DERIVED LANGLANDS IV: NOTES ON $\mathcal{M}_c(G)$-INDUCED REPRESENTATIONS

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Contents

1. Extending the definition of admissibility

If G is a locally profinite group and k is an algebraically closed field then a k-representation of G is a vector space V with a left, k-linear G-action. Let $\mathcal{M}_c(G)$ denote the poset of pairs (H, ϕ) where H is a subgroup of G, which is compact open and $\phi : H \longrightarrow k^*$ is a continuous character.

A representation V is called smooth ([7] p.13) if

$$V = \bigcup_{K \subset G,\ K \text{ compact,open}} V^K.$$

V is called admissible ([7] p.13) if $\dim_k(V^K) < \infty$ for all compact open subgroups K.

Notice that, if $v_i \in V^{K_i}$ for $i = 1, 2$ with K_i compact open then a linear combination $a_1v_1 + a_2v_2 \in V^{K_1 \bigcap K_2}$ for $a_i \in k$ with $K_1 \bigcap K_2$ compact open so that

$$V = \bigcup_{K \subset G,\ K \text{ compact,open}} V^K = \mathrm{Span}_{K \subset G,\ K \text{ compact,open}}\ V^K.$$

The smooth representations of G form an abelian category.

Define a subspace of V, denoted by $V^{(H,\phi)}$, for $(H, \phi) \in \mathcal{M}_c(G)$ by

$$V^{(H,\phi)} = \{v \in V \mid g \cdot v = \phi(g)v \text{ for all } g \in H\}.$$

Hence $V^K = V^{(K,1)}$ if 1 denotes the trivial character.

We shall say that V is $\mathcal{M}_c(G)$-smooth[1] if

$$V = \mathrm{Span}_{(H,\phi) \in \mathcal{M}_c(G)}\ V^{(H,\phi)}.$$

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[1]In [23] I made a childish typographical slip-up, writing $\bigcup$ instead of Σ in this definition. Hence this time I have written Span. In my current time-shortage ([13] and [14]) the reader will probably find me making a number of (I hope inconsequential) slip ups.

In addition we shall say that V is $\mathcal{M}_c(G)$-admissible if $\dim_k V^{(H,\phi)} < \infty$ for all $(H,\phi) \in \mathcal{M}_c(G)$.

Proposition 1.1.
Let G be a locally profinite group and let k be an algebraically closed field. Let V be a k-representation of G. Suppose that every continuous, k-valued character of a compact open subgroup of G has finite image. Then V is $\mathcal{M}_c(G)$-admissible if and only if it is admissible.

Let us begin by recalling, from ([22] Chapter Two §1), induced and compactly induced smooth representations.

Definition 1.2. *Smooth induction*
Let G be a locally profinite group and $H \subseteq G$ a closed subgroup. Thus H is also locally profinite. Let

$$\sigma : H \longrightarrow \mathrm{Aut}_k(W)$$

be a smooth representation of H. Set X equal to the space of functions $f : G \longrightarrow W$ such that (writing simply $h \cdot w$ for $\sigma(h)(w)$ if $h \in H, w \in W$)
(i) $f(hg) = h \cdot f(g)$ for all $h \in H, g \in G$,
(ii) there is a compact open subgroup $K_f \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K_f$.
The (left) action of G on X is given by $(g \cdot f)(x) = f(xg)$ and

$$\Sigma : G \longrightarrow \mathrm{Aut}_k(X)$$

gives a smooth representation of G.
The representation Σ is called the representation of G smoothly induced from σ and is usually denoted by $\Sigma = \mathrm{Ind}_H^G(\sigma)$.

Definition 1.3. *$\mathcal{M}_c(G)$-smooth induction*
Let G be a locally profinite group and $H \subseteq G$ a closed subgroup. Let

$$\sigma : H \longrightarrow \mathrm{Aut}_k(W)$$

be an $\mathcal{M}_c(H)$-smooth representation of H. Set X equal to the space of functions $f : G \longrightarrow W$ such that (writing simply $h \cdot w$ for $\sigma(h)(w)$ if $h \in H, w \in W$)
(i) $f(hg) = h \cdot f(g)$ for all $h \in H, g \in G$,
(ii) Each $f \in X$ may be written as a linear combination of the form $f = \sum_{i=1}^n a_i f_i$ where $a_i \in k$ and for each $1 \leq i \leq n$ there is $(K_{f_i}, \phi_{f_i}) \in \mathcal{M}_c(G)$ such that $f_i(gk) = \phi_{f_i}(k) f_i(g)$ for all $g \in G, k \in K_{f_i}$.
The (left) action of G on X is given by $(g \cdot f)(x) = f(xg)$ and

$$\Sigma_{\mathcal{M}_c} : G \longrightarrow \mathrm{Aut}_k(X)$$

gives an $\mathcal{M}_c(G)$-smooth representation of G.
The representation $\Sigma_{\mathcal{M}_c}$ is called the representation of G $\mathcal{M}_c$-smoothly induced from σ and is will be denoted by $\Sigma_{\mathcal{M}_c} = \mathrm{IND}_H^G(\sigma)$.

We just pause to check the following:

Lemma 1.4. If $f \in X$ in Definition 1.3 and $g \in G$ then $g \cdot f \in X$ also.

Proof:

Suppose that $f \in X$ and $(K_{f_i}, \phi_{f_i}) \in \mathcal{M}_c(G)$ satisfy the second condition of Definition 1.3. Define $g((K_{f_i}, \phi_{f_i}))$ to be the pair consisting of the group $gK_{f_i}g^{-1}$ with character

$$(g^{-1})^*(\phi_{f_i}) : gK_{f_i}g^{-1} \longrightarrow k^*$$

where, for $k \in K_{f_i}$, we set $(g^{-1})^*(\phi_{f_i})(gkg^{-1}) = \phi_{f_i}(k)$. Therefore

$$\begin{aligned}
(g \cdot f_i)(g'gkg^{-1}) &= f_i(g'gkg^{-1}g) \\
&= f_i(g'gk) \\
&= \phi_{f_i}(k) f_i(g'g) \\
&= (g^{-1})^*(\phi_{f_i})(gkg^{-1}) f_i(g'g) \\
&= (g^{-1})^*(\phi_{f_i})(gkg^{-1})(g \cdot f_i)(g'),
\end{aligned}$$

as required. Clearly action by $g \in G$ preserves condition (i) of Definition 1.3 □

1.5. *The c-IND variation*

Inside X let X_c denote the set of functions which are compactly supported modulo H. This means that the image of the support

$$\operatorname{supp}(f) = \{g \in G \mid f(g) \neq 0\}$$

has compact image in $H \backslash G$. Alternatively there is a compact subset $C \subseteq G$ such that $\operatorname{supp}(f) \subseteq H \cdot C$.

The $\Sigma_{\mathcal{M}_c}$-action on X preserves X_c, since $\operatorname{supp}(g \cdot f) = \operatorname{supp}(f)g^{-1} \subseteq HCg^{-1}$, and we obtain $X_c = c - \operatorname{IND}_H^G(W)$, the compact induction of W from H to G [2].

Lemma 1.6.

(i) If K is an open subgroup of G and V is an $\mathcal{M}_c(G)$-smooth representation then $\operatorname{Res}_H^G(V)$ is an $\mathcal{M}_c(H)$-smooth representation.

(ii) If $\chi : G \longrightarrow k^*$ is a continuous character and V is an $\mathcal{M}_c(G)$-admissible representation then $V \otimes_k k_\chi$ is also an $\mathcal{M}_c(G)$-admissible representation. Herre k_χ is k acted upon by G via the character χ.

Proof:

(i) If $v \in V$ lies in $V^{(J,\phi)}$ for some J compact open in G then $v \in V^{(H \bigcap J, \phi)}$ and $H \bigcap J$ is compact open in H. Therefore V is $\mathcal{M}_c(H)$-smooth.

(ii) This follows from the relation $(V \otimes_k k_\chi)^{(H,\phi)} = V^{(H, \phi \otimes \chi^{-1})}$.

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[2] I have not yet bothered to check the following: Note that this condition requires, in Definition 1.3, at first sight, that f is compactly supported modulo H but not necessarily the individual f_i's.

Remark 1.7. Admissibility is not inherited by subgroups. For example, if H is a compact open subgroup of the centre of G and if V has a central character $\underline{\phi}$ then $\mathrm{Res}_H^G(V)^{(H,\underline{\phi})} = V$.

Lemma 1.8.
Let

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

be a short exact sequence of G-representations.
If K is compact open and $(K, \phi) \in \mathcal{M}_c(G)$ then

$$0 \longrightarrow V_1^{(K,\phi)} \longrightarrow V_2^{(K,\phi)} \longrightarrow V_3^{(K,\phi)} \longrightarrow 0$$

is exact.

Proof:

$$0 \longrightarrow V_1 \otimes_k k_{\phi^{-1}} \longrightarrow V_2 \otimes_k k_{\phi^{-1}} \longrightarrow V_3 \otimes_k k_{\phi^{-1}} \longrightarrow 0$$

is an exact sequence of K-representations. Applying K-fixed points to this sequence is exact, because K is compact, and the result is the short exact sequence of k-vector spaces in the statement of the lemma. □

Corollary 1.9.
Let

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

be a short exact sequence of G-representations.
Then V_2 is $\mathcal{M}_c(G)$-admissible if and only if V_1 and V_3 are both $\mathcal{M}_c(G)$-admissible.

Proposition 1.10.
Let V be a smooth representation of G. Suppose that K is a compact open subgroups and that $(K, \phi) \in \mathcal{M}_c(G)$. Then $\dim(V^{(K,\phi)}) < \infty$.

Proof:
By ([8] Proposition 2.1.4 p.20), the restriction of V to K is the direct sum of finite-dimensional irreducible K-representations, each appearing with finite multiplicity. If this K-representation is $\oplus_\alpha V_\alpha$ then $V^{(K,\phi)} = \oplus_{\alpha,\ V_\alpha = k_\phi} V_\alpha$ which is finite-dimensional. □

Theorem 1.11.
Let

$$0 \longrightarrow V_1 \stackrel{i}{\longrightarrow} V_2 \stackrel{j}{\longrightarrow} V_3 \longrightarrow 0$$

be a short exact sequence of G-representations. Then V_2 is $\mathcal{M}_c(G)$-smooth if and only if V_1 and V_3 are both $\mathcal{M}_c(G)$-smooth.

Proof:
Assume that V_2 is $\mathcal{M}_c(G)$-smooth. Therefore

$$V_2 = \mathrm{Span}_{(H,\phi)\in\mathcal{M}_c(G)}\ V_2^{(H,\phi)}.$$

Since $j(V_2^{(H,\phi)}) \subseteq V_3^{(H,\phi)}$ and j is surjective we see that

$$V_3 = \mathrm{Span}_{(H,\phi)\in\mathcal{M}_c(G)}\ V_3^{(H,\phi)}.$$

Now suppose that $0 \neq w \in V_1$ satisfies $w = u_1 + u_2 + \ldots + u_n$ with $u_i \in V_2^{(H_i,\phi_i)}$ with $(H_i, \phi) \in \mathcal{M}_c(G)$. We may assume that $H = H_1 = \ldots = H_n$ by taking intersections. We shall proceed by induction on n and start the induction by Lemma 1.8. By induction we may assume that $w = u_1 + u_2 + \ldots + u_n$ with $u_i \in V_2^{(H,\phi_i)}$ with $(H, \phi) \in \mathcal{M}_c(G)$ and the ϕ_i's are pairwise distinct. We have $j(u_1+u_2+\ldots+u_n) = j(i(w)) = 0$. Choose $g \in G$ such that $\phi_1(g) \neq \phi_2(g)$. Then

$$i(g \cdot w) = \phi_1(g)u_1 + \phi_2(g)u_2 + \ldots + \phi_n(g)u_n$$

so that

$$i(g \cdot w - \phi_1(g)w) = (\phi_2(g) - \phi_1(g)u_2 + \ldots + (\phi_n(g) - \phi_1(g)u_n.$$

By induction, since

$$j((\phi_2(g) - \phi_1(g)u_2 + \ldots + (\phi_n(g) - \phi_1(g)u_n) = 0,$$

we have

$$g \cdot w - \phi_1(g)w \in \mathrm{Span}_{2\leq s\leq n}\ V_1^{(H,\phi_s)}.$$

Similarly

$$g \cdot w - \phi_2(g)w \in \mathrm{Span}_{s=1 \text{ or } 3\leq s\leq n}\ V_1^{(H,\phi_s)}.$$

Therefore

$$\phi_2(g)w - \phi_1(g)w \in \mathrm{Span}_{1\leq s\leq n}\ V_1^{(H,\phi_s)}$$

and, since $\phi_2(g) \neq \phi_1(g)$ we see that

$$w \in \mathrm{Span}_{1\leq s\leq n}\ V_1^{(H,\phi_s)},$$

as required. Therefore V_1 and V_3 are $\mathcal{M}_c(G)$-smooth.

The converse follows from Lemma 1.8. □

Corollary 1.12.

$\mathcal{M}_c(G)$-smooth representations form an abelian category of which $\mathcal{M}_c(G)$-admissible representations form an abelian subcategory.

Theorem 1.13. Let H be a closed subgroup of G and let (τ, U) be an $\mathcal{M}_c(H)$-smooth H-representation. Then

(i) $\mathrm{IND}_H^G(\tau)$ is $\mathcal{M}_c(G)$-smooth.

(ii) The map $\Lambda : \mathrm{IND}_H^G(\tau) \longrightarrow U$ defined by $\Lambda(f) = f(1)$ is a surjective H-map.

(iii) The restriction of Λ to any non-zero G-subspace of $\mathrm{IND}_H^G(\tau)$ is non-trivial.

(iv) If $H\backslash G$ is compact and (τ, U) is $\mathcal{M}_c(H)$-admissible then $\mathrm{IND}_H^G(\tau)$ is $\mathcal{M}_c(G)$-admissible.

(v) If (π, V) is $\mathcal{M}_c(G)$-smooth then composition with Λ induces an isomorphism of the form

$$(\Lambda \cdot -) : \mathrm{Hom}_G(V, IND_H^G(\tau)) \stackrel{\cong}{\longrightarrow} \mathrm{Hom}_H(V, U).$$

Proof:

Part (i) is immediate from the definition.

For part (ii) we observe that $\Lambda(h \cdot f) = f(h) = \tau(h)f(1) = \tau(h)(\Lambda(f))$. Given $u \in U$ choose a compact open subgroups K_i of G such that $(K \bigcap H, \phi_i) \in \mathcal{M}_c(H)$ and $u = \sum_i\ u_i$ with $u_i \in U^{(K \bigcap H, \phi_i)}$. Define $f_i \in IND_H^G(\tau)^{(K \bigcap H, \phi_i)}$ by $f_i(g) = \tau(h)u_i)$ if $g = hk, h \in H, k \in K$ and $f_i(g) = 0$ otherwise. This is compactly supported modulo H and $\Lambda(f_i) = u_i$ so $u = \Lambda(\sum_i f_i)$, as required.

For part (iii) suppose that V is a non-trivial G-subspace of $\mathrm{IND}_H^G(\tau)$ and choose $f \in V$ such that $f(g) \neq 0$ for some $g \in G$. Therefore $g \cdot f \in V$ also and $\Lambda(g \cdot f) = f(g)$.

For part (iv) suppose that K is a compact open subgroup of G such that $(K, \phi) \in \mathcal{M}_c(G)$. Suppose that X is a finite subset of G and that U_0 is a finite dimensional subspace of U. Set

$$\mathcal{I}((K, \phi)X, U_0) = \{f \in IND_H^G(\tau)^{(K,\phi)} \mid f(X) \subseteq U_0,\ \mathrm{supp}(f) \subseteq H \cdot X \cdot K\},$$

which is clearly finite-dimensional.

Now suppose that (τ, U) is $\mathcal{M}_c(H)$-admissible. Let $(K, \phi) \in \mathcal{M}_c(G)$ and choose a finite set X such that $H \cdot X \cdot K = G$. Let $L = \bigcap_{x \in X}\ xKx^{-1}$ so that $L \bigcap H$ is a compact open subspace of H. For each $x \in X$ we have a character $(x^{-1})^*(\phi) : xKx^{-1} \bigcap H \longrightarrow k^*$ Take $U_0 = \mathrm{Span}_{x \in X}\ U^{(L \bigcap H, (x^{-1})^*(\phi))}$ and suppose that $f \in IND_H^G(\tau)^{(K,\phi)}$. Therefore, if $h \in H, k \in K, x \in X$, then $f(hxk) = \phi(k)\tau(h)(f(x)$. Now suppose that we have any element of $z \in L \bigcap H$ then there is $x \in X$ and $k \in K$ such that $z = xkx^{-1} \in H$. Therefore $\tau(xkx^{-1})(f(x)) = f(xkx^{-1}x) = \phi(k)f(x)$ so that $f(x) \in U_0$. Therefore $IND_H^G(\tau)^{(K,\phi)} \subseteq \mathcal{I}((K, \phi)X, U_0)$ is finite-dimensional, as required.

For part (v) define the map in the reverse direction

$$\Phi : \mathrm{Hom}_H(V, U) \longrightarrow \mathrm{Hom}_G(V, IND_H^G(\tau))$$

by $(\Phi(f)(v))(g) = f(\pi(g)v)$ which is a function of $g \in G$ which on gk satisfies $(\Phi(f)(v))(gk) = f(\pi(g)\pi(k)v) = \phi(k)f(\pi(g)\pi(k)v) = \phi(k)(\Phi(f)(v))(g)$ if $v \in V^{(K,\phi)}$ so that the $\mathcal{M}_c(G)$-smoothness of V ensures tha $\Phi(f)$ maps V into $IND_H^G(\tau)$. Also composing with Λ gives $v \mapsto f(v)$ so that $\Phi \cdot (\Lambda \cdot -) = 1$. Also $(\Lambda \cdot -) \cdot \Phi = 1$ as in the classical case ([8] Theorem 2.4.1(e) p.27).

2. Analogue of Jacquet's Theorem for $\mathcal{M}_c(G)$-admissibility

In this section I shall establish for $\mathcal{M}_c(G)$-admissibility results which are analogous to those of ([20] Chapter III, §2.3), which is the source of any notation that I forget to elucidate.

Let (P, A) be a p-pair - $P = MN$. Hence P is a parabolic subgroup and N is its unipotent radical and $P = MN$ is its Levi decomposition.

For $\chi : N \longrightarrow k^*$ let $V_\chi(N)$ be the subspace of V spanned by elements of the form $\pi(n)(v) - \chi(n)v$ as $v \in V$ varies[3]. The proof of ([20] Lemma 2.2.1 p.82; also a particular case is [6] p.461 Proposition 4.4.3) shows that

Lemma 2.1. $v \in V_\chi(N)$ if and only if there is a compact open subset $N(v)$ of N such that

$$\int_{N(v)} \chi^{-1}(n)\pi(n)(v)dn = 0.$$

The element $m \in M$ maps $V_\chi(N)$ to $V_{(m^{-1})^*(\chi)}(N)$ where $(m^{-1})^*(\chi)(n) = \chi(m^{-1}nm)$ and $n \in N$ maps $V_\chi(N)$ to itself. In the classical case $\chi = 1$ and in that case I shall write $V(N)$ rather than $V_1(N)$.

Let τ be an $\mathcal{M}_c(P)$-smooth representation of P which is trivial on N in which P acts on a vector space U. Set $\rho = \mathrm{IND}_P^G(\tau)$, as in Theorem 1.13. Therefore, by Theorem 1.13(i) ρ is $\mathcal{M}_c(G)$-smooth and, since $P \backslash G$ is compact, by Theorem 1.13(iv) ρ is $\mathcal{M}_c(G)$-admissible if τ is $\mathcal{M}_c(P)$-admissible.

Following the method of ([20] Chapter III, §2.3) we shall prove a series of lemmas in order to establish the following result :

Theorem 2.2. *(Analogue of Jacquet's Theorem)*
Let X_0 be a non-zero $\mathcal{M}_c(G)$-admissible G-subrepresentation of $\mathrm{IND}_P^G(\tau)$. Then $\Delta(X_0)$ is a non-zero $\mathcal{M}_c(M)$-admissible M-subrepresentation of U.

Let (P_0, A_0) is a minimal p-pair such that $(P, A) > (P_0, A_0)$ and $\{K_j, 1 \leq j \leq \infty\}$ is a sequence of compact open subsets of G such that

(i) $\{K_j, 1 \leq j \leq \infty\}$ is a fundamental sequences of neighbourhoods of the identity in G,

(ii) Let (P, A) be a p-pair of G with Levi decomposition $P = MN$. If (P, A) is standard with respect to $P_0, A_0)$ then $K_j = \overline{N}_j M_j N_j = N_j M_j \overline{N}_j$ with $N_j = N \bigcap K_j, M_j = M \bigcap K_j, \overline{N}_j = \overline{N} \bigcap K_j$.

Write $U_0 = \Delta(X_0)$ and define, for $1 \leq j \leq \infty$,

$$\begin{aligned} X_0(j, \phi) &= X_0^{(K_j, \phi)} \\ &= \{f \in X_0 \mid \rho(k)f = f(- \cdot k) = \phi(k) \cdot f \text{ for all } k \in K_j\} \\ &= \{f \in X_0 \mid \phi(k)^{-1}\rho(k)f = f \text{ for all } k \in K_j\}. \end{aligned}$$

Also

$$\begin{aligned} U_0(j, \phi) &= U_0^{(M_j, \phi)} \\ &= \{u \in U_0 \mid \tau(m)u = \phi(m)u \text{ for all } m \in M_j\} \\ &= \{u \in U_0 \mid \phi(m)^{-1}\tau(m)u = u \text{ for all } m \in M_j\} \end{aligned}$$

[3]When $\chi = 1$ [20] writes $V(N)$ sometimes as $V(P)$ but still meaning the subspace of V generated by elements of the form $n \cdot v - v$ with $n \in N, v \in V$. See ([20] p.82).

Lemma 2.3. $\Delta(X_0(j,\phi)) \subseteq U_0(j,\phi)$.

Proof:
If $m \in M_j$ then $m \in K_j$. Now, by ([20] Lemma 2.3.2), $\tau(m)(f(1)) = f(m) = \phi(m)\Delta(f)$. $\square$

Lemma 2.4.
For any $j > 0$, $\Delta(X_0(j,\phi))$ is stable under τ restricted to A.

Proof:
Since $\Delta(X_0(j,\phi))$ is finite-dimensional the set

$$S = \{a \in A \mid \tau(a)\Delta(X_0(j,\phi)) \subseteq \Delta(X_0(j,\phi))\}$$

is a group. Since A is the group generated by $A^+(t)$ for any $t > 0$, in the notation of ([20] Chapter 0), it is sufficient to show that there exists $t > 0$ such that $a^{-1} \in S$ for all $a \in A^+(t)$.

Choose t such that $a\overline{N}_j a^{-1} \subseteq \overline{N}_j$ ($\overline{N}$ is the opposite of N, which is therefore also normalised by M) provided that $a \in A^+(t)$.

For $a \in A^+(t)$ and $f \in X_0(j,\phi)$ set

$$\begin{aligned} f_{(a,j)} &= \int_{K_j} \phi(k)^{-1}\rho(ka^{-1})dkf \\ &= \int_{N_j}\int_{M_j}\int_{\overline{N}_j} (\phi(nm\overline{n})^{-1}\rho(nm\overline{n}a^{-1})\cdot f)dndmd\overline{n}, \end{aligned}$$

which, assuming normalised measures, $f_{(a,j)} \in X(j,\phi)$ since we have averaged over K_j.

Notice that $\phi(n) = 1$ for $n \in N_j$, if $\Delta(f) \neq 0$, since $\tau(n)$ acts trivially on U.

Applying Δ we obtain, as a centralises with M_j ([8] p.11),

$$\begin{aligned} \Delta(f_{a,j}) &= f_{a,j}(1) \\ &= \int_{M_j}\int_{\overline{N}_j} \phi(a)\phi(m\overline{n}a^{-1})^{-1}\rho(m\overline{n}a^{-1})dmd\overline{n}f(1) \\ &= \int_{\overline{N}_j} \phi(a)d\overline{n}f(a^{-1}(a\overline{n}a^{-1})) \\ &= \phi(a)f(a^{-1}) \text{ since } a\overline{N}_j a^{-1} \subseteq \overline{N}_j \\ &= \phi(a)\tau(a^{-1})\Delta(f), \end{aligned}$$

as required. $\square$

Lemma 2.5. $U_0(j,\phi) \subseteq \Delta(X_0(j,\phi))$.

Proof:
We must show that, if $u_0 \in U_0(j,\phi)$, then there exists $f_0 \in X_0(j,\phi)$ such that $\Delta(f_0) = u_0$.

There exists j' and $\phi_i : K_{j'} \longrightarrow k^*$ for $1 \leq i \leq N$, with $f_i \in X(j',\phi_i)$ such that $u_0 = \sum_{i=1}^{N} \Delta(f_i)$.

Suppose that N is minimal integer occurring in the above type of relation. If $K_{j'} \subseteq K_j$ then $M_{j'} \subseteq M_j$ and for $m \in M_{j'}$ we have

$$\begin{aligned} \textstyle\sum_{i=1}^{N} \phi(m) f_i(1) &= \phi(m) u_0 \\ &= \tau(m) u_0 \\ &= \textstyle\sum_{i=1}^{N} (\rho(m) f_i)(1) \\ &= \textstyle\sum_{i=1}^{N} f_i(m) \\ &= \textstyle\sum_{i=1}^{N} \phi_i(m) f_i(1). \end{aligned}$$

Therefore we can eliminate one of the f_i's unless each $f_i \in X_0(j', \phi)$, but such an eliniation would contradict minimality.

If $K_j \subset K_{j'}$ a similar minimality argument shows that we may assume $f_i \in X_0(j', \phi)$ for each i, which implies the result in this case.

We continue, assuming at $K_{j'} \subseteq K_j$ and that $f_i \in X_0(j', \phi)$ for each i.

Let $\lambda \in C_c(M//M_j)$ be the characteristic function of M_j multiplied by the reciprocal of the Haar measure of M_j ([20] p.30). We have

$$\begin{aligned} u_0 &= \textstyle\int_{M_j} \lambda(m) \phi(m)^{-1} \tau(m) dm u_0 \\ &= \textstyle\sum_{i=1}^{N} \Delta(\int_{M_j} \lambda(m) \phi(m)^{-1} \rho(m) dm f_i) \\ &= \textstyle\sum_{i=1}^{N} \int_{M_j} \lambda(m) \phi(m)^{-1} f_i(m) dm. \end{aligned}$$

Assume that $f_i \in X_0(j', \phi)$ and choose $t > 0$ so that, if $a \in A^+(t)$, then $a\overline{N}_j a^{-1} \subseteq \overline{N}_{j'}$. Fix $a \in A^+(t)$. Set $\lambda_a(m) = \lambda(am)$ for $m \in M$ and $\lambda'_a(x) = \lambda_a(m)$ if $x = km \in K_j M$, 0 if $x \notin K_j M$. Then $\lambda'_a \in C_c(K_j \backslash G)$, so define $f_{1,i}$ to be the convolution of λ'_a with $\phi^{-1} f_i$ so that $f_{1,i} = (\lambda'_a) * \phi^{-1} f_i \in X_0(j, \phi)$.

Then

$$\begin{aligned} ((\lambda'_a) * \phi^{-1} f_i)(1) &= \textstyle\int_G (\lambda'_a(x) \phi(x)^{-1} f(x) dx \\ &= \textstyle\int_{K_j} \phi(xa^{-1})^{-1} f_i(xa^{-1}) dx \\ &= \phi(a) \textstyle\int_{\overline{N}_j} \phi(axa^{-1})^{-1} f_i(axa^{-1}) dx \\ &= \phi(a) \tau(a) \textstyle\int_{\overline{N}_j} f_i(a\overline{n}a^{-1}) d\overline{n} \\ &= \phi(a) \tau(a) (f_i(1)), \end{aligned}$$

since $a\overline{N}_j a^{-1} \subseteq \overline{N}_{j'}$. □

Lemmas 2.3, 2.4 and 2.5 establish Theorem 2.2. In particular, combining Theorem 2.2 and Theorem 1.13(iv) we obtain the following result.

Corollary 2.6.

In the notation of §§2.2-2.5, (τ, U) is $\mathcal{M}_c(M)$-admissible if and only if $\mathrm{IND}_H^G(\tau)$ is $\mathcal{M}_c(G)$-admissible.

Theorem 2.7. If V is an $\mathcal{M}_c(G)$-admissible representation then $V/V(P)$ is an $\mathcal{M}_c(M)$-admissible representation[4].

Proof:

By Frobenius reciprocity (see Theorem 1.13(v)) we have an isomorphism of the form

$$(\Lambda \cdot -) : \mathrm{Hom}_G(V, IND_P^G(V/V(P))) \stackrel{\cong}{\longrightarrow} \mathrm{Hom}_P(V, V/V(P))$$

and we denote by T the G-map on the left which corresponds the the canonical quotient of P-representations on the right. Setting $X_0 = T(V) \subset IND_P^G(V/V(P))$, which is $\mathcal{M}_c(G)$-admissible by Corollary 1.9, implies that $\Delta(X_0) = V/V(P)$ is an $\mathcal{M}_c(M)$-admissible representation, by Theorem 2.2. $\square$

3. Related remarks and questions

If χ is a character of P which is trivial on M then it gives an M-fixed character on N and conversely an an M-fixed character on N χ extends to $\chi(mn) = \chi(n))$ on P, which is trivial on M.

In this case $V/V_{\hat{\chi}}(P)$ of Lemma 2.1 is a P-representation[5].

If $\chi : P \longrightarrow k^*$ is a continuous character which is trivial on M and (U, τ) is an $\mathcal{M}_c(P)$-admissible representation then so is $U' = U \otimes \chi$. Therefore, by Theorem 1.13(iv), $X' = \mathrm{IND}_P^G(U \otimes \chi)$ is an $\mathcal{M}_c(G)$-admissible representation.

In [22], [23], [24] and [25] one is often concerned with the subcategory of admissible representations with a fixed central character $\underline{\phi}$ and $\mathcal{M}_{cmc.\underline{\phi}}(G)$-admissibility, where $\mathcal{M}_{cmc.\underline{\phi}}(G)$ is the poset of pairs (H, ϕ) where $Z(G) \subseteq H$ and ϕ is a continuous character extending $\underline{\phi}$ and H is compact open modulo the centre. There is an analogue of $IND_P^G(-)$ in this setting and presumably the obvious analogue of the classical Jacquet theory.

For the classical definition can one modify the Jacquet proof to show V G-admissible implies $V/V_\chi(P)$ is P-admissible?

If that works then can one try to prove it for $\mathcal{M}_c(G)$-admissibility?

A question left in suspense in [23] was the derivation of the general formula for generators of the hyperHecke algebra in term of convolution products. I gave the formula in the case when all characters on compact opens have finite image (as in the classical case). It seems to me that, with a little more expertise with Mahler's theorem (see [18]) my previous proof might be promoted to the general case? Unfortunately, (see [13] and [14]) this may take rather too long, in the absence of access to "big-hitting Langlands

[4]Recall that, following the ambiguous notation of [20], $V(P)$ is also denoted by $V(N)$.

[5]Similarly, recall that, following [20], we adopt the ambiguous notation of writing $V_{\hat{\chi}}(P)$ and $V_{\hat{\chi}}(N)$ for the same subspace of V.

professionals". This one will have to wait until the unlikely event of my next trip to Toronto, Paris or Princeton.

There is another dangling, incompletely sketched problem, vaguely conjectured in [24], concerning the Hopf-like algebra structure of the hyperHecke algebra of general linear groups. An additional discussion of this algebra is skectched in [25].

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