ADMISSIBLE REPRESENTATIONS OF GL_nK AND THEIR MONOMIAL RESOLUTIONS

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Contents

1. Introduction

This document is a sketch of a method of constructing all admissible representations of GL_2K of a local field (representations defined over any algebraically closed field) and all automorphic representations when K is a number field in terms of uniquely defined chain complexes in the homotopy category of monomial complexes. This construction makes sense for all GL_nK but my current knowledge of Bruhat-Tits buldings runs only to Conjecture 4.3. In the later sections I sketch how the definition of epsilon factors and L-functions and Galois descent should go.

I am sending out this essay to a few mathematicians who might be interested in getting themselves, their postdocs or their students involved. I am intending to prepare the 300-page typescript of the details of results so far into a monograph fit for publication. The only drawbacks being that, as a retired professor in the UK, I have neither access to resources nor do I give many hours per days to the development of this project.

2. Monomial resolutions

2.1. *Monomial resolutions of admissible representations*

Let k be an algebraically closed field with a topology, not necessarily of characteristic zero.

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Let G be an arbitrary locally compact Hausdorff group such that the compact open subgroups form a basis for the neighborhoods of the identity. In particular we are thinking of locally p-adic Lie groups such as $GL_n K$ where K is a p-adic local field. The subgroups of such a group which we shall concentrate on are the subgroups $H \subseteq G$ which are compact open modulo the centre $Z(G)$. This means that the image of H in $G/Z(G)$, with the induced topology, is compact open. Since we shall be emphasising $GL_n K$ whose centre is the group K^* consisting of the scalar matrices a good example of a compact open modulo the centre subgroup is $K^* \cdot GL_n \mathcal{O}_K \subset GL_n K$ where $\mathcal{O}_K$ denotes the valuation ring of K.

If $H \subseteq G$ is a compact modulo the centre subgroup then we write $\hat{H}$ for the multiplicative group of continuous group homomorphisms $\hat{H} = \mathrm{Hom}_{cts}(H, k^*)$. When $k = \mathbb{C}$ then $\mathbb{C}^*$ has the discrete topology. The example $\hat{K^*}$ is easy to describe since there is a topological isomorphism

$$K^* \cong \mathcal{O}_K^* \times \mathbb{Z}\langle \pi_K \rangle$$

where π_K is a uniformiser of K. A continuous homomorphism to $\mathbb{C}^*$ in this case means a homomorphism which is of finite order when restricted to the group of units $\mathcal{O}_K^*$ and on the infinite cyclic group generated by π_K it is given by $\pi_K^n \mapsto x^n$ for some $x \in \mathbb{C}^*$.

Let $\underline{\phi} : Z(G) \longrightarrow k^*$ be a fixed choice of continuous central character.

Let $\mathcal{M}_{G,\underline{\phi}}$ denote the set of pairs (J, ϕ) with $J \subseteq G$ a compact modulo the centre subgroup containing $Z(G)$ and $\phi \in \hat{J}$ such that $\mathrm{Res}^J_{Z(G)}(\phi) = \underline{\phi}$. The set $\mathcal{M}_{G,\underline{\phi}}$ is endowed with the usual partial order in which $(J, \phi) \leq (J', \phi')$ if and only if $J \subseteq J'$ and $Res^{J'}_J(\phi') = \phi$. Also G acts on $\mathcal{M}_{G,,\underline{\phi}}$ (on the left) by the formula $g(J, \phi) = (gJg^{-1}, (g^{-1})^*(\phi))$ where $(g^{-1})^*(\phi)(gjg^{-1}) = \phi(j)$. The G-stabiliser of (J, ϕ) under this action is denoted by $N_G(J, \phi)$ and the G-orbit of (J, ϕ) is denoted by $(J, \phi)^G$. Associated to (J, ϕ) is the $k[J]$-module k_ϕ given by k on which J acts via $j(z) = \phi(j) \cdot z$ for all $z \in k$.

An G-Line Bundle[1] is a continuous $k[G]$-module M together with a decomposition into the direct sum of one-dimensional subspaces

$$M = \oplus_{\alpha \in A} M_\alpha$$

where the M_α's are permuted by the G-action and the stabiliser of each M_α is a pair $(H_\alpha, \phi_\alpha) \in \mathcal{M}_{G,\underline{\phi}}$. In particular each H_α is compact modulo the centre. The subspaces M_α's are called the Lines of M. The pair (H_α, ϕ_α) is called the stabilising pair of M_α.

For a pair $(H, \phi) \in \mathcal{M}_{G,\underline{\phi}}$ the G-Line Bundle denoted by $\underline{\mathrm{Ind}}_H^G(\phi)$ is given the direct sum of lines L_g for $g \in G/H$ where L_g is k upon which $ghg^{-1} \in gHg^{-1}$ acts by the formula $ghg^{-1} \cdot z = \phi(h)z$ and $g \in G$ sends $L_{g'}$ to $L_{gg'}$ by the identity map on k. As in the case of linear representations this construction

[1] The capital letters are chosen there to distinguish the Line Bundle from the familiar vector bundle terminology.

with be called the G-Line Bundle induced from (H, ϕ). The stabilising pair of L_g is $g(H, \phi) = (gHg^{-1}, (g^{-1})^*(\phi))$. Since $(H, \phi) \in \mathcal{M}_{G,\underline{\phi}}$ the Line Bundle $\underline{\mathrm{Ind}}_H^G(\phi)$ may be topologised to be continuously isomorphic to the compact induction $c - \mathrm{Ind}_H^G(\phi)$ ([1] p.19).

For each $(J, \phi) \in \mathcal{M}_{G,\underline{\phi}}$ set

$$M^{((J,\phi))} = \oplus_{\alpha \in A,\ (J,\phi) \leq (H_\alpha, \phi_\alpha)}\ M_\alpha,$$

which is a subspace of M called the (J, ϕ)-fixed points of M. A morphism

$$f : M = \oplus_{\alpha \in A}\ M_\alpha \longrightarrow \oplus_{\beta \in B}\ M'_\beta = M'$$

between two G-Line Bundles is a continuous $k[G]$-module homomorphism such that

$$f(M^{((J,\phi))}) \subseteq (M')^{((J,\phi))}$$

for all $(J, \phi) \in \mathcal{M}_{G,\underline{\phi}}$. This defines a category ${}_{k[G,\underline{\phi}]}\mathrm{mon}$ and the union of categories over all $\underline{\phi}$ is just denoted by ${}_{k[G]}\mathrm{mon}$. These are additive categories; this means, for example, that $Hom_{{}_{k[G,\underline{\phi}]}\mathrm{mon}}(M, M')$ is an k-vector space.

A monomial complex is a chain complex of continuous $k[G]$-modules

$$C_* : \qquad \ldots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \ldots$$

in which each C_n is an G-Line Bundle and each d is a morphism.

Let $\pi : G \longrightarrow GL(V)$ be a smooth (aka admissible) representation on a k-vector space, V. That is, for any compact open modulo the centre subgroup K of G the restriction to K is a direct sum of irreducible finite-dimensional representations, each isomorphism class occurring with finite multiplicity.

Let V be an admissible representation of G, not necessarily irreducible, but with central character $\underline{\phi}$. For $(J, \phi) \in \mathcal{M}_{G,\underline{\phi}}$ define $V^{(J,\phi)}$ to be the subspace

$$V^{(J,\phi)} = \{v \in V \mid j(v) = \phi(j) \cdot v \text{ for all } j \in J\}.$$

An exact chain complex of continuous $k[G]$-modules of the form

$$\ldots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \ldots \xrightarrow{d} C_0 \xrightarrow{\epsilon} V \longrightarrow 0$$

is called a monomial resolution of V if each C_n is an G-Line Bundle, each d is a morphism and, in addition, $\epsilon(C_0^{((J,\phi))}) \subseteq V^{(J,\phi)}$ for each $(J, \phi) \in \mathcal{M}_{G,\underline{\phi}}$ and, furthermore, the complex of k-vector spaces

$$\ldots \xrightarrow{d} C_n^{((J,\phi))} \xrightarrow{d} C_{n-1}^{((J,\phi))} \xrightarrow{d} \ldots \xrightarrow{d} C_0^{((J,\phi))} \xrightarrow{\epsilon} V^{(J,\phi)} \longrightarrow 0$$

is exact for each $(J, \phi) \in \mathcal{M}_{G,\underline{\phi}}$.

Our first main result is the following.

Theorem 2.2.

Let V be an admissible representation of GL_2K with central character $\underline{\phi}$ as in §2.1. Then V admits a monomial resolution which is unique up to chain homotopy in the category ${}_{k[GL_2K,\underline{\phi}]}\mathrm{mon}$.

2.3. *Some subgroups of GL_2K*

Let K be a p-adic local field with valuation $v_K : K^* \longrightarrow \mathbb{Z}$. We have homomorphisms

$$\det : GL_2K \longrightarrow K^* \text{ and } v_K \cdot \det : GL_2K \longrightarrow \mathbb{Z}.$$

Following ([4] p.75) we may define subgroups of GL_2K denoted by SL_2K, GL_2K^0 and GL_2K^+ by

$$SL_2K = \mathrm{Ker}(\det),\ GL_2K^0 = \mathrm{Ker}(v_K \cdot \det),\ GL_2K^+ = \mathrm{Ker}(v_K \cdot \det \text{ modulo } 2)$$

so that

$$SL_2K \subset GL_2K^0 \subset GL_2K^+ \subset GL_2K.$$

As explained in ([4] pp.78/79) and in terms of Tits buildings (i.e. BN-pairs etc) in ([4] p.91) each of the first three groups acts transitively on the vertices of the tree and act on a 1-simplex between adjacent vertices simplicially (i.e. any element sending the 1-simplex to itself does so point-wise).

The following result summarises the properties of the monomial resolutions for representations of GL_2K.

Theorem 2.4.

Let V be an admissible representation of G with central character $\underline{\phi}$ as in §2.1, where G is one of the subgroups of GL_2K given by $G = SL_2K, GL_2K^0$ or GL_2K^+. Then V admits a monomial resolution which is unique up to chain homotopy in the category ${}_{k[G,\underline{\phi}]}\mathrm{mon}$.

Remark 2.5. In order to describe the construction of the monomial resolution in Theorems 2.2 and 2.4 I need to introduce the functorial bar-monomial resolution for finite-dimensional representations of groups which are finite modulo the centre. This will be introduced in §3 and the construction which proves Theorems 2.2 and 2.4 will be sketched in §4.

The important point to note is that the construction generalises to all GL_nK and when I know more about the Bruhat-Tits building of GL_nK for $n \geq 3$ it is likely that this knowledge will result in a generalisation of Theorem 2.2 to all GL_nK, at the moment this is merely a conjecture which appears in §4.

3. The bar-monomial resolution

3.1. For the moment let G be a group which is finite modulo the centre with central character $\underline{\phi}$. Let W be a k-vector space and let M be a left $k[G,\underline{\phi}]$-Line Bundle. Define another $k[G,\underline{\phi}]$-Line Bundle on the vector space $W \otimes M$ by letting G act only on the M-factor, $g(w \otimes m) = w \otimes gm$, and by setting the Lines of $W \otimes M$ to consist of the one-dimensional subspaces $\langle w \otimes L \rangle$ where w is a non-zero vector of W, running through a choice of basis for W, and L is a Line of M. Therefore, if M' is another $k[G,\underline{\phi}]$-Line Bundle we have a natural isomorphism

$$\mathrm{Hom}_{k[G,\underline{\phi}]-mon}(W \otimes M, M') \xrightarrow{\cong} W \otimes \mathrm{Hom}_{k[G,\underline{\phi}]-mon}(M, M')$$

providing that W is finite-dimensional.

Let ${}_{k[G,\underline{\phi}]}\mathrm{mod}$ denote the category of finite dimensional k-modules on which G acts with central character $\underline{\phi}$. Let V be a representation of G in ${}_{k[G,\underline{\phi}]}\mathrm{mod}$ and let S be a Line Bundle given by a direct sum of some $\underline{\mathrm{Ind}}_H^G(\phi)$'s with $(H,\phi) \in \mathcal{M}_{G,\underline{\phi}}$.

If $\mathcal{V}$ is the forgetful functor from Line Bundles to G-modules set

$$W_i = \mathrm{Hom}_{k[G,\underline{\phi}]-mod}(\mathcal{V}(S), V) \otimes \mathrm{Hom}_{k[G,\underline{\phi}]-mon}(S,S)^{\otimes^i},$$

assuming that W_i is finite-dimensional.

We have left $k[G,\underline{\phi}]$-monomial morphisms, defined by the obvious formulae,

$$d_0, d_1, \ldots, d_i : W_i \otimes S \longrightarrow W_{i-1} \otimes S$$

for $i \geq 1$ and a left $k[G,\underline{\phi}]$-module homomorphism

$$\epsilon : \mathrm{Hom}_{k[G,\underline{\phi}]-mod}(\mathcal{V}(S), V) \otimes S \longrightarrow V$$

given by $\epsilon(f \otimes s) = f(s)$. Let d be given by the alternating sum $d = \sum_{j=0}^{i} (-1)^j d_j$ and choice S to be the direct sum of one copy of each $\underline{\mathrm{Ind}}_H^G(\phi)$ for $(H,\phi) \in \mathcal{M}_{G,\underline{\phi}}$.

Theorem 3.2.

In the notation of §3.1 the complex

$$\ldots \stackrel{d}{\longrightarrow} W_i(S) \otimes S \stackrel{d}{\longrightarrow} W_{i-1}(S) \otimes S \ldots \stackrel{d}{\longrightarrow} W_0 \otimes S \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is a canonical left $k[G,\underline{\phi}]$-monomial resolution of V, which is natural with respect to homomorphisms of groups G.

4. Constructing admissible representations via monomial resolutions

4.1. Let G be a locally p-adic Lie group such as $G = GL_nK$ for K a local field. Let Y be a simplicial complex upon which G acts simplicially and in which the stabiliser $H_\sigma = \mathrm{stab}_G(\sigma)$ is compact, open modulo the centre of G. An example of this is GL_nK acting on a suitable subdivision of its building. Let V be an irreducible, admissible representation of G over k. For each simplex σ we have a kH_σ-bar-monomial resolution

$$W_{*,H_\sigma} \longrightarrow V \longrightarrow 0,$$

obtained as the direct sum of bar-monomial resolutions of the finite-dimensional irreducible summands of V restricted to H_σ. Form the graded k-vectorspace which in degree m is equal to

$$\underline{M}_m = \oplus_{\alpha+n=m} W_{\alpha, H_{\sigma^n}}.$$

If σ^{n-1} is a face of σ^n there is an inclusion $H_{\sigma^n} \subseteq H_{\sigma^{n-1}}$. Therefore there is a monomial chain map

$$i_{H_{\sigma^n}, H_{\sigma^{n-1}}} : W_{*,H_{\sigma^n}} \longrightarrow W_{*,H_{\sigma^{n-1}}}$$

such that

$$i_{H_{\sigma^{n-1}},H_{\sigma^{n-2}}} i_{H_{\sigma^n},H_{\sigma^{n-1}}} = i_{H_{\sigma^n},H_{\sigma^{n-2}}}.$$

If σ^{n-1} is a face of σ^n let $d(\sigma^{n-1},\sigma^n)$ denote the incidence degree of σ^{n-1} in σ^n; this is ± 1. In the simplicial chain complex of Y

$$d(\sigma^n) = \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1},\sigma^n)\sigma^{n-1}.$$

For $x \in W_{\alpha,H_{\sigma^n}}$ write

$$d_Y(x) = \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1},\sigma^n)\ i_{H_{\sigma^n},H_{\sigma^{n-1}}}(x).$$

Let $d_{\sigma^n} : W_{\alpha,H_{\sigma^n}} \longrightarrow W_{\alpha-1,H_{\sigma^n}}$ denote the differential in the monomial resolution.

Define $\underline{d} : \underline{M}_m \longrightarrow \underline{M}_{m-1}$ when $m = \alpha + n$ by

$$\underline{d}(x) = d_Y(x) + (-1)^n d_{\sigma^n}(x).$$

Therefore we have

$$\begin{aligned}
&\underline{d}(\underline{d}(x)) \\
&= \underline{d}\big(\textstyle\sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1},\sigma^n)\ i_{H_{\sigma^n},H_{\sigma^{n-1}}}(x)\big) + \underline{d}((-1)^n d_{\sigma^n}(x)) \\
&= \textstyle\sum_{\substack{\sigma^{n-2} \text{ face of } \sigma^{n-1} \\ \sigma^{n-1} \text{ face of } \sigma^n}} d(\sigma^{n-2},\sigma^{n-1})\ i_{H_{\sigma^{n-1}},H_{\sigma^{n-2}}}(\ d(\sigma^{n-1},\sigma^n)\ i_{H_{\sigma^n},H_{\sigma^{n-1}}}(x)) \\
&\qquad + \textstyle\sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1},\sigma^n)\ i_{H_{\sigma^n},H_{\sigma^{n-1}}}((-1)^n d_{\sigma^n}(x)) \\
&\qquad + (-1)^{n-1} \textstyle\sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1},\sigma^n)\ d_{\sigma^{n-1}}(i_{H_{\sigma^n},H_{\sigma^{n-1}}}(x)) \\
&\qquad + (-1)^n d_{\sigma^n}((-1)^n d_{\sigma^n}(x)) \\
&= \textstyle\sum_{\substack{\sigma^{n-2} \text{ face of } \sigma^{n-1} \\ \sigma^{n-1} \text{ face of } \sigma^n}} d(\sigma^{n-2},\sigma^{n-1})\ \ d(\sigma^{n-1},\sigma^n)\ i_{H_{\sigma^n},H_{\sigma^{n-2}}}(x) \\
&\qquad + (-1)^n \textstyle\sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1},\sigma^n)\ i_{H_{\sigma^n},H_{\sigma^{n-1}}}(d_{\sigma^n}(x)) \\
&\qquad + (-1)^{n-1} \textstyle\sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1},\sigma^n)\ i_{H_{\sigma^n},H_{\sigma^{n-1}}}(d_{\sigma^n}(x)) \\
&\qquad + d_{\sigma^n}(d_{\sigma^n}(x)) \\
&= \textstyle\sum_{\substack{\sigma^{n-2} \text{ face of } \sigma^{n-1} \\ \sigma^{n-1} \text{ face of } \sigma^n}} d(\sigma^{n-2},\sigma^{n-1})\ \ d(\sigma^{n-1},\sigma^n)\ i_{H_{\sigma^n},H_{\sigma^{n-2}}}(x) \\
&= 0
\end{aligned}$$

because, as is well-known, for each pair (σ^n, σ^{n-2}) the sum

$$\sum_{\substack{\sigma^{n-2} \text{ face of } \sigma^{n-1} \\ \sigma^{n-1} \text{ face of } \sigma^n}} d(\sigma^{n-2}, \sigma^{n-1}) \ \ d(\sigma^{n-1}, \sigma^n) = 0.$$

Theorem 4.2. *Existence of monomial resolutions for GL_2K*
$(\underline{M}_*, \underline{d})$ is a kG-monomial complex when $G = GL_nK$ and Y is the building of G.

When $G = GL_2K$ $(\underline{M}_*, \underline{d})$ is a monomial resolution of the admissible representation V and, as such, is unique up to chain homotopy equivalence.

Conjecture 4.3. *Existence of monomial resolutions for GL_nK*
For $n \geq 3$, K local and $G = GL_nK$

$$\longrightarrow \underline{M}_i \stackrel{d}{\longrightarrow} \underline{M}_{i-1} \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} \underline{M}_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is a kG-monomial resolution. That is, for each $(H, \phi) \in \mathcal{M}_G$

$$\longrightarrow \underline{M}_i^{((H,\phi))} \stackrel{d}{\longrightarrow} \underline{M}_{i-1}^{((H,\phi))} \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} \underline{M}_0^{((H,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(H,\phi)} \longrightarrow 0$$

is an exact sequence of k-vector spaces.

Remark 4.4. From the monomial resolution one recovers V as the zero-th homology.

5. Relation of the monomial resolution with π_K-adic levels

5.1. In this section suppose that V is defined on a vector space over k, an algebraically closed field of characteristic zero.

For $n \geq 1$ consider the compact open modulo the centre subgroup $J_n = K^* \cdot U_n$ where $U_n = 1 + \pi_K^n M_2 \mathcal{O}_K$. In this section we shall assume that n is large enough such that the restriction of the central character $\underline{\phi}$ to $K^* \bigcap U_n$ is trivial. In this case we shall denote by

$$\underline{\phi} : K^* \cdot U_n \longrightarrow k^*$$

the character which is given by $\underline{\phi}$ on K^* is trivial on U_n.

Theorem 5.2.
In the situation of §5.1

$$\underline{M}_*^{((K^* \cdot U_n, \underline{\phi}))} \longrightarrow V^{(K^* \cdot U_n, \underline{\phi})}$$

is a $K^* \cdot U_n/U_n$-monomial resolution, whose chain homotopy class contains a finitely generated $K^* \cdot U_n/U_n$-monomial resolution of finite length.

Proof
This follows from the fact that $\underline{M}_* \longrightarrow V \longrightarrow 0$ is a monomial resolution of V. □

Remark 5.3. If Conjecture 4.3 were true then the analogue of Theorem 4.2 for GL_nK would be true for all $n \geq 1$.

6. Epsilon factors and L-functions

6.1. If V is an admissible representation of GL_2K and $\underline{M}_* \longrightarrow V$ is a monomial resolution as in Theorem 4.2 one may construct epsilon factors for V by applying an integral to each Line given by an integral for character values which in the finite finite case specialises to the Kondo Gauss sums. These integrals respect induction from one compact, open modulo the centre subgroup to another.

I am **assuming** an analogue of the result concerning wild epsilon factors modulo p-power roots of unity [3] holds for all but a finite set of Lines with the result that a well-defined epsilon factor modulo p-power roots of unity is defined by a finite product of Kondo-style Gauss sums. Here I ought to mention that I slightly disagree with a fundamental result in [3] (see [5]) so the epsilon factor I propose may only be well defined up to ± 1 times a p-power root of unity.

I have yet to develop the approach of Tate's thesis to each Line to get the L-functions.

These methods would apply to GL_nK if Conjecture 4.3 holds.

7. Galois descent for GL_2K

7.1. Suppose that K is a p-adic local field and $\rho : GL_2K \longrightarrow GL(V)$ is a complex, irreducible admissible representation and that k is the complex field. Let K/F be a Galois extension and suppose that $z^*(\rho)$ is equivalent to ρ for each $z \in \mathrm{Gal}(K/F)$. Therefore for $z \in \mathrm{Gal}(K/F)$ there exists $X_z \in GL(V)$ such that

$$X_z \rho(g) X_z^{-1} = \rho(z(g))$$

for all $g \in GL_2K$. Therefore if $z, z_1 \in \mathrm{Gal}(K/F)$ replacing g by $z_1(g)$ gives

$$X_z \rho(z_1(g)) X_z^{-1} = \rho(z z_1(g))$$

and so

$$X_z \rho(z_1(g)) X_z^{-1} = X_z X_{z_1} \rho(g) X_{z_1}^{-1} X_z^{-1} = X_{zz_1} \rho(g) X_{zz_1}^{-1}.$$

By Schur's Lemma $X_{z_1}^{-1} X_z^{-1} X_{zz_1}$ is a scalar matrix and so

$$f(z, z_1) = X_{z_1}^{-1} X_z^{-1} X_{zz_1}$$

is a function from $\mathrm{Gal}(K/F) \times \mathrm{Gal}(K/F)$ to $\mathbb{C}^*$. In fact, f is a 2-cocycle.

By a result of Tate $H^2(F; \mathbb{C}^*) = 0$ if K is local or global. Therefore there exists a finite Galois extension E/F such that the 2-cocycle induced by f

$$f' : \mathrm{Gal}(E/F) \times \mathrm{Gal}(E/F) \longrightarrow \mathbb{C}^*$$

is a coboundary $f' = dF$. Then $z \mapsto X_z F(z)$ is a homomorphism from $\mathrm{Gal}(E/F) \longrightarrow GL(V)$.

Recall that the semi-direct product $\mathrm{Gal}(E/F) \propto GL_2K = G$ is given by the set $\mathrm{Gal}(E/F) \times GL_2K$ with the product defined by

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 h_2, g_1 h_1(g_2)).$$

The map

$$\tilde{\rho} : \mathrm{Gal}(E/F) \propto GL_n K \longrightarrow GL(V)$$

which sends (z, g) to $\rho(g)X_z F(z)$ is an irreducible admissible representation extending ρ. Any two such extensions differ by twisting via a homomorphism $\mathrm{Gal}(E/F) \longrightarrow \mathbb{C}^*$ for some E.

The action of $\mathrm{Gal}(K/F)$ on $K \oplus K$ preserves the lattice $L = \mathcal{O}_K \oplus \mathcal{O}_K$ and $L' = \mathcal{O}_K \oplus \pi_K \mathcal{O}_K$ and their stabilisers under the tree-action H_1 and H_2 Therefore the Galois action fixes the canonical fundamental domain on the tree and the semi-direct product acts on the tree of GL_2K, extending the action of GL_2K.

Replacing H_1 and H_2 by $\mathrm{Gal}(E/K) \propto H_1$ and $\mathrm{Gal}(E/K) \propto H_2$ yields the following result.

Theorem 7.2.

There exists a monomial resolution of $\tilde{\rho}$ which is unique up to chain homotopy and satisfies the analogue of Theorem 4.2.

7.3. *The Galois descent yoga*

Take ρ and form the monomial resolution of $\tilde{\rho}$ as in Theorem 7.2. Quotient out the monomial complex by the Lines whose stabiliser group is not sub-conjugate in the semi-direct product to $\mathrm{Gal}(E/F) \times GL_nF$. This is a monomial complex for the semi-direct product which originates, via induction, with $\mathrm{Gal}(E/F) \times GL_nF$.

In **ONE** case of finite general linear groups this yoga is equivalent to Shintani descent. See [6].

I conjecture (on the basis of only one case!!) that Galois base change for admissible representations of GL_n of local fields can be described in terms of the above yoga with monomial resolutions.

8. Restricted tensor products and global automorphic representations

8.1. Automorphic representations of GL_2F where F is a number field are constructed by the tensor product theorem. For convenience let $F = \mathbb{Q}$, the rationals, so that we can just refer to [2]. Here is the tensor product theorem in that case - the reader is referred to [2] for details. Suffice to say that the key to the construction is the fact that $V^{(GL_2\mathbb{Z}_p, 1)}$ is one-dimensional for almost all primes p.

Theorem 8.2. *Tensor product theorem ([2] Vol. I Theorem 10.8.2 pp. 407)*

Let (π, V) denote an irreducible admissible $(\mathcal{U}(gl_2\mathbb{C}), K_\infty) \times GL_2\mathbb{A}_{fin}$-module. Let $\{q_1, \ldots, q_m\}$ be the finite set of primes where π is ramified. Let $S = \{\infty, q_1, \ldots, q_m$. Then there exists

(i) an irreducible admissible $(\mathcal{U}(gl_2\mathbb{C}), K_\infty)$-module (π_∞, V_∞),

(ii) an irreducible admissible representation (π_p, V_p) of $GL_2\mathbb{Q}_p$ for each finite prime p,

(iii) a non-zero vector $v_p^0 \in V_p^{GL_2\mathbb{Z}_p}$ for each prime $\notin S$ such that

$$\pi \cong \bigotimes_{v \leq \infty}{}' \pi_v.$$

The factors are unique ([2] Vol. I Theorem 10.8.12 pp. 412).

8.3. *Restricted tensor products of monomial resolutions*

The restricted tensor product construction of Theorem 8.1 makes sense when applied to monomial resolutions of local admissible irreducible representation of $GL_2\mathbb{Q}_p$. One just restricts the tensor product of the monomial resolutions to lie in $M_*^{((\mathbb{Q}_p^* \cdot GL_2 Z_p, \underline{\phi}))}$ for primes for which $V^{(GL_2\mathbb{Z}_p, 1)}$ is one-dimensional.

This constructs a global monomial resolution, unique up to chain homotopy, for each global automorphic representation of $GL_2\mathbb{Q}$ or , more generally, for GL_2F for any number field, F. Handling the Archimedean places in the usual manner (see [2]) gives a construction of automorphic representations of GL_2F via monomial resolutions.

References

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[6] V.P. Snaith: Galois descent of representations, posted on my Sheffield webpage January 2013.