theta}: F_E \rightarrow St(A)$ be given by $s_{E_{ij}^\mu} \mapsto X_{ij}^\mu$. Then $\hat{\theta}$ is surjective and by construction the following diagram commutes:

$$\begin{array}{ccc} F_E & \xrightarrow{\hat{\theta}} & St(A) \\ & \searrow \phi|_{F_E} & \downarrow \psi \\ & & E(A). \end{array}$$

Hence $\hat{\theta}(R_E) \subset \ker(\psi)$ which is, as we have said, central. Hence $\hat{\theta}([F_E, R_E]) = \{e\}$ so $\hat{\theta}$ induces a well defined map $\theta: F_E/[F_E, R_E] \rightarrow St(A)$ mapping $s_{E_{ij}^\mu} \mapsto X_{ij}^\mu$.

Note that $[F_E, R_E] \subset R_E$ so ϕ_{F_E} induces a well defined map $\phi: F_E/[F_E, R_E] \rightarrow E(A)$ and the following diagram also commutes:

$$\begin{array}{ccc} F_E/[F_E, R_E] & \xrightarrow{\theta} & St(A) \\ & \searrow \phi & \downarrow \psi \\ & & E(A) \end{array}$$

where ϕ here is understood to denote the map induced by the restriction.

The kernel of θ is contained in $\ker(\psi\theta) = R_E/[F_E, R_E]$, so is central in $F_E/[F_E, R_E]$. Thus θ is a central extension. From [20] pp.48–51 applied to θ we get:

Lemma 3.9. *i) The element $[s_{E_{ik}^1}, s_{E_{kj}^\mu}] \in F_E/[F_E, R_E]$ is independent of $k \neq i, j$.
ii) The map θ has a section c , given by $c(X_{ij}^\mu) = [s_{E_{ik}^1}, s_{E_{kj}^\mu}]$, $k \neq i, j$.*

In particular c respects the Steinberg relations. Thus each $c(R_i) = e \in F_E/[F_E, R_E]$ and may be expressed as a product of commutators $[u, v]^{\pm 1}$, with $v \in R_E$.

We apply c to (5) and take conjugation by $c(w_i)$ inside the commutators to get

$$[c(a'), c(b')] = C_3$$

where C_3 has the required form of a product of commutators, each involving a relator. We have $\phi c(a') = \psi\theta c(a') = \psi(a') = a$ and $\phi c(b') = \psi\theta c(b') = \psi(b') = b$, so $K_a = c(a')s_a^{-1}$ and $K_b = c(b')s_b^{-1}$ are relators. We have

$$[c(a'), c(b')] = [K_a s_a, K_b s_b] = [K_a, s_a s_b s_a^{-1}](s_a s_b s_{ab}^{-1})(s_b s_a s_{ab}^{-1})^{-1}[s_b K_a s_a s_b^{-1}, K_b].$$

Let $C_1 = [K_a, s_a s_b s_a^{-1}]$ and $C_2 = [s_b K_a s_a s_b^{-1}, K_b]$. Then we have constructed (4) as

$$C_1(s_a s_b s_{ab}^{-1})(s_b s_a s_{ab}^{-1})^{-1}C_2 = [c(a'), c(b')] = C_3. \quad (6)$$

3.5 Computing the free group identity

We shall now describe how to implement the strategy of §3.4. The first step is to find $a', b' \in St(A)$. For λ a unit in A , let $Y_{ij}^\lambda = X_{ij}^\lambda X_{ji}^{-(\lambda^{-1})} X_{ij}^\lambda$. Factorizing g_{ij}^λ into matrices of the form E_{ij}^μ and replacing each E_{ij}^μ with X_{ij}^μ , we get $Y_{ij}^\lambda Y_{ij}^{-1}$, which we will denote h_{ij}^λ . Thus $\psi(h_{ij}^\lambda) = g_{ij}^\lambda$ and in particular $a' = h_{12}^t, b' = h_{13}^{-t}$.

In [20], pp. 71–75, it is shown that the commutator $[h_{12}^t, h_{13}^{-t}]$ may be reduced via the Steinberg relations to e . This proof depends on the identities $h_{13}^{-t} Y_{12}^\lambda (h_{13}^{-t})^{-1} = Y_{12}^{-t\lambda}$ for $\lambda = t$ or -1 , and $Y_{12}^t Y_{12}^{-1} Y_{12}^{-t} = Y_{12}^{-t^2}$, which themselves depend on several identities proved in [20], Lemma 9.2. The proof of each of these is given by taking words in the X_{ij}^μ and simplifying them using the Steinberg identities.

Thus the entire proof may be written out as a sequence of equalities in the Steinberg group, where at each step a simplification is made using one of the Steinberg identities. Of course such a proof would be extremely long, as each step in proving an identity needs to be repeated every time the identity is used to prove a consequential identity. If the consequential identity is used several times to prove an identity higher up the chain of consequences then one can appreciate how the length of such a proof grows exponentially with the length of the proof given in [20].

Thus we have a long chain of equalities in the Steinberg group $[h_{12}^t, h_{13}^{-t}] = \dots = e$, where at each step we essentially factor off a conjugate of one of the relators $S_{ij}^{\mu, \nu}, T_{ijk}^{\mu, \nu}, U_{ijkl}^{\mu, \nu}$ or their inverses. Next we must write the entire proof out again, this time not suppressing these factors, so we have a sequence of equalities in the free group on the letters X_{ij}^μ . This long nested sequence of operations, together with the vast amount of data needed to store all these factors, made it natural to employ a computer to construct the resulting identity:

$$[a', b'] = [h_{12}^t, h_{13}^{-t}] = \dots = \prod_{i=1}^m (w_i R_i^{\pm 1} w_i^{-1}) \quad (7)$$

where the R_i are words of the form $S_{ij}^{\mu,\nu}$, $T_{ijk}^{\mu,\nu}$ or $U_{ijkl}^{\mu,\nu}$ and $m = 392$. The sequence of letters in the product would fill about 40 pages. We note that none of the relations used required us to include extra indices, so only 1, 2, and 3 were used.

We next apply the homomorphism c . Explicitly, this means replacing each X_{ij}^μ in (7) by $[s_{E_{i4}^1}, s_{E_{4j}^\mu}]$. This is now a free group identity in $F_E \subset F_{E(A)}$:

$$[c(a'), c(b')] = \prod_{i=1}^m (c(w_i) c(R_i)^{\pm 1} c(w_i)^{-1}). \quad (8)$$

From [20], pp. 48-51 we know that $c: St(A) \rightarrow F_E/[F_E, R_E]$ is a well defined homomorphism. Thus as the words $S_{ij}^{\mu,\nu}$, $T_{ijk}^{\mu,\nu}$ and $U_{ijkl}^{\mu,\nu}$ represent $e \in St(A)$, we have that the words $c(S_{ij}^{\mu,\nu})$, $c(T_{ijk}^{\mu,\nu})$, $c(U_{ijkl}^{\mu,\nu}) \in F_E$ represent $e \in F_E/[F_E, R_E]$.

The proofs of these three identities ([20] pp.49-51) can be written as a sequence of equalities in $F_E/[F_E, R_E]$, where at each step we factor off a word in $[F_E, R_E]$. As before, by not suppressing these factors we obtain identities in the free group F_E , equating the $c(S_{ij}^{\mu,\nu})$, $c(T_{ijk}^{\mu,\nu})$, $c(U_{ijkl}^{\mu,\nu}) \in F_E$ with elements of $[F_E, R_E]$. For example:

$$\begin{aligned} c(U_{ijkl}^{\mu,\nu}) &= [s_{E_{i4}^1}, s_{E_{4j}^\mu}], [s_{E_{k4}^1}, s_{E_{4l}^\nu}] \\ &= [L', s_{E_{ij}^\mu} [s_{E_{k4}^1}, s_{E_{4l}^\nu}] (s_{E_{ij}^\mu})^{-1}] [L'', s_{E_{k4}^1} [s_{E_{ij}^\mu}, s_{E_{4l}^\nu}] s_{E_{4l}^\nu} (s_{E_{k4}^1})^{-1}] \\ &\quad [L''', s_{E_{k4}^1} s_{E_{4l}^\nu} (s_{E_{k4}^1})^{-1} (s_{E_{4l}^\nu})^{-1} (s_{E_{k4}^1})^{-1}] \end{aligned}$$

where L' is the relator $[s_{E_{i4}^1}, s_{E_{4j}^\mu}] (s_{E_{ij}^\mu})^{-1}$, L'' is the relator $[s_{E_{k4}^1}, s_{E_{4l}^\nu}]$ and L''' is the relator $s_{E_{k4}^1} [s_{E_{ij}^\mu}, s_{E_{4l}^\nu}] (s_{E_{k4}^1})^{-1}$. Thus $c(U_{ijkl}^{\mu,\nu})$ may be expressed as the product of 3 commutators, each involving a relator. Similarly, we may derive expressions for $c(S_{ijkl}^{\mu,\nu})$, $c(T_{ijkl}^{\mu,\nu})$ expressing them as the product of 4 and 18 commutators respectively, all involving a relator. During the expansion of $T_{ijkl}^{\mu,\nu}$ it was necessary to introduce the index 5, so from now on we work with 5×5 matrices.

Each $c(R_i)$ in (8) may now be expressed as a product: $c(R_i) = \prod_{j=1}^{m_i} [L_{ij}, y_{ij}]^{\pm 1}$, so

$$\begin{aligned} (c(w_i) c(R_i)^{\pm 1} c(w_i)^{-1}) &= c(w_i) \left(\prod_{j=1}^{m_i} [L_{ij}, y_{ij}]^{\pm 1} \right)^{\pm 1} c(w_i)^{-1} \\ &= \left(\prod_{j=1}^{m_i} [c(w_i) L_{ij} c(w_i)^{-1}, c(w_i) y_{ij} c(w_i)^{-1}]^{\pm 1} \right)^{\pm 1} \end{aligned}$$

and from (8)

$$[c(a'), c(b')] = \prod_{i=1}^m \left(\prod_{j=1}^{m_i} [c(w_i) L_{ij} c(w_i)^{-1}, c(w_i) y_{ij} c(w_i)^{-1}]^{\pm 1} \right)^{\pm 1} = C_3.$$

Here C_3 is the product of 2392 commutators, each involving a lengthy relator and word. Conversely $C_1 = [K_a, s_a s_b s_a^{-1}]$ and $C_2 = [s_b K_a s_a s_b^{-1}, K_b]$ are much shorter.

In order to apply the operation W to both sides of (6) we must rewrite the relator in each commutator in C_1, C_2, C_3 as a product of words of the form $(s_x s_y s_{xy}^{-1})^{\pm 1}$. We may use the inductive process from the proof of Lemma 3.5 to do this. The number of terms of the form $(s_x s_y s_{xy}^{-1})^{\pm 1}$ needed to express each relator will be approximately the length of the relator, and each such term will add 2 terms to the 3-cycle produced by the application of W (one coming from the relator and the other from its inverse).

We wrote a computer algorithm to work through the left and right hand sides of (6) (which we will denote u_1, u_2 respectively) applying W . Clearly this would produce millions of terms. Whenever a new term was produced, we stored it in memory, and thereafter merely kept a running total of the coefficient on it. Thus we obtained $W(u_2)$ as a 3-chain with 11123 terms, which between them involve 3691 distinct 5×5 matrices. Subtracting from the much smaller $W(u_1)$ we apply Theorem 3.8 to get:

$$d'(W(v_1) - W(v_2)) = \{a, b\}.$$

We next ran an algorithm on $W(u_1) - W(u_2)$ searching for boundaries of 4-cycles which could be added or subtracted to shorten it. Let Z_1 denote the result after adding

and subtracting those boundaries. This has merely 6844 distinct terms, involving between them 3265 matrices. Thus we have attained the desired boundary relation:

$$d'Z_1 = [a|b] - [b|a]$$

Remark 4. The operator d' was applied to Z_1 by computer, to independently verify this identity.

Remark 5. A computer file containing an explicit description of the universal chain Z_1 is available in either PDF or L^AT_EX format².

4 Computing the universal Borel class

As an independent result we show how to compute the universal Borel class b_n evaluated in the homology group $H_{2n+1}(GL(\mathbb{C}))$ by expanding the Karoubi-Hamida integral of [14]. This approach allows the calculation of the Borel regulator map for any number field after evaluation of the Hurewicz homomorphism. In particular for $n = 1$ and F a cyclotomic field, we can compute $V_1(F)$ (Theorem 2.11).

This section is organised as follows. We begin by recalling the definition of the Karoubi-Hamida integral (§4.1). We expand this integral in an arbitrary odd dimension as a infinite series (§4.2). The formula requires a certain matrix A to have norm less than 1; in §4.3 we explain how to guarantee this condition. We then simplify the formula in dimension 3 (§4.4) in a way that can be implemented straightaway in a computer algorithm. Finally we discuss some computational aspects of this algorithm (§4.5).

4.1 Karoubi-Hamida's integral

By a result of Hamida [14], the universal Borel class b_m has the following description as an integral of differential forms. Let $n = 2m + 1$ and let $X_0, \dots, X_n \in GL_N(\mathbb{C})$, for some $N \geq 2n + 1$ (cf. Remark 1). Let Δ_n be the standard n -simplex

$$\Delta_n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum_i x_i = 1 \right\}. \quad (9)$$

Define for every point $\mathbf{x} = (x_0, \dots, x_n) \in \Delta_n$

$$\nu(\mathbf{x}) = x_0 X_0^* X_0 + \dots + x_n X_n^* X_n, \quad (10)$$

where X^* denotes the conjugate transpose of X . Thus ν is a matrix of 0-forms (complex functions) on the n -manifold Δ_n . For any \mathbf{x} and any non-zero vector $\vec{u} \in \mathbb{C}^N$, we have $\vec{u}^* \nu(\mathbf{x}) \vec{u}$ a positive real number. That is $\nu(\mathbf{x})$ is positive definite hermitian and in particular invertible. Consider the matrix of differential n -forms $(\nu^{-1} d\nu)^n$, where ν^{-1} denotes matrix inversion, d is the exterior derivative applied to each entry of ν and we multiply individual differential forms using the wedge product. Define

$$\varphi(X_0, X_1, \dots, X_n) = \text{Tr} \int_{\Delta_n} (\nu^{-1} d\nu)^n, \quad (11)$$

the trace of a matrix of integrals of differential n -forms.

Theorem 4.1 (Hamida [14]). *Let $m \geq 1$ and $n = 2m + 1$. The map defined on tuples in the homogeneous bar resolution, sending*

$$(X_0, \dots, X_n) \mapsto \frac{(-1)^{m+1}}{2^{3m+1} (\pi i)^m} \varphi(X_0, X_1, \dots, X_n) \quad (12)$$

is a cocycle representing the universal Borel class $b_m: H_{2m+1}(GL(\mathbb{C})) \rightarrow \mathbb{R}(m)$.

²<http://www.personal.soton.ac.uk/rsg1y09/universalchain.pdf>
<http://www.personal.soton.ac.uk/rsg1y09/universalchain.tex>

Remark 6. The cocycle (12) is homogeneous and unitarily normalized [14], that is,

$$\varphi(X_0g, \dots, X_n g) = \varphi(X_0, \dots, X_n) \quad \text{for all } g \in GL_N(\mathbb{C}), \quad (13)$$

$$\varphi(u_0 X_0, \dots, u_n X_n) = \varphi(X_0, \dots, X_n) \quad \text{for all } u_i \in U_N(\mathbb{C}). \quad (14)$$

In particular, we can assume $X_n = 1$ by (13), and all the X_i to be positive definite hermitian matrices by (14), via the polar decomposition: every invertible matrix X can be written as $X = UP$ where U is unitary and P is positive definite hermitian. Indeed, $X^*X = P^*P = P^2$; compare with (10).

4.2 The infinite series formula

Our goal is to make the computation of the Karoubi-Hamida integral (11) explicit. Namely, we will transform the integral into an infinite series whose value we can arbitrarily approximate.

Step 1 *Express the integrand in terms of n coordinates rather than $n + 1$*

We have a homeomorphism from the n -simplex in \mathbb{R}^n

$$\Delta^n = \{(y_1, y_2, \dots, y_n) \mid y_i \geq 0 \text{ for all } i, \sum_{j=1}^n y_j \leq 1\}$$

to $\Delta_n \subset \mathbb{R}^{n+1}$ given by the map

$$(y_1, \dots, y_n) \mapsto (1 - \sum_{j=1}^n y_j, y_1, y_2, \dots, y_n).$$

Therefore, in terms of the y -coordinates, we have

$$\nu = X_0^* X_0 + y_1 (X_1^* X_1 - X_0^* X_0) + \dots + y_n (X_n^* X_n - X_0^* X_0). \quad (15)$$

Step 2 *Change of variables*

Next, we perform a change of variables by means of the map

$$T: [0, 1] \times \Delta^{n-1} \longrightarrow \Delta^n$$

given by

$$T(t, s_1, s_2, \dots, s_{n-1}) = (s_1 t, s_2 t, \dots, s_{n-1} t, 1 - t).$$

For each fixed non-zero value of t the corresponding horizontal $(n - 1)$ -simplex is mapped diffeomorphically onto its image, while $\{0\} \times \Delta^{n-1}$ is collapsed to the vertex $(0, 0, \dots, 0, 1)$. The Jacobian of T is equal to $(-1)^n t^{n-1}$. Therefore when $t \neq 0$ we have $J(T) > 0$ if n is even and $J(T) < 0$ if n is odd. For any compact n -manifold with boundary $M \subset [0, 1] \times \Delta^{n-1}$ having image $T(M)$ in the n -simplex and continuous map $f: T(M) \rightarrow \mathbb{R}$ we have the substitution rule [4, p. 28]

$$\begin{aligned} \int_M (f \circ T)(t, s_1, \dots, s_{n-1}) t^{n-1} dt ds_1 \dots ds_{n-1} \\ = \int_{T(M)} f(y_1, \dots, y_n) dy_1 \dots dy_n, \end{aligned}$$

since the absolute value of the Jacobian is just t^{n-1} .

We shall be interested in the integral $\text{Tr} \int_{\Delta^n} ((\nu)^{-1} d\nu)^n$ which may be written as the limit of integrals over manifolds of the form $T(M)$ as they tend towards the n -simplex. (For example, when $M_\alpha = [\alpha, 1] \times \Delta^{n-1}$ and $\alpha \rightarrow 0^+$.) Therefore we can compute this integral as a limit of corresponding integrals over $M \subset [0, 1] \times \Delta^{n-1}$, provided that this limit exists. However the integral over M , involving the Jacobian of T , is merely

the integral over M where $\nu \circ T$ is ν written in terms of t, s_1, \dots, s_{n-1} and $d\nu$ is also computed in these coordinates. Thus from (15) and the definition of T we have

$$\begin{aligned}\nu \circ T &= X_0^* X_0 + \sum_{j=1}^{n-1} t s_j (X_j^* X_j - X_0^* X_0) + (1-t)(X_n^* X_n - X_0^* X_0) \\ &= X_n^* X_n + tA(s_1, \dots, s_{n-1}),\end{aligned}$$

where

$$A(s_1, s_2, \dots, s_{n-1}) = (X_0^* X_0 - X_n^* X_n) + \sum_{j=1}^{n-1} s_j (X_j^* X_j - X_0^* X_0). \quad (16)$$

Therefore we shall compute

$$\mathrm{Tr} \int_M ((\nu')^{-1} d\nu')^n (-1)^n t^{n-1}$$

where $\nu' = \nu \circ T$, that is,

$$\nu'(t, s_1, \dots, s_n) = X_n^* X_n + tA(s_1, \dots, s_{n-1})$$

for all $(t, s_1, \dots, s_{n-1}) \in M$ where $M \subset [0, 1] \times \Delta^{n-1}$ is an arbitrary compact n -manifold with boundary.

Step 3 Assume $X_n = 1$

Taking $g = X_n^{-1}$ in (13) we may from now on assume $X_n = 1$, and hence

$$\begin{aligned}\nu(t, s_1, \dots, s_{n-1}) &= 1 + tA(s_1, \dots, s_{n-1}), \\ A(s_1, \dots, s_{n-1}) &= (X_0^* X_0 - 1) + \sum_{j=1}^{n-1} s_j (X_j^* X_j - X_0^* X_0).\end{aligned} \quad (17)$$

(We write 1 for the identity matrix whenever there is no possibility of confusion.) Therefore $d\nu = dt A + t dA$, with $dA = \sum_{j=1}^{n-1} ds_j (X_j^* X_j - X_0^* X_0)$ and as M varies, we must examine the integral

$$\mathrm{Tr} \int_M t^{n-1} ((1 + tA)^{-1} (dt A + t dA))^n. \quad (18)$$

Step 4 *Commuting factors and cyclic permutations*

We have

$$\begin{aligned}\mathrm{Tr} \int_M t^{n-1} ((1 + tA)^{-1} (dt A + t dA))^n \\ = \mathrm{Tr} \int_M t^{n-1} ((1 + tA)^{-1} dt A + (1 + tA)^{-1} t dA)^n.\end{aligned}$$

Write $Y = (1 + tA)^{-1} dt A$ and $Z = (1 + tA)^{-1} t dA$. The 1-form dt commutes with the 0-forms $(1 + tA)^{-1}$ and t , and anticommutes with the 1-form dA . In the expansion of $(Y + Z)^n$, any monomial involving more than one Y will vanish, as it contains $dt dt = 0$. In addition, $Z^n = 0$ as it is an n -form on $n - 1$ variables s_1, \dots, s_{n-1} . Consequently,

$$\mathrm{Tr} \int_M (Y + Z)^n = \sum_{j=0}^{n-1} \mathrm{Tr} \int_M Z^j Y Z^{n-1-j}. \quad (19)$$

Next we observe that if W_1, \dots, W_n are $m \times m$ matrix-valued functions then

$$\mathrm{Tr} (W_1 dx_1 W_2 dx_2 \dots W_n dx_n) = (-1)^{n-1} \mathrm{Tr} (W_2 dx_2 \dots W_n dx_n W_1 dx_1) \quad (20)$$

because the trace of a product of n matrices is invariant under cyclic permutations but the n -form $dx_1 \dots dx_n$ changes by the sign of the n -cycle, which is $(-1)^{n-1}$. Accordingly

each integral in the sum (19) equals $(-1)^{n-1}$ the next one. This implies that if n is even the sum is zero, that is,

$$\varphi(X_0, X_1, \dots, X_{2n}) = \text{Tr} \int_{\Delta^{2n}} (\nu^{-1} d\nu)^{2n} = 0.$$

For the rest of this section we shall assume that $n \geq 3$ is odd (for the case $n = 1$ see Remark 8). In this case all the summands in (19) are equal and consequently

$$\begin{aligned} & \text{Tr} \int_M t^{n-1} ((1+tA)^{-1} (dtA + t dA))^n \\ &= n \text{Tr} \int_M t^{n-1} (1+tA)^{-1} dtA ((1+tA)^{-1} t dA)^{n-1} \\ &= n \text{Tr} \int_M t^{2n-2} (1+tA)^{-1} dtA ((1+tA)^{-1} dA)^{n-1}. \end{aligned}$$

Now A commutes with $1+tA$ and hence with its inverse. Thus the previous integral equals

$$\begin{aligned} & n \text{Tr} \int_M t^{2n-2} dtA (1+tA)^{-1} ((1+tA)^{-1} dA)^{n-1} \\ &= n \text{Tr} \int_M t^{2n-2} dtA (1+tA)^{-2} (dA (1+tA)^{-1})^{n-2} dA. \quad (21) \end{aligned}$$

Step 5 *Invert $1+tA$*

Recall the geometric series formula for a matrix A [15, 5.6.16]: if $\|\cdot\|$ is a matrix norm and $\|A\| < 1$ then $1-A$ is invertible and $\sum_{k=0}^{\infty} A^k = (1-A)^{-1}$ with respect to $\|\cdot\|$. (A *matrix norm* on $M_N(\mathbb{C})$ is a vector norm which satisfies $\|XY\| \leq \|X\|\|Y\|$.) In order to invert $1+tA$, we assume from now on that $\|A\| < 1$ through the domain of integration. There is a justification, explained in §4.3, which allows us to do so.

Remark 7. By $\|A\| < 1$ we formally mean $\|A(s_1, \dots, s_{n-1})\| < 1$ for all $(s_1, \dots, s_{n-1}) \in \Delta^{n-1}$. Equivalently, we may define $\|A\|$ as the maximum of this function over the compact set Δ^{n-1} .

The geometric series for $-tA$ gives

$$(1+tA)^{-1} = \sum_{k=0}^{\infty} (-tA)^k = \sum_{k=0}^{\infty} (-1)^k t^k A^k,$$

and it follows that

$$(1+tA)^{-2} = \sum_{k=0}^{\infty} (-1)^k (k+1) t^k A^k.$$

Hence under the assumption $\|A\| < 1$ we may express the integral (21) as a convergent infinite series in the following manner.

$$\begin{aligned} & \text{Tr} \int_M ((1+tA)^{-1} (dtA + t dA))^n \\ &= n \text{Tr} \int_M t^{2n-2} dtA (1+tA)^{-2} dA (1+tA)^{-1} dA \dots (1+tA)^{-1} dA \\ &= n \text{Tr} \int_M t^{2n-2} dtA \sum_{m_i \geq 0} (-1)^{m_1} (m_1+1) (tA)^{m_1} dA \dots (-1)^{m_{n-1}} (tA)^{m_{n-1}} dA \\ &= n \text{Tr} \int_M \sum_{m_i \geq 0} (-1)^{\underline{m}} (m_1+1) t^{\underline{m}+2n-2} dtA^{m_1+1} dA A^{m_2} dA \dots A^{m_{n-1}} dA \\ &= n \text{Tr} \int_M \sum_{m_i \geq 0} (-1)^{\underline{m}-1} m_1 t^{\underline{m}+2n-3} dtA^{m_1} dA A^{m_2} dA \dots A^{m_{n-1}} dA, \end{aligned}$$

where \underline{m} is our short notation for $\underline{m}(m_1, \dots, m_{n-1}) = m_1 + \dots + m_{n-1}$.

For $0 < \alpha < 1$ write $M_\alpha = [\alpha, 1] \times \Delta^{n-1}$. The conditional convergence and Fubini theorems guarantee that the original integral over Δ^n equals

$$\begin{aligned} & \lim_{\alpha \rightarrow 0^+} \int_{M_\alpha} \sum_{m_i \geq 0} (-1)^{m-1} m_1 t^{\underline{m}+2n-3} dt A^{m_1} dA A^{m_2} dA \dots A^{m_{n-1}} dA \\ &= \sum_{m_i \geq 0} \frac{(-1)^{m-1} m_1}{\underline{m} + 2n - 2} \int_{\Delta^{n-1}} A^{m_1} dA A^{m_2} dA \dots A^{m_{n-1}} dA, \end{aligned}$$

since

$$\lim_{\alpha \rightarrow 0^+} \int_\alpha^1 t^{\underline{m}+2n-3} dt = \frac{1}{\underline{m} + 2n - 2}.$$

Step 6 *Expand the powers of A*

Let us write

$$A = U_0 + \sum_{j=1}^{n-1} U_j s_j$$

where $U_0 = X_0^* X_0 - I$, $U_j = X_j^* X_j - X_0^* X_0$ for $1 \leq j \leq n-1$ as in Eq. (17). Then

$$dA = \sum_{j=1}^{n-1} U_j ds_j$$

and we can write

$$\begin{aligned} A^{m_1} dA \dots A^{m_{n-1}} dA &= \prod_{i=1}^{n-1} \left(U_0 + \sum_{j=1}^{n-1} U_j s_j \right)^{m_i} \left(\sum_{j=1}^{n-1} U_j ds_j \right) \\ &= \sum_{|\mathbf{l}| \leq |\mathbf{m}|} U(l_1, l_2, \dots, l_{n-1}) s_1^{l_1} s_2^{l_2} \dots s_{n-1}^{l_{n-1}} ds_1 ds_2 \dots ds_{n-1}. \end{aligned}$$

Here the sum is over all nonnegative integer vectors $\mathbf{l} = (l_1, \dots, l_{n-1})$ such that $|\mathbf{l}| = \sum_i l_i \leq \sum_i m_i = |\mathbf{m}|$, and $U(l_1, \dots, l_{n-1})$ is defined as the matrix coefficient of $s_1^{l_1} \dots s_{n-1}^{l_{n-1}}$ in the expansion of the previous line. Since this matrix depends on both \mathbf{l} and $\mathbf{m} = (m_1, \dots, m_{n-1})$, we will sometimes write $U(\mathbf{m}, \mathbf{l})$. It can be described more explicitly:

Lemma 4.2. *The matrix $U(\mathbf{m}, \mathbf{l})$ equals the sum of all the matrices of the form*

$$s \cdot V_1^1 \dots V_{m_1}^1 W_1 V_1^2 \dots V_{m_2}^2 W_2 \dots V_1^{n-1} \dots V_{m_{n-1}}^{n-1} W_{n-1}$$

where

1. $V_j^i \in \{U_k : 0 \leq k \leq n\}$ for all i, j ;
2. (W_1, \dots, W_{n-1}) is a permutation of (U_1, \dots, U_{n-1}) of signature $s \in \{\pm 1\}$;
3. $l_k = \{(i, j) \mid V_j^i = U_k\}$ for each k .

Step 7 *Remove the remaining integral*

Since the matrices $U(\mathbf{m}, \mathbf{l})$ are constant with respect to the variables s_i , we have

$$\begin{aligned} & \int_{\Delta^{n-1}} A^{m_1} dA A^{m_2} dA \dots A^{m_{n-1}} dA \\ &= \sum_{|\mathbf{l}| \leq |\mathbf{m}|} U(\mathbf{m}, \mathbf{l}) \int_{\Delta^{n-1}} s_1^{l_1} s_2^{l_2} \dots s_{n-1}^{l_{n-1}} ds_1 ds_2 \dots ds_{n-1}. \end{aligned}$$

Lemma 4.3.

$$\int_{\Delta^{n-1}} s_1^{l_1} \dots s_{n-1}^{l_{n-1}} ds_1 \dots ds_{n-1} = \frac{l_1! l_2! \dots l_{n-1}!}{(l_1 + l_2 + \dots + l_{n-1} + n - 1)!}.$$

To ease notation we write the number above as $\text{fact}(l_1, \dots, l_{n-1})$ or $\text{fact}(\mathbf{l})$. We leave the proof of the lemma as a multivariable calculus exercise.

On the whole we have proven the following.

Theorem 4.4. Let $n \geq 3$ odd and $X_0, \dots, X_n \in GL_N(\mathbb{C})$. Let

$$A = U_0 + \sum_{j=1}^{n-1} U_j s_j$$

where $U_0 = X_0^* X_0 - I$ and $U_j = X_j^* X_j - X_0^* X_0$ for $1 \leq j \leq n-1$. Suppose that $\|A\| < 1$. Then Hamida's function $\varphi(X_0, \dots, X_n)$ equals the limit of the convergent series

$$n \sum_{|\mathbf{m}|=0}^{\infty} \frac{(-1)^{|\mathbf{m}|-1}}{|\mathbf{m}| + 2n - 2} m_1 \sum_{|\mathbf{l}| \leq |\mathbf{m}|} \text{fact}(\mathbf{l}) \text{Tr}(U(\mathbf{m}, \mathbf{l}))$$

where

1. the outer sum is over all nonnegative integer vectors $\mathbf{m} = (m_1, \dots, m_{n-1})$, that is, the limit when $k \rightarrow \infty$ of the finite sums over $|\mathbf{m}| \leq k$;
2. the (finite) inner sum is over all nonnegative integer vectors $\mathbf{l} = (l_1, \dots, l_{n-1})$ such that $|\mathbf{l}| \leq |\mathbf{m}|$;
3. $\text{fact}(\mathbf{l})$ is as defined in Lemma 4.3;
4. $U(\mathbf{m}, \mathbf{l})$ is as defined in Lemma 4.2.

Remark 8 (Case $n = 1$). In this case we only need Steps 1 to 3 and the geometric series formula to invert $1 + tA$. The map T can be taken to be the identity map, $M = [0, 1]$ and note that the matrix A is constant. From (18) and assuming $\|A\| < 1$,

$$\begin{aligned} \text{Tr} \int_{\Delta^1} (\nu^{-1} d\nu) &= \text{Tr} \int_0^1 (1 + tA)^{-1} A dt \\ &= \text{Tr} \int_0^1 \sum_{m \geq 0} (-1)^m t^m A^m A dt \\ &= \text{Tr} \sum_{m \geq 0} (-1)^m A^{m+1} \int_0^1 t^m dt \\ &= \text{Tr}(\log(1 + A)). \end{aligned}$$

In Section 4.4 we will consider the case $n = 3$ in more detail (the relevant case for the computation of $V_1(F)$). Before that, we explain how to ensure the condition $\|A\| < 1$.

4.3 Controlling the norm

In order to use the geometric series to invert $1 + tA$ (Step 5 in §4.2) we need $\|A\| < 1$ for any matrix norm $\|\cdot\|$. We ensure this condition for the spectral norm by using a homological trick. First we briefly discuss matrix norms.

Let $\|\cdot\|$ a *matrix norm*, that is, a vector norm in $M_N(\mathbb{C})$ which satisfies $\|XY\| \leq \|X\| \|Y\|$. Our main example will be the *spectral norm* [15, 5.6.6]

$$\|X\|_2 = \max \left\{ \sqrt{\lambda} : \lambda \text{ is an eigenvalue of } X^* X \right\}.$$

(Note that $X^* X$ is positive semidefinite (hermitian) and hence all eigenvalues are real and nonnegative.) This norm satisfies

- (i) $\|X^*\| = \|X\|$ for all $X \in M_N(\mathbb{C})$;
- (ii) $\|X^* X\| = \|X\|^2$ for all $X \in M_N(\mathbb{C})$;
- (iii) if X is hermitian then $\|X\| = \max \{|\lambda| : \lambda \text{ is an eigenvalue of } X\}$;
- (iv) if X is positive semidefinite then $\|X\| = \lambda_{\max}(X)$ the maximum eigenvalue.

If X is hermitian, all its eigenvalues are real and we will write them in increasing order as

$$\lambda_{\min}(X) = \lambda_1(X) \leq \lambda_2(X) \leq \dots \leq \lambda_n(X) = \lambda_{\max}(X).$$

Lemma 4.5. Let X, Y be hermitian matrices. Then $\lambda_{\max}(X+Y) \leq \lambda_{\max}(X) + \lambda_{\max}(Y)$ and $\lambda_{\min}(X+Y) \geq \lambda_{\min}(X) + \lambda_{\min}(Y)$.

This Lemma follows from Theorem 4.3.1 in [15].

If X is a matrix of functions over a set $\Delta \subset \mathbb{R}^n$ such that $X(\mathbf{s})$ is hermitian for each $\mathbf{s} \in \Delta$, we define

$$\begin{aligned}\lambda_{\max}(X) &= \sup_{\mathbf{s} \in \Delta} \lambda_{\max}(X(\mathbf{s})), \\ \lambda_{\min}(X) &= \inf_{\mathbf{s} \in \Delta} \lambda_{\min}(X(\mathbf{s})), \\ \|X\| &= \sup_{\mathbf{s} \in \Delta} \|X(\mathbf{s})\|.\end{aligned}$$

This definition is consistent with our previous notation (Remark 7). Note that if $X(\mathbf{s})$ is positive definite for all \mathbf{s} then $\|X\| = \lambda_{\max}(X)$.

Let $X_0, \dots, X_n \in GL_N(\mathbb{C})$ and consider for each $\mathbf{s} = (s_1, \dots, s_{n-1}) \in \Delta^{n-1}$

$$A(\mathbf{s}) = X_0^* X_0 - I + \sum_{j=1}^n s_j (X_j^* X_j - X_0^* X_0).$$

We may write $A(\mathbf{s}) = H(\mathbf{s}) - I$ where

$$H(\mathbf{s}) = X_0^* X_0 \left(1 - \sum_{j=1}^n s_j\right) + \sum_{j=1}^n s_j X_j^* X_j$$

is a positive definite hermitian matrix (a positive linear combination of positive definite hermitian matrices). Define

$$\begin{aligned}\lambda_{\max} &= \max_{0 \leq j \leq n} \lambda_{\max}(X_j^* X_j) = \max_{i,j} \lambda_i(X_j^* X_j) \quad \text{and} \\ \lambda_{\min} &= \min_{0 \leq j \leq n} \lambda_{\min}(X_j^* X_j) = \min_{i,j} \lambda_i(X_j^* X_j).\end{aligned}$$

Lemma 4.6. *We have*

- (i) $\|H\| = \lambda_{\max}(H) = \lambda_{\max}$, $\lambda_{\min}(H) = \lambda_{\min}$;
- (ii) $\|A\| = \max\{|\lambda_{\max} - 1|, |\lambda_{\min} - 1|\}$;
- (iii) $\|A\| < 1$ if and only if $\lambda_{\max} < 2$.

Proof. Using Lemma 4.5 we have

$$\begin{aligned}\lambda_{\max}(H(\mathbf{s})) &\leq \left(1 - \sum_{j=1}^n s_j\right) \lambda_{\max}(X_0^* X_0) + \sum_{j=1}^n s_j \lambda_{\max}(X_j^* X_j) \leq \lambda_{\max} \quad \text{and} \\ \lambda_{\min}(H(\mathbf{s})) &\geq \left(1 - \sum_{j=1}^n s_j\right) \lambda_{\min}(X_0^* X_0) + \sum_{j=1}^n s_j \lambda_{\min}(X_j^* X_j) \geq \lambda_{\min}\end{aligned}$$

for all $\mathbf{s} = (s_1, \dots, s_{n-1}) \in \Delta^{n-1}$. This proves part (i).

For part (ii), we have

$$\|A\| = \sup_{\mathbf{s}} \|H(\mathbf{s}) - I\| = \sup_{\mathbf{s}} \max_i |\lambda_i(H(\mathbf{s}) - I)| = \sup_{\mathbf{s}, i} |\lambda_i(H(\mathbf{s})) - 1|.$$

The supremum of the distances between a point and the points of a bounded subset of \mathbb{R} equals the distance to either the supremum or the infimum of the set,

$$\|A\| = \max\{|\sup_{\mathbf{s}, i} (\lambda_i(H(\mathbf{s}))) - 1|, |\inf_{\mathbf{s}, i} (\lambda_i(H(\mathbf{s}))) - 1|\} = \max\{|\lambda_{\max} - 1|, |\lambda_{\min} - 1|\},$$

using part (i).

Part (iii) follows from (ii) observing that $0 < \lambda_{\min} \leq \lambda_{\max}$. □

We now explain how to guarantee the condition $\|A\| < 1$ by rescaling the matrices X_i . Let $\mu > 0$. The boundary of the tuple $(X_0, \dots, X_n, \mu^{-1}I)$ is

$$\sum_{i=0}^n (-1)^i (X_0, \dots, \widehat{X}_i, \dots, X_n, \mu^{-1}I) + (-1)^{n+1} (X_0, \dots, X_n).$$

Since Hamida's function φ is a cocycle, it vanishes on boundaries and thus

$$\begin{aligned}\varphi(X_0, \dots, X_n) &= \sum_{i=0}^n (-1)^{n+i} \varphi(X_0, \dots, \widehat{X}_i, \dots, X_n, \mu^{-1}I) \\ &= \sum_{i=0}^n (-1)^{n+i} \varphi(\mu X_0, \dots, \widehat{\mu X}_i, \dots, \mu X_n, I),\end{aligned}$$

the last equality coming from multiplying by a diagonal matrix with μ in the diagonal (Eq.(13) in Remark 6).

We now prove that if μ is small enough, the matrix A associated to any of the tuples on the right-hand side satisfies $\|A\| < 1$ for the spectral norm. Hence the value of Hamida's function φ at (X_0, \dots, X_n) can be computed as the alternating sum of the values at tuples satisfying the hypotheses of Theorem 4.4.

Proposition 4.7. *Let $X_0, \dots, X_n \in GL_N(\mathbb{C})$ and $\mu > 0$. Then*

$$\varphi(X_0, \dots, X_n) = \sum_{i=0}^n (-1)^{n+i} \varphi(\mu X_0, \dots, \widehat{\mu X}_i, \dots, \mu X_n, I).$$

Let $\lambda_{\max} = \max_{0 \leq j \leq n} \lambda_{\max}(X_j^* X_j)$. If $0 < \mu < \sqrt{\frac{2}{\lambda_{\max}}}$ then for each tuple on the right-hand side, the associated matrix A satisfies $\|A\| < 1$.

Proof. We are left with the proof of the second statement. Let A be the matrix associated to $(\mu X_0, \dots, \widehat{\mu X}_i, \dots, \mu X_n, I)$ for some $0 \leq i \leq n$. That is,

$$A(\mathbf{s}) = \mu^2 X_0^* X_0 - I + \mu^2 \sum_{j=1}^i s_j (X_j^* X_j - X_0^* X_0) + \mu^2 \sum_{j=i+1}^n s_{j-1} (X_j^* X_j - X_0^* X_0),$$

for each $\mathbf{s} = (s_1, \dots, s_{n-1}) \in \Delta^{n-1}$. Let us write $A(\mathbf{s}) = H(\mathbf{s}) - I$. By Lemma 4.6(i) we have

$$\lambda_{\max}(H) = \max_{j \neq i} \lambda_{\max}(\mu^2 X_j^* X_j) \leq \mu^2 \lambda_{\max} < 2.$$

The result follows now from Lemma 4.6(iii). \square

For computational purposes (cf. §4.5) we will be interested in minimizing $\|A\|$. We record the relevant result here, followed by a remark.

Lemma 4.8. *Let $X_0, \dots, X_n \in GL_N(\mathbb{C})$, $\lambda_{\max} = \max_{0 \leq j \leq n} \lambda_{\max}(X_j^* X_j)$, $\lambda_{\min} = \max_{0 \leq j \leq n} \lambda_{\min}(X_j^* X_j)$. Given $\mu > 0$ write A_μ for the matrix associated to the tuple $(\mu X_0, \dots, \mu X_n)$. Then the function $\mu \mapsto \|A_\mu\|$ reaches a minimum value $\frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}$ at $\mu = \sqrt{\frac{2}{\lambda_{\max} + \lambda_{\min}}}$.*

Proof. By Lemma 4.6(ii)

$$\|A_\mu\| = \max\{|\mu^2 \lambda_{\max} - 1|, |\mu^2 \lambda_{\min} - 1|\}.$$

The maximum of the distances to 1 reaches a minimum when both points are equidistant

$$\mu^2 \lambda_{\max} - 1 = 1 - \mu^2 \lambda_{\min} \Leftrightarrow \mu = \sqrt{\frac{2}{\lambda_{\max} + \lambda_{\min}}}.$$

For this value of μ

$$\|A_\mu\| = \mu^2 \lambda_{\max} - 1 = \frac{2\lambda_{\max}}{\lambda_{\max} + \lambda_{\min}} - 1 = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}. \quad \square$$

Remark 9. Note that $\mu = \sqrt{\frac{2}{\lambda_{\max} + \lambda_{\min}}} \leq \sqrt{\frac{2}{\lambda_{\max}}}$ and hence Proposition 4.7 applies for this choice of scaling factor μ . Moreover, for all tuples on the right-hand side except at most two, the associated matrix A will reach its minimum norm.

4.4 The infinite series for $n = 3$

In this section we simplify and rearrange the formula in Theorem 4.4 for $n = 3$; this is the case relevant for the computation of $V_1(F)$. The resulting formula can be implemented as a computer algorithm. The impatient reader may skip over the next calculations to Theorem 4.11.

For $n = 3$ Theorem 4.4 gives the expression

$$3 \sum_{m_1, m_2 \geq 0} \frac{(-1)^{m_1+m_2-1}}{m_1+m_2+4} m_1 \sum_{l_1+l_2 \leq m_1+m_2} \text{fact}(l_1, l_2) \text{Tr}(U(m_1, m_2, l_1, l_2)). \quad (22)$$

Recall from Lemma 4.2 that the matrix $U(m_1, m_2, l_1, l_2)$ is a sum of words of the form

$$s \cdot V_1^1 \dots V_{m_1}^1 W_1 V_1^2 \dots V_{m_2}^2 W_2.$$

for $s \in \{\pm 1\}$. We exploit the symmetry between (m_1, m_2) and (m_2, m_1) , and the invariance of the trace under cyclic permutations to simplify (22). We will only need to consider traces of matrices of the form $U_1 \omega_1 U_2 \omega_2$, and in this way we can disregard the sign s of the permutation. Hence we define:

Definition 4.9. The matrix $\tilde{U}(m_1, m_2, l_1, l_2)$ is the sum of all the matrices of the form

$$U_1 V_1^1 \dots V_{m_1}^1 U_2 V_1^2 \dots V_{m_2}^2 \quad (23)$$

where $V_j^i \in \{U_k : 0 \leq k \leq n\}$ for all i, j and $l_k = \{(i, j) \mid V_j^i = U_k\}$ for $k = 1, 2$.

To ease notation let us write $c = \frac{(-1)^{m_1+m_2-1}}{m_1+m_2+4}$ whenever m_1 and m_2 are clear from the context.

Lemma 4.10.

$$\sum_{m_1, m_2 \geq 0} c m_1 \sum_{|\mathbf{l}| \leq |\mathbf{m}|} \text{fact}(\mathbf{l}) \text{Tr}(U(\mathbf{m}, \mathbf{l})) = \sum_{m_1, m_2 \geq 0} c(m_2 - m_1) \sum_{|\mathbf{l}| \leq |\mathbf{m}|} \text{fact}(\mathbf{l}) \text{Tr}(\tilde{U}(\mathbf{m}, \mathbf{l})).$$

Proof. Fix $m_1, m_2 \geq 0$. Let ω_1 and ω_2 be arbitrary words on letters U_i , $0 \leq i \leq n-1$ of length m_1 respectively m_2 . On the LHS we have the four words

$$\omega_1 U_1 \omega_2 U_2, \quad \omega_1 U_2 \omega_2 U_1, \quad \omega_2 U_1 \omega_1 U_2 \quad \text{and} \quad \omega_2 U_2 \omega_1 U_1.$$

The trace of the first and last, and of the second and third are the same, say t and t' . Hence we have the four summands $c m_1 t$, $-c m_1 t'$, $c m_2 t'$ and $-c m_2 t$, times the factorial coefficient (\mathbf{l} is the same for all words). Suppose that $m_1 \neq m_2$. Then on the RHS we have two words,

$$U_1 \omega_1 U_2 \omega_2 \quad \text{and} \quad U_1 \omega_1 U_2 \omega_2.$$

By a cyclic permutation, the traces are t' and t respectively. The corresponding summands are then $c(m_2 - m_1) t'$ and $c(m_1 - m_2) t$, multiplied by the same factorial coefficient. Finally, the case $m_1 = m_2$ give 0 on both sides. \square

Now consider a matrix $U_1 U_{i_1} \dots U_{i_k}$ with $i_j \in \{0, 1, 2\}$ and $k \geq 1$. Write n_i for the total number of letters equal to U_i , $i = 1, 2$ among the U_{i_j} . How many times does this matrix appear in an expression of the form (23)? For each $i_j = 2$ it appears in $\tilde{U}(m_1, m_2, n_1, n_2 - 1)$ for $m_1 = j - 1$ and $m_2 = k - j$, and coefficient

$$\frac{(-1)^{m_1+m_2-1}}{m_1+m_2+4} (m_2 - m_1) \text{fact}(l_1, l_2) = \frac{(-1)^{k-2}}{k+3} (k - 2j + 1) \text{fact}(n_1, n_2 - 1).$$

All in all we have proven the following. Write χ_a for the characteristic function on an integer a (so $\chi_a(a) = 1$ and $\chi_a(i) = 0$ if $i \neq a$).

Theorem 4.11 (Infinite series for $n = 3$). *Let $X_0, X_1, X_2, X_3 \in GL_N(\mathbb{C})$ and $A = U_0 + s_1 U_1 + s_2 U_2$ defined as in Theorem 4.4 for $n = 3$. Suppose that $\|A\| < 1$. Then Hamida's function $\varphi(X_0, X_1, X_2, X_3)$ equals the limit of the convergent series*

$$3 \sum_{k=1}^{\infty} \frac{(-1)^k}{k+3} \sum_{i_1, \dots, i_k=0}^2 d \text{fact}(n_1, n_2 - 1) \text{Tr}(U_1 U_{i_1} \dots U_{i_k}), \quad (24)$$

where n_1, n_2 and d depend on each tuple (i_1, \dots, i_k) as $n_1 = \sum_j \chi_1(i_j)$, $n_2 = \sum_j \chi_2(i_j)$ and $d = \sum_j \chi_2(i_j) (k - 2j + 1)$, and $\text{fact}(l_1, l_2)$ if defined in Lemma 4.3 for $l_2 \geq 0$ and as 0 if $l_2 = -1$.

Remark 10. For an arbitrary 3-cycle (X_0, X_1, X_2, X_3) we have (see §4.3)

$$\begin{aligned} \varphi(X_0, X_1, X_2, X_3) &= -\varphi(\mu X_1, \mu X_2, \mu X_3, I) + \varphi(\mu X_0, \mu X_2, \mu X_3, I) \\ &\quad - \varphi(\mu X_0, \mu X_1, \mu X_3, I) + \varphi(\mu X_0, \mu X_1, \mu X_2, I). \end{aligned}$$

For μ small enough (see Proposition 4.7 or Remark 9), each tuple on the right hand side will satisfy the conditions of Theorem 4.11.

4.5 Computational aspects

We have implemented Theorem 4.11 as a computer algorithm. It takes as input matrices $X_0, X_1, X_2, X_3 \in GL_N(\mathbb{C})$ and outputs the partial sum of (24) up to a given k . These partial sums converge to Hamida's function $\varphi(X_0, X_1, X_2, X_3)$ and, if we quantify the error, we can in principle compute the universal Borel class b_1 evaluated at $H_3(GL_N(\mathbb{C}))$ to any prescribed degree of accuracy (Theorem 4.1). This, together with the input chain $qZ_1 - Z_2(q)$ (see §2.3) allow the computation of $V_1(F)$ for any cyclotomic field to any required precision.

However, the computational complexity of our algorithm is quite large (exponential on k) and could potentially prevent the algorithm from finishing in a reasonable time. Note that for each term in a homological cycle (for the chain $qZ_1 - Z_2(q)$ we have $6844 + 3q$ terms) we compute 4 instances of the infinite series in Theorem 4.11 (see Remark 10). For each series, we add successive terms until the error is smaller than the target error. The k^{th} summand of the series in Theorem 4.11 involves conducting 3^k different multiplications of $k + 1$ matrices of size N (for the chain $qZ_1 - Z_2(q)$ we have $N = 5$).

Remark 11. For a general $n > 3$ odd we expect an infinite series analogous to the one in Theorem 4.11. Thus for each term in a homological cycle we would have n instances of the infinite series, and in turn the k^{th} summand in a series would involve n^k multiplications of $k + 1$ matrices of size typically $2n + 1$.

On the other hand, the series (24) is dominated by a convergent power series (Corollary 4.15) and thus the error decreases exponentially fast with k , so we may not need to compute many terms to approximate the value within a given error interval.

The rest of this section is devoted to bounding the error in computing the infinite series (24) after k iterations. The impatient reader may refer to Proposition 4.14. and Corollary 4.15.

Let $\|\cdot\|$ be the spectral norm.

Lemma 4.12. *Let X be a hermitian matrix of size N . Then*

$$|\text{Tr}(X)| \leq N\|X\|.$$

Proof. If $\lambda_1, \dots, \lambda_N$ are the eigenvalues of X then $\|X\| = \max_i |\lambda_i|$ and hence

$$|\text{Tr}(X)| = \left| \sum_i \lambda_i \right| \leq N\|X\|. \quad \square$$

Lemma 4.13. *Let X be a matrix of size N of continuous functions over a compact subset $\Delta \subset \mathbb{R}^n$. Suppose that $X(\mathbf{s})$ is hermitian for each $\mathbf{s} \in \Delta$ and define $\|X\| = \max_{\mathbf{s} \in \Delta} \|X(\mathbf{s})\|$. Then*

$$\left| \text{Tr} \int_{\Delta} X(\mathbf{s}) d\mathbf{s} \right| \leq \text{vol}(\Delta) N \|X\|.$$

Proof. By elementary properties of integration and Lemma 4.12,

$$\left| \text{Tr} \int_{\Delta} X(\mathbf{s}) d\mathbf{s} \right| = \left| \int_{\Delta} \text{Tr}(X(\mathbf{s})) d\mathbf{s} \right| \leq \int_{\Delta} |\text{Tr}(X(\mathbf{s}))| d\mathbf{s} \leq \text{vol}(\Delta) N \|X\|. \quad \square$$

Proposition 4.14. *Let a_k be the k^{th} summand of the series in Theorem 4.11, and keep the same hypothesis and notation. Then*

$$|a_k| \leq N \|U_1\| \|U_2\| k \|A\|^{k-1}.$$

Proof. In order to attain a neat upper bound depending on $\|A\|$, we retrace our steps in §4.4 and §4.2 to write

$$a_k = \sum_{m_1+m_2=k-1} \frac{(-1)^k}{k+3} m_1 \operatorname{Tr} \int_{\Delta^2} A^{m_1} dA A^{m_2} dA,$$

since a_k involves all the words of total length $k+1$. Recall that $dA = U_1 ds_1 + U_2 ds_2$. Then

$$\operatorname{Tr} \int_{\Delta^2} A^{m_1} dA A^{m_2} dA = \operatorname{Tr} \int_{\Delta^2} A^{m_1} U_1 A^{m_2} U_2 ds_1 ds_2 - \operatorname{Tr} \int_{\Delta^2} A^{m_1} U_2 A^{m_2} U_1 ds_1 ds_2.$$

By the triangle inequality and the Lemma 4.13,

$$\begin{aligned} \left| \operatorname{Tr} \int_{\Delta^2} A^{m_1} dA A^{m_2} dA \right| &\leq \operatorname{vol}(\Delta^2) N (\|A^{m_1} U_1 A^{m_2} U_2\| + \|A^{m_1} U_2 A^{m_2} U_1\|) \\ &\leq N \|U_1\| \|U_2\| \|A\|^{k-1}. \end{aligned}$$

There are k pairs of nonnegative integers (m_1, m_2) such that $m_1 + m_2 = k - 1$. Hence

$$|a_k| \leq k \frac{1}{k+3} k N \|U_1\| \|U_2\| \|A\|^{k-1}.$$

Observing that $\frac{k}{k+3} \leq 1$ finishes the proof. \square

Corollary 4.15. *Let $C = N \|U_1\| \|U_2\|$, $\rho = \|A\| < 1$ and $b_k = k \rho^{k-1}$ for each $k \geq 1$. Then*

$$\left| \sum_{k=l+1}^{\infty} a_k \right| \leq C \sum_{k=l+1}^{\infty} b_k = C \rho^l \left(\frac{1+l(1-\rho)}{(1-\rho)^2} \right).$$

Note that the expression on the left is the absolute error of approximating $\sum_{k=1}^{\infty} a_k$ by $\sum_{k=1}^l a_k$, the output of the algorithm after l iterations.

Remark 12. Recall that Lemma 4.6(ii) allows us to have an explicit value for $\rho = \|A\|$.

5 A numerical computation of $V_1(F)$

Finally, we discuss the result of computing $V_1(F)$ for a particular cyclotomic field F . Let $F = \mathbb{Q}(\omega)$ with ω a primitive cube root of unity. There is just one pair of conjugate embeddings $\psi_1, \bar{\psi}_1: F \hookrightarrow \mathbb{C}$, and one pair of conjugate primitive 3rd roots of unity, $\omega, \omega^2 \in F$. Thus $V_1(F)$ is the absolute value of the determinant of the one by one matrix whose entry (see §2.3) is:

$$L_{11} = b_1 ((3Z_1 - Z_2(3))|_{t=w}) / i$$

This has 6844 terms coming from Z_1 and 9 terms coming from $Z_2(3)$. Our program first removed terms containing repeated matrices, as they do not contribute to L_{11} , and combined terms with the same matrices in permuted order. This left 3450 distinct terms. We applied the series formula of Theorem 4.11 to these terms, running our algorithm for 12 hours on a 2.4GHz computer. We obtained:

$$V_1(F) = 0.02 + \epsilon, \text{ where } |\epsilon| < 0.01.$$

Thus we know that $V_1(F) \neq 0$ and from Theorem 2.10 we have $R_1(F) = V_1(F) / \operatorname{ind}(F)$ with $V_1(F)$ lying in the range above, and $\operatorname{ind}(F) \in \mathbb{Z}$.

For this number field the Borel regulator $R_1(F)$ is known to be $-2^k 3 \zeta_F^*(-1)$ for some $k \in \mathbb{Z}$, and $\zeta_F^*(-1) = -0.02692 \dots$ (see §5.2 in [8] for details).

If $\operatorname{ind}(F) = 1$, then we would have $R_1(F) = V_1(F)$, and our calculation would suggest that $k = -2$, as this would give $R_1(F) = \frac{3}{4} \zeta_F^*(-1) = 0.02019 \dots$, although the error ϵ is just big enough to allow $k = -3$. Of course reducing the size of the error ϵ would eliminate this possibility.

The large computational times have prevented us from obtaining a better approximation of $V_1(F)$ in this case, and from computing other cases. However, faster implementations of the theoretical foundations which we have laid down here, will lead to more accurate results for more cyclotomic fields.

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