A FREEMAN DYSON ANECDOTE

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On 10 March 2015 I attended the inaugural professorial lecture of my son-in-law Francesco Mezzadri. During the lecture Francesco referred to the following event, taking place in the IAS refectory, where Freeman Dyson speedily solved a problem posed at lunch by someone else. Both in the original venue and at the inaugural lecture this anecdote must have set the more competitive mathematicians in the audience into a computational frenzy aimed at out-speeding the reputedly the speedy FD!

X: " Is there an integer which becomes twice itself when the right-most digit is moved to the left-most position?"

FD: "That's easy! There exists one and the smallest such has 18 digits!"

Sounds impressive does it not? But, one-up-manship bluffing is not unknown that the IAS. What do you think?

Recall that any trained pure mathematician would know that 19 is a prime number and that for any prime p the multiplicative group of the finite field $\mathbb{F}_p$ is cyclic of order $p-1$.

For myself - and I am sure for quite a few of Francesco's audience - I immediately rephrased the problem into the form of an equation $(**)$ (given in a few lines time) and smugly settled back thinking that the $\mathbb{F}_{19}$-fact which I have just mentioned immediately allowed one to blurt out FD's pronouncement. However, as we shall see, mine was the "bluffers' reasoning".

Suppose that the number in question has the form $a_1a_2\ldots a_n$ when written to the base 10 so that

$$a_1a_2\ldots a_{n-1}a_n = a_n + 10 \times (a_1a_2\ldots a_{n-1})$$

and

$$\begin{aligned} & a_na_1a_2\ldots a_{n-2}a_{n-1} \\ & = a_1a_2\ldots a_{n-2}a_{n-1} + 10^{n-1} \times a_n \\ & = 2a_n + 20 \times (a_1a_2\ldots a_{n-1}). \end{aligned}$$

Write $x = a_1a_2\ldots a_{n-1}$ so that

$$(*) : \qquad x + 10^{n-1} \times a_n = 2a_n + 20x$$

or

$$(**) : \qquad x = \frac{a_n \times (10^{n-1} - 2)}{19}.$$

Date: 18 March 2015.

This implies that either there is no integer y such that 19 divides $10^y - 2$ in which case, since 19 cannot divide a number smaller that 10 like a_n, there would be no integers x and a_n satisfying $(*)$. If, on the other hand, y is the smallest positive integer such that 19 divides $10^y - 2$ then there is a positive integer r - called the order of 10 and dividing $18 = 19 - 1$ - such that 19 divides $10^r - 1$. Then all the positive integers t such that 19 divides $10^t - 2$ are precisely $y, y + r, y + 2r, y + 3r, \ldots$.

Next observe that, if z is a positive integer, $10z$ is divisible by 19 if and only if z is. But $10^{18} - 1$ is divisible by 19 - this is an example of the Little Fermat Theorem - so

$$10 \times (10^{17} - 2) = 10^{18} - 1 + 1 - 20 = (10^{18} - 1) + 19$$

is divisible by 19 so $y = 17$ and, since r divides 18 and is at least as large as y, we must have $r = 18$.

Therefore the values of n for which we can solve $(*)$ lie within the sequence $17, 35, 53, 71, \ldots$.

Now consider[1] $X = 5263157894736842$ so that $20X$ is given by the addition sum

$$\begin{array}{cccccccccccccccccc} & 5 & 2 & 6 & 3 & 1 & 5 & 7 & 8 & 9 & 4 & 7 & 3 & 6 & 8 & 4 & 2 & 0 \\ + & 5 & 2 & 6 & 3 & 1 & 5 & 7 & 8 & 9 & 4 & 7 & 3 & 6 & 8 & 4 & 2 & 0 \\ \hline 1 & 0 & 5 & 2 & 6 & 3 & 1 & 5 & 7 & 8 & 9 & 4 & 7 & 3 & 6 & 8 & 4 & 0 \end{array}$$

so that

$$20X + 2 = 10^{17} + X$$

or equivalently that

$$19X = 10^{17} - 2.$$

Next we know that x is a 17-digit number given by $(**)$ so the possibilities are $a_{18} = 2, 3, 4, 5, 6, 7, 8, 9$. Each of these values for a_{18} works but $a_{17} = 1$ does not. There is a quick algebraic way to see this. We have the equation

$$x = a_n \times \frac{(10^{17} - 2)}{19}$$

where the fraction is known to be an integers and $1 \leq a_n \leq 9$ is the last digit of the base-10 number $xa_n = 10x + a_n$. Hence twice this number is

$$20x + 2a_n = 19x + x + 2a_n = 10^{17} \times a_n - 2a_n + x + 2a_n = 10^{17} \times a_n + x$$

which is the base-10 number with 18 digits $a_n x$ if x has 17 digits but not when x has only 16 digits. When $a_n = 2, 3, 4, 5, 6, 7, 8, 9$ the former occurs but for $a_n = 1$ it is the latter.

[1]The number X and the solutions to FD's problem which appear in the table below are very easy to find without a calculator. I shall explain the algorithm at the very end of this essay.

If you are still sceptical we have the following table:

X	52631578947368421
$2X$	105263157894736842
$3X$	157894736842105263
$4X$	210526315789473684
$5X$	263157894736842105
$6X$	315789473684210526
$7X$	368421052631578947
$8X$	421052631578947368
$9X$	473684210526315789

Then we have

$$2 \times 52631578947368421 = 105263157894736842 \neq 152631578947368421$$

$$2 \times 105263157894736842 = 2\underline{10526315789473684}$$

$$2 \times 157894736842105263 = 3\underline{15789473684210526}$$

$$2 \times 210526315789473684 = 4\underline{21052631578947368}$$

$$2 \times 263157894736842105 = 5\underline{26315789473684210}$$

$$2 \times 315789473684210526 = 6\underline{31578947368432052}$$

$$2 \times 368421052631578947 = 7\underline{3684210526315794}$$

$$2 \times 421052631578947368 = 8\underline{42105263157894736}$$

$$2 \times 473684210526315789 = 9\underline{47368421052631578}$$

Therefore there are exactly eight 18-digit integers which satisfy $(**)$.

Are there any more? Yes indeed! There are precisely 8 integers with 36 digits satisfying $(**)$, precisely 8 more with 54 digits. The complete list is that for each integer of the form $17 + 18k + 1$ with $k = 0, 1, 2, 3, \ldots$ there are precisely 8 integers satisfying $(**)$. This follows from some algebra analogous to the case when $k = 0$.

For example suppose that $k = 1$ and that

$$x = a_n \times \frac{(10^{35} - 2)}{19}$$

then the integer $\frac{(10^{35}-2)}{19}$ has 34 digits and so does x when $a_n = 1$ but x has 35 digits for $a_n = 2, 3, 4, 5, 6, 7, 8, 9$. In this case twice the base-10 number xa_n is

$$20x + 2a_n = 19x + x + 2a_n = 10^{35} \times a_n - 2a_n + x + 2a_n = 10^{35} \times a_n + x$$

which is the 36-digit base-10 integer when $a_n = 2, 3, 4, 5, 6, 7, 8, 9$. The case when $k \geq 2$ is analysed by the same sort of algebra.

Finally, I promised an algorithm for finding the solutions in the table. Start with any one of the final digits $2, 3, 4, 5, 6, 7, 8, 9$. Take 2 for example. We want a number ending in 2 looking like

$$a \ldots\ tuvwyxz2$$

such that twice it looks like

$$2a \ldots\ tuvwyxz.$$

Since twice the first number ends in $2 \times 2 = 4$ we must have $z = 4$. Then we want

$$a \ldots\ tuvwyx42$$

such that twice it looks like

$$2a \ldots\ tuvwyx4$$

which shows that $x = 8$. Then we want

$$a \ldots\ tuvwy842$$

such that twice it looks like

$$2a \ldots\ tuvwy84$$

and twice 8 is 16 so that $y = 6$. Then we want

$$a \ldots\ tuvw6842$$

such that twice it looks like

$$2a \ldots\ tuvw684$$

and twice 6 is 12 so we get $w = 2$ except that we "carried 1" from $2 \times 8 = 10 + 6$ so $w = 2 + 1 = 3$. Then we want

$$a \ldots\ tuv36842$$

such that twice it looks like

$$2a \ldots\ tuv3684.$$

Similarly $2 \times 3 = 6$ plus the "carried 1" from $2 \times 6 = 10 + 2$ gives $v = 7$. So far we have found that we want

$$a \ldots\ tu736842$$

such that twice it looks like

$$2a \ldots\ tu73684.$$

Proceeding in this manner, carefully keeping track of "carried digits" until we find a digit equal to 2 appears, yields the integer appearing in the $2X$ row of the table.

In a far away corner of the IAS refectory there was a table reserved for combinatorialists. In this long ago days - prior to Reid using combinatorics to solve the invariant subspace problem and Gowers using combinatorics to classify von Neumann algebras - coming "out" as a combinatorialist (an umbral calculus addict or something similar) would have been greeted with howls of derision, particularly in Princeton. Accordingly the group at the combinatorialist's table listened in silence to the Dysonian conversation. One will never know whether they beat the local speed record. However, listening in did inspire the following mathematically related exchange.

The combinatorialists' discussion concerned whole numbers of the form $a_1a_2\ldots a_n$, written to the base 10 with digits $a_1, a_2, \ldots$ etc such that $a_1 \neq 0$ and all the other a_i's are one of $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

One of their number, referred to simply as HG, made a "bluff". The HG bluff asserts that there is no whole number which is doubled when one swaps the 1st and last digits. That is, the equation

$$2 \times (a_1a_2\ldots a_n) = a_na_2a_3\ldots a_{n-1}a_1$$

is never true.

HG's bluff is correct and below I shall explain the reason, which was immediately given by one of the ostracised combinatorial brethren.

Let us start with a very small example. The equation $2 \times (a_1a_2) = a_2a_1$ implies that a_1 is even. That is, being non-zero, $a_1 = 2, 4, 6$ or 8.

$\underline{a_1 = 2}$: $2 \times (2a_2) = a_22$ implies that $a_2 = 1$ or $a_2 = 6$. However $2 \times (21) = 42 \neq 12$ and $2 \times (26) = 52 \neq 62$.

$\underline{a_1 = 4}$: $2 \times (4a_2) = a_24$ implies that $a_2 = 2$ or $a_2 = 7$. However $2 \times (42) = 84 \neq 24$ and $2 \times (47) = 94 \neq 74$.

$\underline{a_1 = 6}$: $2 \times (6a_2)$ has 3 digits but a_26 has only 2.

$\underline{a_1 = 6}$: $2 \times (8a_2)$ has 3 digits but a_28 has only 2.

Now in general suppose that for n bigger than or equal to 3 we have

$$2 \times (a_1a_2\ldots a_n) = a_na_2a_3\ldots a_{n-1}a_1.$$

Again we must have a_1 even but $a_1 = 6$ and $a_1 = 8$ will not work because the left side has more digits than the right.

There $a_1 = 2$ or $a_1 = 4$.

$\underline{a_1 = 2}$: Therefore we have

$$2 \times (2a_2\ldots a_n) = a_na_2a_3\ldots a_{n-1}2$$

and $a_n = 1$ or $a_n = 6$. However if

$$2 \times (2a_2\ldots a_{n-1}1) = 1a_2a_3\ldots a_{n-1}2$$

the number on the left is strictly bigger than $4 \times (100\ldots 0)$ (1 followed by $n-1$ zeroes while the number on the right is strictly less than $2 \times (100\ldots 0)$. Therefore $a_n = 6$.

$\underline{a_1 = 4}$: Therefore we have

$$2 \times (4a_2\ldots a_n) = a_na_2a_3\ldots a_{n-1}4$$

and $a_n = 2$ or $a_n = 7$. However if

$$2 \times (4a_2 \ldots a_{n-1}2) = 2a_2a_3 \ldots a_{n-1}4$$

the number on the left is strictly bigger than $8 \times (100 \ldots 0)$ (1 followed by $n-1$ zeroes) while the number on the right is strictly less than $3 \times (100 \ldots 0)$. Therefore $a_n = 7$.

Hence we are left with either

$$2 \times (2a_2 \ldots a_{n-1}6) = 6a_2a_3 \ldots a_{n-1}2$$

where the number on the left is strictly less than $5 \times (100 \ldots 0)$ and on the right is strictly bigger than $6 \times (100 \ldots 0)$ or

$$2 \times (4a_2 \ldots a_{n-1}7) = 7a_2a_3 \ldots a_{n-1}4$$

where the number on the left is strictly greater than $8 \times (100 \ldots 0)$ and on the right is strictly less than $8 \times (100 \ldots 0)$.

This discussion covers all the cases, so HG's bluff is correct!

Having completed the proof to his colleagues satisfaction, the speaker finished his third and final chocolate donut and left to drive back to Rutgers in his uninsurable 1976 Ford Pinto estate.