GALOIS EXERCISES WITH THE SECOND CHERN CLASS

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Abstract. We examine examples of the behaviour of a formula for the second Chern class of irreducible representations related by Shintani base change.

Contents

1. The second Chern class and 4-cocycle $\tilde{c}_4$

Let G be a finite group with a subgroup H and suppose that $\tilde{\lambda} : H \longrightarrow \mathbb{C}^*$ is a homomorphism. The coboundary of $\tilde{\lambda}$ is a homology class $[\lambda] \in H^2(H;\mathbb{Z})$ represented by a 2-cocycle λ on the inhomogeneous bar resolution of H.

The second cohomology class of the complex representation $\mathrm{Ind}_H^G(\tilde{\lambda})$ is denoted by $c_2(\mathrm{Ind}_H^G(\tilde{\lambda})) \in H^4(G;\mathbb{Z})$.

As described in §2 there is a homomorphism

$$\Phi_{\underline{x}} : G \longrightarrow \Sigma_m \int H$$

where $m = [G : H]$.

In §2 we describe an explicit 4-cocycle

$$\tilde{c}_4 \in \mathrm{Hom}_{\mathbb{Z}[\Sigma_m \int H]}(\underline{B}_4\Sigma_m \int H, \mathbb{Z})$$

which gives rise to a cohomology class

$$\Phi^*[\tilde{c}_4] \in H^4(G;\mathbb{Z})$$

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whose relation to the second Chern class is given by

$$c_2(\mathrm{Ind}_H^G(\tilde{\lambda})) = c_{2,0}(\lambda) + c_{1,1}(\lambda) + c_{0,2}(\lambda)$$

where
(a) $c_{0,2}(\lambda) = \Phi^*[\tilde{c}_4]$,
(b) $c_{1,1}(\lambda) = c_1(\mathrm{Ind}_H^G(1)) \cdot \mathrm{Trace}_H^G(\lambda)$.
(c) $c_{2,0}(\lambda) = c_2(\mathrm{Ind}_H^G(1))$.

The above formulae originate in [3] and [4] and the explicit 4-cocycle was derived in [14] by following the theory discovered by Lenny Evens. The formula simplifies when we invert 6 to

$$c_2(\mathrm{Ind}_H^G(\tilde{\lambda})) = c_{2,0}(\lambda) \in H^4(G; \mathbb{Z}[1/6])$$

because 6 kills the low-dimensional integral cohomology of symmetric groups which contain $c_1(\mathrm{Ind}_H^G(1))$ and $c_2(\mathrm{Ind}_H^G(1))$ ([7], [8]).

The explicit $2n$-cocycle $\tilde{c}_{2n} \in H^{2n}(\Sigma_m \int H; \mathbb{Z})$ analogous to $\tilde{c}_4$ has a similar form the finding of which I leave as an exercise to the reader. The relation between $\tilde{c}_{2n}$ and the n-th Chern class has more terms in it (see [4]).

2. A 4-cocycle $\tilde{c}_4$ and group actions

Now suppose that G is a finite subgroup with subgroup H such that $x_1, \ldots, x_m$ are coset representatives for $G/H = \{x_1 H, \ldots, x_m H\}$.

Therefore there is a homomorphism

$$\pi_H^G : G \longrightarrow \Sigma_m$$

such that

$$g x_i = x_{\pi_H^G(g)(i)} h_i(g)$$

where $h_i(g) \in H$. Since

$$\begin{aligned} g(g'(x_i)) &= g x_{\pi_H^G(g')(i)} h_i(g') \\ &= x_{\pi_H^G(g)(\pi_H^G(g')(i))} h_{\pi(g')(i)}(g) h_i(g') \\ &= x_{(\pi_H^G(gg')(i)} h_{\pi(g')(i)}(g) h_i(g') \end{aligned}$$

we see that

$$h_i(gg') = h_{\pi(g')(i)}(g) h_i(g').$$

If we set

$$\Phi_H^G(g) = (\pi_H^G(g), h_1(g), h_2(g), \ldots, h_n(g)) \in \Sigma_m \int H$$

we find that

$$\Phi_H^G(gg')$$

$$= (\pi_H^G(gg'), h_1(gg'), h_2(gg'), \ldots, h_n(gg'))$$

$$= (\pi_H^G(g)\pi_H^G(g'), h_{\pi_H^G(g')(1)}(g)h_1(g'), h_{\pi_H^G(g')(2)}(g)h_2(g'), \ldots, h_{\pi_H^G(g')(n)}(g)h_n(g'))$$

$$= (\pi_H^G(g), h_1(g), \ldots, h_n(g)) \cdot (\pi_H^G(g'), h'_1(g'), \ldots, h'_n(g'))$$

$$= \Phi_H^G(g)\Phi_H^G(g')$$

so that

$$\Phi_H^G : G \longrightarrow \Sigma_m \int H$$

is a homomorphism, depending (up to conjugacy) on the choice of coset representatives. See [14] for the conventions concerning the multiplication in the semi-direct product $\Sigma_m \int H$[1].

Let $\underline{B}_* G$ denote the inhomogeneous bar resolution of G ([6] p.212 et seq; [11]). Suppose that $\lambda \in \mathrm{Hom}_{\mathbb{Z}[H]}(\underline{B}_2 H, \mathbb{Z})$ is a 2-cocycle. Then from [14] we have a 4-cocycle

$$\tilde{c}_4 \in \mathrm{Hom}_{\mathbb{Z}[\Sigma_m \int H]}(\underline{B}_4 \Sigma_m \int H, \mathbb{Z})$$

sending the 4-chain ($\hat{\sigma}, \sigma, \sigma', \sigma'', \sigma''' \in \Sigma_m$ and $\hat{h}_j, h_j, h'_j.h''_j, h'''_j \in H$)

$$z = (\hat{\sigma}, \hat{h}_1, \ldots)[(\sigma, h_1, \ldots)|(\sigma', h'_1, \ldots)|(\sigma'', h''_1, \ldots)|(\sigma''', h'''_1, \ldots)]$$

to

$$\tilde{c}_4(z) = \sum_{1 \leq i \neq j \leq m} \lambda[h_{\sigma^{-1}(i)}|h'_{(\sigma\sigma')^{-1}(i)}] \cdot \lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(j)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(j)}],$$

which defines a cohomology class

$$[\tilde{c}_4] \in H^4(\Sigma_m \int H; \mathbb{Z}) \cong H_3(\Sigma_m \int H; \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H_3(\Sigma_m \int H; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}).$$

Let J be a finite group acting (on the left) upon G and preserving the subgroup H. Suppose that $\phi \in J$ and that $\tilde{\lambda} : H \longrightarrow \mathbb{C}^*$ is a homomorphism such that $\phi^*(\tilde{\lambda}) = \tilde{\lambda}$.

We have an isomorphism of cohomology groups

$$\partial : H^1(H; \mathbb{C}^*) \cong \mathrm{Hom}(H, \mathbb{C}^*) \xrightarrow{\cong} H^2(H; \mathbb{Z}).$$

We shall verify that there is a 2-cocycle λ representing $\partial(\tilde{\lambda})$ which is also invariant under ϕ. Using the inhomogeneous bar resolution to describe

[1]The product in the semi-direct product $J \propto G$ differs from that chosen for the multiplication in the wreath product.

$H^*(H; M)$, when H acts trivially on M, the cohomology is the homology of the chain complex

$$M \longrightarrow \mathrm{Map}(H, M) \longrightarrow \mathrm{Map}(H \times H, M) \longrightarrow \mathrm{Map}(H \times H \times H, M) \longrightarrow \ldots.$$

On the level of this chain complex $\partial(\tilde{\lambda})$ is given by a representative $\lambda : H \times H \longrightarrow \mathbb{Z}$ defined in the following manner. Write $\tilde{\lambda}(h) = e^{2\pi\sqrt{-1}x(h)}$ where $x(h)$ is a real number in the range $0 \leq x(h) < 1$. Lift λ to $\hat{\lambda} : H \longrightarrow \mathbb{C}$ such that $e^{\hat{\lambda}(h)} = \tilde{\lambda}(h)$ given by $\hat{\lambda}(h) = 2\pi\sqrt{-1}x(h)$. Therefore

$$d(\hat{\lambda}) : H \times H \longrightarrow \mathbb{C}$$

is given by

$$d(\hat{\lambda})(h_1, h_2) = \hat{\lambda}(h_2) - \hat{\lambda}(h_1 h_2) + \hat{\lambda}(h_1) = 2\pi\sqrt{-1}(x_2 - [x_1 + x_2] + x_1)$$

where $[x_1 + x_2] = x_1 + x_2$ if $0 \leq x_1 + x_2 < 1$ and $[x_1 + x_2] = x_1 + x_2 - 1$ if $1 \leq x_1 + x_2 < 2$. By definition

$$\lambda(h_1, h_2 = d(\hat{\lambda})(h_1, h_2)/2\pi\sqrt{-1}$$

so that

$$\lambda(h_1, h_2) = \begin{cases} 0 & \text{if } 0 \leq x_1 + x_2 < 1 \\ \\ 1 & \text{if } 1 \leq x_1 + x_2 < 2. \end{cases}$$

Since the ϕ-action fixes $\tilde{\lambda}$ and hence also x_1, x_2 we see that it fixed $\lambda(h_1, h_2)$, too.

Since J acts on G/H there is a homomorphism $\phi \mapsto S_\phi \in \Sigma_m$.

As an important special case we shall first study the case where the J-action gives a permutation of the *coset representatives* i.e. $\phi(x_i) = x_{S_\phi(i)}$.

We have, as introduced earlier, $g x_i = x_{\pi_H^G(g)(i)} h_i(g)$ and acting by ϕ yields

$$\phi(g) x_{S_\phi(i)} = x_{S(\pi_H^G(g)(i))} \phi(h_i(g)) = x_{(S \pi_H^G(g) S^{-1})(S(i)))} \phi(h_i(g)).$$

Setting $S_\phi(i) = j$ we have

$$\phi(g) x_j = x_{(S_\phi \pi_H^G(g) S_\phi^{-1})(j))} \phi(h_{S_\phi^{-1}(j)}(g)).$$

From ([14] §5) we have

$$\Phi_{\underline{x}} : GL_2\mathbb{F}_{q^n} \longrightarrow \Sigma_m \int H$$

given by

$$\Phi_{\underline{x}}(g)) = (\pi_H^G(g), h_1(g), h_2(g), \ldots, h_m(g)) \in \Sigma_m \int H$$

so that

$$\Phi_{\underline{x}}(\phi(g)) = (S_\phi \pi_H^G(g) S_\phi^{-1}, \phi(h_{S_\phi^{-1}(1)}(g)), \ldots, \phi(h_{S_\phi^{-1}(m)}(g))) \in \Sigma_m \int H.$$

Multiplication in the semi-direct product is given by

$$(\sigma, h_1, \ldots, h_n) \cdot (\sigma', h'_1, \ldots, h'_n) = (\sigma\sigma', h_{\sigma'(1)} h'_1, h_{\sigma'(2)} h'_2, \ldots)$$

so that

$$(S_\phi, 1, \ldots) \cdot (\pi_H^G(g), \phi(h_1(g)), \ldots, \phi(h_m(g))) \cdot (S_\phi^{-1}, 1, \ldots)$$

$$= (S_\phi, 1, \ldots) \cdot (\pi_H^G(g) S_\phi^{-1}, \phi(h_{S_\phi^{-1}(1)}(g)), \ldots, \phi(h_{S_\phi^{-1}(m)}(g)))$$

$$= (S_\phi \pi_H^G(g) S_\phi^{-1}, \phi(h_{S_\phi^{-1}(1)}(g)), \ldots, \phi(h_{S_\phi^{-1}(m)}(g))).$$

Therefore $\Phi_{\underline{x}}$extends to a homomorphism on the of semi-direct product with J, also denoted by $\Phi_{\underline{x}}$,

$$\Phi_{\underline{x}} : J \propto G \longrightarrow \Sigma_m \int H$$

given by $\phi \mapsto S_\phi$. Here the multiplication in $J \propto G$ is given by

$$(\phi, g)(\phi', g') = (\phi\phi', g\phi(g')).$$

On the inhomogeneous bar resolution in dimension four $\Phi_{\underline{x}}$ maps

$$g_0[g_1|g_2|g_3|g_4]$$

to, temporarily denoting π_H^G simply by π,

$$(\pi(g_0), h_1(g_0), ...)[(\pi(g_1), h_1(g_1), ...)|(\pi(g_2), h_1(g_2), ...)|$$
$$(\pi(g_3), h_1(g_3), ...)|(\pi(g_4), h_1(g_4), ...)]$$

so that in the notation for the 4-cocycle $\tilde{c}_4$ we have

$$\hat{\sigma} = \pi(g_0),\ \sigma = \pi(g_1),\ \sigma' = \pi(g_2), \sigma'' = \pi(g_3), \sigma''' = \pi(g_4)$$

and

$$\hat{h}_i = h_i(g_0),\ h_i = h_i(g_1),\ h'_i = h_i(g_2),\ h''_i = h_i(g_3),\ h'''_i = h_i(g_4).$$

For the 4-cycle which starts by sending $g_0[g_1|g_2|g_3|g_4]$ to

$$(\pi(g_0), h_1(\phi(g_0)), ...)[(\pi(g_1), h_1(\phi(g_1)), ...)|(\pi(g_2), h_1(\phi(g_2)), ...)\ \ldots]$$

we have

$$\hat{\sigma} = \pi(g_0),\ \sigma = \pi(g_1),\ \sigma' = \pi(g_2),\ \sigma'' = \pi(g_3),\ \sigma''' = \pi(g_4)$$

and

$$\hat{h}_i(g_0) = h_i(\phi(g_0)),\ h_i(g_1) = h_i(\phi(g_1)),\ h'_i(g_2) = h_i(\phi(g_2)),$$

$$h''_i(g_3) = h_i(\phi(g_3)),\ h'''_i(g_4) = h_i(\phi(g_4)).$$

From the formula, introduced earlier, defining the 4-cocycle

$$\tilde{c}_4 \in \mathrm{Hom}_{\mathbb{Z}[\Sigma_m \int H]}(\underline{B}_4 \Sigma_m \int H, \mathbb{Z})$$

sending the 4-chain ($\hat{\sigma}, \sigma, \sigma', \sigma'', \sigma''' \in \Sigma_n$ and $\hat{h}_j, h_j, h'_j . h''_j, h'''_j \in H$)

$$z = (\hat{\sigma}, \hat{h}_1, \ldots)[(\sigma, h_1, \ldots)|(\sigma', h'_1, \ldots)|(\sigma'', h''_1, \ldots)|(\sigma''', h'''_1, \ldots)]$$

to

$$\tilde{c}_4(z) = \sum_{1\leq i\neq j\leq m} \lambda[h_{\sigma^{-1}(i)}|h'_{(\sigma\sigma')^{-1}(i)}] \cdot \lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(j)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(j)}].$$

The two 4-chains

$$(\hat{\sigma}, \hat{h})[(\sigma, h)|(\sigma', h')|(\sigma'', h'')|(\sigma''', h''')]$$

introduced above differ only in the fact that ϕ has been applied to the h's. If the 2-cocycle λ is fixed by ϕ then $\tilde{c}_4$ agrees on these two 4-chains.

Therefore, for $g \in G$,

$$\tilde{c}_4(\Phi_{\underline{x}}(\phi(g_0))[\Phi_{\underline{x}}(\phi(g_1)) \ \ldots]) = (S - S^{-1})^* \tilde{c}_4(\Phi_{\underline{x}}(g_0)[\Phi_{\underline{x}}(g_1) \ \ldots]).$$

Hence the 4-cocycle on G extends to the semi-direct product $J \propto G$ by sending the action of ϕ to $(S_\phi - S_\phi^{-1})$.

The above discussion establishes the following result:

Theorem 2.1.

Let J be a finite group acting on the left of the finite group G and preserving the subgroup $H \subseteq G$. Suppose that there exists a set of coset representatives for G/H which are permuted by the J-action. Let $\lambda \in \mathrm{Hom}_{\mathbb{Z}[H]}(\underline{B}_2 H, \mathbb{Z})$ be a J-fixed 2-cocycle defined on the inhomogeneous bar resolution, where all groups act trivially on $\mathbb{Z}$. Then the 4-cocycle, which was introduced above,

$$\Phi_{\underline{x}}^* \tilde{c}_4 \in \mathrm{Hom}_{\mathbb{Z}[G]}(\underline{B}_4 G, \mathbb{Z})$$

extends to a 4-cocycle in the bar resolution of the semi-direct product $J \propto G$, by the explicit formula given above.

In Theorem 2.1 we considered the special situation where the homomorphism $\phi \mapsto S_\phi$ could be realised by the action of ϕ permuting the coset representatives of G/H. Now we compare this with the general situation in which all we know is that ϕ permutes the cosets G/H.

Since $\phi^*(\tilde{\lambda}) = \tilde{\lambda}$ for all $\phi \in J$ we may extend $\tilde{\lambda}$ on the semi-direct product to give a homomorphism, also denoted by $\tilde{\lambda}$,

$$\tilde{\lambda} : J \propto H \longrightarrow \mathbb{C}^*$$

given by $\tilde{\lambda}(\phi, h) = \tilde{\lambda}(h)$.

Therefore we have an induced representation $\mathrm{Ind}_{J\propto H}^{J\propto G}(\hat{\lambda})$ whose associated $\tilde{c}_4$ we shall calculate. The cosets satisfy

$$J \propto G/J \propto H = \{(1, x_i)J \propto H \ |1 \leq i \leq m\}$$

where the $x_i \in G$ are the coset representatives for G/H.

Therefore $gx_i = x_{\pi_H^G(i)} h_i(g)$ with $g \in G$ and $h_i(g) \in H$. We also have $\phi(1, x_i) = (1, x_{S_\phi(i)})(\underline{j}_i(\phi), \underline{h}_i(\phi))$ with $\phi \in J$ and $(\underline{j}_i(\phi), \underline{h}_i(\phi)) \in J \propto H$. Here $S : J \longrightarrow \Sigma_m$ is a homomorphism.

Since, in the semi-direct product $J \propto G$, we have

$$(\phi, 1)(1, g) = (\phi, \phi(g)) = (1, \phi(g))(1, \phi)$$

so that

$$S_\phi \cdot \pi_H^G(g) = \pi_H^G(\phi(g)) \cdot S_\phi \in \Sigma_m.$$

Therefore we have a homomorphism

$$\Phi_{\underline{x}} : J \propto G \longrightarrow \Sigma_m \int (J \propto H)$$

given by

$$\Phi_{\underline{x}}(1, g) = (\pi_H^G(g), (1, h_1(g)), (1, h_2(g)), \ldots, (1, h_m(g))) \in \Sigma_m \int (J \propto H)$$

and

$$\Phi_{\underline{x}}(1, \phi^{-1}(g)) = (\pi_H^G(\phi^{-1}(g)), (1, h_1(\phi^{-1}(g))), (1, h_2(\phi^{-1}(g))), \ldots, (1, h_m(\phi^{-1}(g))))$$

and

$$\Phi_{\underline{x}}(\phi, 1)) = (S_\phi, (\underline{j}_1(\phi), \underline{h}_1(\phi)), \ldots, (\underline{j}_m(\phi), \underline{h}_m(\phi))) \in \Sigma_m \int (J \propto H).$$

Therefore

$$\begin{aligned}
&\Phi_{\underline{x}}(\phi, g) \\
&= \Phi_{\underline{x}}(\phi, 1)\Phi_{\underline{x}}(1, \phi^{-1}(g)) \\
&= (S_\phi, (\underline{j}_1(\phi), \underline{h}_1(\phi)), \ldots, (\underline{j}_m(\phi), \underline{h}_m(\phi)))\times \\
&(\pi_H^G(\phi^{-1}(g)), (1, h_1(\phi^{-1}(g))), (1, h_2(\phi^{-1}(g))), \ldots, (1, h_m(\phi^{-1}(g)))) \\
&= (S_\phi \pi_H^G(\phi^{-1}(g)), (\underline{j}_{\pi_H^G(\phi^{-1}(g))(1)}(\phi), \underline{h}_{(\pi_H^G(\phi^{-1}(g))(1))} h_1(\phi^{-1}(g))), \ldots, \\
&\qquad \ldots, (\underline{j}_{\pi_H^G(\phi^{-1}(g))(m)}(\phi), \underline{h}_{(\pi_H^G(\phi^{-1}(g))(m))} h_m(\phi^{-1}(g)))).
\end{aligned}$$

Starting in the inhomogenous bar resolution in dimension four with the 4-chain

$$(\phi_0, g_0)[(\phi_1, g_1)|(\phi_2, g_2)|(\phi_3, g_3)|(\phi_4, g_4)]$$

we have, in the notation for the 4-cocycle $\tilde{c}_4$ associated to $\mathrm{Ind}_{J\propto H}^{J\propto G}(\hat{\lambda})$,

$$\hat{\sigma} = S_{\phi_0}\pi_H^G(\phi_0^{-1}(g_0)), \sigma = S_{\phi_1}\pi_H^G(\phi_1^{-1}(g_1)), \sigma' = S_{\phi_2}\pi_H^G(\phi_2^{-1}(g_2)),$$

$$\sigma'' = S_{\phi_3}\pi_H^G(\phi_3^{-1}(g_3)), \sigma''' = S_{\phi_4}\pi_H^G(\phi_4^{-1}(g_4))$$

and

$$\hat{h}_i = (\underline{j}_{\pi_H^G(\phi_0^{-1}(g_0))(i)}(\phi_0), \underline{h}_{(\pi_H^G(\phi_0^{-1}(g_0))(i))} h_i(\phi_0^{-1}(g_0))),$$

$$h_i = (\underline{j}_{\pi_H^G(\phi_1^{-1}(g_1))(i)}(\phi_1), \underline{h}_{(\pi_H^G(\phi_1^{-1}(g_1))(i))} h_i(\phi_1^{-1}(g_1))),$$

$$h_i' = (\underline{j}_{\pi_H^G(\phi_2^{-1}(g_2))(i)}(\phi_2), \underline{h}_{(\pi_H^G(\phi_2^{-1}(g_2))(i))} h_i(\phi_2^{-1}(g_2))),$$

$$h_i'' = (\underline{j}_{\pi_H^G(\phi_3^{-1}(g_3))(i)}(\phi_3), \underline{h}_{(\pi_H^G(\phi_3^{-1}(g_3))(i))} h_i(\phi_3^{-1}(g_3))),$$

$$h_i''' = (\underline{j}_{\pi_H^G(\phi_4^{-1}(g_4))(i)}(\phi_4), \underline{h}_{(\pi_H^G(\phi_4^{-1}(g_4))(i))} h_i(\phi_4^{-1}(g_4))).$$

Theorem 2.2.

Let J be a finite group acting on the left of the finite group G and preserving the subgroup $H \subseteq G$. Let $\lambda \in \mathrm{Hom}_{\mathbb{Z}[H]}(\underline{B}_2 H, \mathbb{Z})$ be a J-fixed 2-cocycle, derived as the coboundary of the homomorphism $\hat{\lambda}$, defined on the inhomogeneous bar resolution, where all groups act trivially on $\mathbb{Z}$. Then the 4-cocycle, which was introduced above, associated to the induced representation $\mathrm{Ind}_{J \propto H}^{J \propto G}(\hat{\lambda})$,

$$\Phi_{\underline{x}}^* \tilde{c}_4 \in \mathrm{Hom}_{\mathbb{Z}[G]}(\underline{B}_4 J \propto G, \mathbb{Z})$$

is a 4-cocycle in the bar resolution of the semi-direct product $J \propto G$, given by the explicit formula for $\tilde{c}_4$ using the parameter-values

$$\hat{\sigma}, \sigma, \sigma', \sigma'', \sigma''', \hat{h}_i, h_i, h_i', h_i'', h_i'''$$

listed immediately above.

The following result is clear from the formulae.

Corollary 2.3.

The 4-cocycles of Theorem 2.1 and Theorem 2.2 coincide in the special case of Theorem 2.1.

3. $\mathrm{Irr}(GL_2\mathbb{F}_q)$

In the [12] Chapter Three) the irreducible Weil representation $r(\Theta)$ is constructed from a character $\Theta : \mathbb{F}_{q^2}^* \longrightarrow \mathbb{C}^*$ which is not fixed by the Frobenius of $\mathbb{F}_{q^2}/\mathbb{F}_q$. I shall recall other constructions of $r(\Theta)$ in an Appendix. Before proceeding further, we shall now construct the remaining irreducible representations of GL_2F_q.

Suppose that we are given characters of the form

$$\chi, \chi_1, \chi_2 : F_q^* \longrightarrow \mathbf{C}^*$$

then we clearly have a one-dimensional representation, $L(\chi)$, given by

$$L(\chi) = \chi \cdot det : GL_2F_q \xrightarrow{det} F_q^* \xrightarrow{\chi} \mathbf{C}^*.$$

If χ_1 and χ_2 are *distinct* define

$$Inf_T^B(\chi_1 \otimes \chi_2) : B \longrightarrow \mathbf{C}^*$$

by inflating $\chi_1 \otimes \chi_2$ from the diagonal torus, T, to the Borel subgroup, B, of upper triangular matrices. That is

$$Inf_T^B(\chi_1 \otimes \chi_2)(\left(\begin{array}{cc} \alpha & \beta \\ 0 & \delta \end{array} \right)) = \chi_1(\alpha)\chi_2(\delta).$$

Define a $(q+1)$-dimensional representation, $R(\chi_1, \chi_2)$, by

$$R(\chi_1, \chi_2) = Ind_B^{GL_2 F_q}(Inf_T^B(\chi_1 \otimes \chi_2)).$$

When $\chi = \chi_1 = \chi_2$ we have

$$Inf_T^B(\chi \otimes \chi) = Res_B^{GL_2 F_q}(L(\chi)) : B \longrightarrow \mathbf{C}^*$$

so that there is a canonical surjection of the form

$$Ind_B^{GL_2 F_q}(Inf_T^B(\chi \otimes \chi)) \longrightarrow Ind_{GL_2 F_q}^{GL_2 F_q}(L(\chi)) = L(\chi).$$

Therefore we may define a q-dimensional representation, $S(\chi)$, by means of the following split short exact sequence of representations

$$0 \longrightarrow S(\chi) \longrightarrow Ind_B^{GL_2 F_q}(Inf_T^B(\chi \otimes \chi)) \longrightarrow L(\chi) \longrightarrow 0.$$

Theorem 3.1. *(*[12] *Theorem 3.2.4)* A complete list of all the irreducible representations of $GL_2 F_q$ is given by

(i) $L(\chi)$ for $\chi : F_q^* \longrightarrow \mathbf{C}^*$,

(ii) $S(\chi)$ for $\chi : F_q^* \longrightarrow \mathbf{C}^*$,

(iii) $R(\chi_1, \chi_2) = R(\chi_2, \chi_1)$ for any pair of distinct characters

$$\chi_1, \chi_2 : F_q^* \longrightarrow \mathbf{C}^*$$

and

(iv) $r(\Theta) = r(F^*(\Theta))$ for any character $\Theta : F_{q^2}^* \longrightarrow \mathbf{C}^*$ which is distinct from its Frobenius conjugate, $F^*(\Theta)$.

4. Shintani base change

4.1. Let $\Sigma \in \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$ denote the Frobenius automorphism. Let us recall the main result of [10] which, for our notation used in [13] for the semi-direct product, is stated in the following form:

Theorem 4.2. *(*[10] *Theorem 1; see also Lemmas 2.7 and 2.11)*

(i) Let ρ be a finite-dimensional complex irreducible representation of $GL_n \mathbb{F}_q$. Then there exists an irreducible representation $\tilde{\rho}$ of the semi-direct product $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \propto GL_n \mathbb{F}_{q^m}$ which satisfies, for all $g \in GL_n \mathbb{F}_{q^m}$,

$$\chi_{\tilde{\rho}}(\Sigma, g) = \epsilon \chi_\rho([g\Sigma(g) \ldots \Sigma^{m-1}(g)])$$

where $\epsilon = \pm 1$ is independent of g. Here $[g\Sigma(g) \ldots \Sigma^{m-1}(g)]$ denotes the unique conjugacy class in $GL_n \mathbb{F}_q$ given by the intersection of the conjugacy class of $g\Sigma(g) \ldots \Sigma^{m-1}(g)$ in $GL_n \mathbb{F}_{q^m}$ with $GL_n \mathbb{F}_q$.

(ii) The Shintani base change correspondence (see [13] Appendix I, §4), which is a bijection,

$$Sh : \mathrm{Irr}(GL_n\mathbb{F}_{q^m})^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)} \stackrel{\cong}{\longrightarrow} \mathrm{Irr}(GL_n\mathbb{F}_q)$$

is given by, in the case where ϵ may be chosen to equal 1 in part (i),

$$Sh(\mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_n\mathbb{F}_{q^m}}_{GL_n\mathbb{F}_{q^m}}(\tilde{\rho})) = \rho.$$

When $\epsilon = -1$ is the only possibility there is an extension, denoted by ρ', of ρ to $\mathrm{Gal}(\mathbb{F}_{q^{2m}}/\mathbb{F}_q) \propto GL_n\mathbb{F}_{q^m}$ and

$$\chi_{\rho'}(\Sigma, g) = \chi_\rho([g\Sigma(g)\ldots\Sigma^{m-1}(g)])$$

specifies $\chi(\rho)$ in this case.

In this Theorem χ_ρ denotes the character function of ρ. In part (ii) of the theorem it should be noted that $\tilde{\rho}$ is an irreducible of the first kind because the $\chi_{\tilde{\rho}}(\Sigma, g)$'s are not identically zero ([10] Lemma 1.1(i)) and therefore $\mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_n\mathbb{F}_{q^m}}_{GL_n\mathbb{F}_{q^m}}(\tilde{\rho})$ is an irreducible representation.

Given $\tilde{\rho}$ as in part (i) of the theorem write $\tilde{\rho}(z, 1) = X_z$ for $z \in \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$ and $\tilde{\rho}(1, g) = \hat{\rho}(g)$ for $g \in GL_n\mathbb{F}_{q^m}$. Since $(1, g)(z, 1) = (z, g)$ we have

$$X_z\hat{\rho}(g) = \hat{\rho}(z(g))X_z$$

so that $\chi_{\tilde{\rho}}(\Sigma, g) = \mathrm{Trace}(\tilde{\rho}(1, g)X_\Sigma)$ (see [10] Theorem 1)[2].

For $\tilde{\rho}$ and $\hat{\rho}$ as in Theorem 4.2 the matrix X_Σ will satisfy $X_\Sigma^m = 1$. However, as mentioned in the statement of ([10] Theorem 1), for a general Galois invariant $\hat{\rho}$ there exists a choice satisfying $X_\Sigma^m = \pm 1$. When $X_\Sigma^m = 1$ the extension $\tilde{\rho}$ of $\hat{\rho}$ may be constructed as in Theorem 4.2 but when $X_\Sigma^m = -1$ the extension of $\hat{\rho}$ must be a representation of $\mathrm{Gal}(\mathbb{F}_{q^{2m}}/\mathbb{F}_q) \propto GL_n\mathbb{F}_{q^m}$.

Given a choice of $\hat{\rho}$ the irreducible extension $\tilde{\rho}$ to the semi-direct product, which we may take to be $\mathrm{Gal}(\mathbb{F}_{q^{2m}}/\mathbb{F}_q) \propto GL_n\mathbb{F}_{q^m}$ in general, is unique up to twists by Galois characters.

5. Examples of Shintani base change and c_2

Example 5.1.

Suppose that $r(\Theta) \in \mathrm{Irr}(GL_2\mathbb{F}_{q^n})^{\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)}$ so that $\Theta : \mathbb{F}^*_{q^{2n}} \longrightarrow \mathbb{C}^*$ and $F(\Theta) \neq \Theta$ where $F \in \mathrm{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^n})$ is the Frobenius. Since $r(\Theta)$ is Galois invariant we have ([12] p.102)

$$\Sigma(\mathrm{Res}^{\mathbb{F}^*_{q^{2n}}}_{\mathbb{F}^*_{q^n}}(\Theta)) = \mathrm{Res}^{\mathbb{F}^*_{q^{2n}}}_{\mathbb{F}^*_{q^n}}(\Theta)$$

where $\Sigma \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is the Frobenius. Therefore there is a unique character $\tilde{\Theta} : \mathbb{F}^*_q \longrightarrow \mathbb{C}^*$ such that for all $z \in \mathbb{F}^*_{q^n}$

$$\Theta(z) = \tilde{\Theta}(N(z))$$

[2]The formula of [10] differs from mine because we have used different formulae for the multiplication in a semi-direct product.

where $N : \mathbb{F}_{q^n}^* \longrightarrow \mathbb{F}_q^*$ is the norm. Since $\mathrm{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_q) \cong \mathbb{Z}/2n$ we shall denote the Frobenius in this group by Σ also. Since $\Sigma(\Theta)$ and Θ must be distinct on $\mathbb{F}_{q^{2n}}^*$ we must have $\Sigma(\Theta) = F(\Theta) = \Sigma^n(\Theta)$ and so $\Sigma^{n-1}(\Theta) = \Theta$. Since $\langle \Sigma^{n-1} \rangle$ must be a proper subgroup of $\langle \Sigma \rangle$ we see that n must be off and

$$\mathbb{Z}/n \cong \langle \Sigma^{n-1} \rangle = \mathrm{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^2}).$$

Therefore there exists a unique $\overline{\Theta} : \mathbb{F}_{q^2}^* \longrightarrow \mathbb{C}^*$ such that $\Theta(w) = \overline{\Theta}(N(w))$ for all $w \in \mathbb{F}_{q^{2n}}^*$. If $z \in \mathbb{F}_q^*$ and $s \in \mathbb{F}_{q^n}^*$ satisfy $N(s) = z$ one finds that $\overline{\Theta}(z) = \tilde{\Theta}(s)$ ([12] p.102).

Shintani base change in this case satisfies

$$Sh(r(\Theta)) = r(\overline{\Theta}).$$

From the short exact sequence of Theorem 6.1 the second Chern class of $r(\Theta)$ is given by

$$c_2(r(\Theta)) = c_2(\mathrm{Ind}_{H_{\mathbb{F}_{q^n}}}^{GL_2\mathbb{F}_{q^n}}(\Theta \otimes \Psi)) - c_2(\mathrm{Ind}_{\mathbb{F}_{q^{2n}}^*}^{GL_2\mathbb{F}_{q^n}}(\Theta))$$

$$-c_1(r(\Theta)) \bigcup c_1(\mathrm{Ind}_{\mathbb{F}_{q^{2n}}^*}^{GL_2\mathbb{F}_{q^n}}(\Theta))$$

where, by the discussion of §1, we have

$$c_2(\mathrm{Ind}_{H_{\mathbb{F}_{q^n}}}^{GL_2\mathbb{F}_{q^n}}(\Theta \otimes \Psi)) = \Phi^*[\tilde{c}_{4,1}] + c_1(\mathrm{Ind}_{H_{\mathbb{F}_{q^n}}}^{GL_2\mathbb{F}_{q^n}}(1)) \cdot \mathrm{Trace}_{H_{\mathbb{F}_{q^n}}}^{GL_2\mathbb{F}_{q^n}}(\lambda_1)$$

$$+c_2(\mathrm{Ind}_{H_{\mathbb{F}_{q^n}}}^{GL_2\mathbb{F}_{q^n}}(1))$$

where $\Phi^*[\tilde{c}_{4,1}]$ is the appropriate 4-cocycle introduced in §2 and λ_1 is the 1-dimensional cohomology class given by the character $\Theta \otimes \Psi$. Similarly we have

$$c_2(\mathrm{Ind}_{\mathbb{F}_{q^{2n}}^*}^{GL_2\mathbb{F}_{q^n}}(\Theta)) = \Phi^*[\tilde{c}_{4,2}] + c_1(\mathrm{Ind}_{\mathbb{F}_{q^{2n}}^*}^{GL_2\mathbb{F}_{q^n}}(1)) \cdot \mathrm{Trace}_{\mathbb{F}_{q^{2n}}^*}^{GL_2\mathbb{F}_{q^n}}(\lambda_2)$$

$$+c_2(\mathrm{Ind}_{\mathbb{F}_{q^{2n}}^*}^{GL_2\mathbb{F}_{q^n}}(1)).$$

Shintani base change of $r(\Theta)$ is $r(\overline{\Theta})$ whose second Chern class is given by

$$c_2(r(\overline{\Theta})) = c_2(\mathrm{Ind}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(\overline{\Theta} \otimes \overline{\Psi})) - c_2(\mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2\mathbb{F}_q}(\overline{\Theta}))$$

$$-c_1(r(\overline{\Theta})) \bigcup c_1(\mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2\mathbb{F}_q}(\overline{\Theta}))$$

where, by the discussion of §1, we have

$$c_2(\mathrm{Ind}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(\overline{\Theta} \otimes \overline{\Psi})) = \Phi^*[\tilde{c}_{4,3}] + c_1(\mathrm{Ind}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(1)) \cdot \mathrm{Trace}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(\lambda_3)$$

$$+c_2(\mathrm{Ind}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(1))$$

where $\Phi^*[\tilde{c}_{4,3}]$ is the appropriate 4-cocycle introduced in §2 and λ_3 is the 1-dimensional cohomology class given by the character $\overline{\Theta} \otimes \overline{\Psi}$. Similarly we have

$$c_2(\mathrm{Ind}_{\mathbb{F}^*_{q^2}}^{GL_2\mathbb{F}_q}(\overline{\Theta})) = \Phi^*[\tilde{c}_{4,4}] + c_1(\mathrm{Ind}_{\mathbb{F}^*_{q^2}}^{GL_2\mathbb{F}_q}(1)) \cdot \mathrm{Trace}_{\mathbb{F}^*_{q^2}}^{GL_2\mathbb{F}_q}(\lambda_4)$$

$$+c_2(\mathrm{Ind}_{\mathbb{F}^*_{q^2}}^{GL_2\mathbb{F}_q}(1)).$$

The above formulae show how one may extract the ingredients for $c_2(r(\overline{\Theta}))$ from those of the formula for $c_2(r(\Theta))$. Since $H^1(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q); S)$ is trivial for $S = \mathbb{F}^*_{q^{2n}}, H_{\mathbb{F}_{q^n}}, GL_2\mathbb{F}_{q^n}$ the Galois invariants of G/H for any suitable pair of these is the coset space of the Galois invariants of G and H. This implies that Φ^* in $\Phi^*[\tilde{c}_{4,3}]$ and $\Phi^*[\tilde{c}_{4,4}]$ is given by the action of the Galois-fixed points of the coset space used to derive $\Phi^*[\tilde{c}_{4,1}]$ and $\Phi^*[\tilde{c}_{4,2}]$ respectively. Similarly the characters λ_3 and λ_4 are the unique ones which, composed with the norm, give λ_1 and λ_2 respectively.

Example 5.2.
Suppose that $R(\chi_1, \chi_2) \in \mathrm{Irr}(GL_2\mathbb{F}_{q^n})^{\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)}$ but that χ_i is *not* Galois invariant. In this case $\Sigma(\chi_1) = \chi_2$ and $\Sigma(\chi_2) = \chi_1$, where $\Sigma \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is the Frobenius automorphism. Therefore we have a quadratic character

$$\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{Z}/n \longrightarrow \{\pm 1\}$$

given by the Galois permutation of $\{\chi_1, \chi_2\}$. Write $n = 2d$ then $\Sigma^2(\chi_i) = \chi_1$ and there exist characters

$$\overline{\chi}_1, \overline{\chi}_2 : \mathbb{F}^*_{q^2} \longrightarrow \mathbb{C}^*$$

such that $\chi_i(z) = \overline{\chi}_i(N(z))$ for $z \in \mathbb{F}^*_{q^n}$ and N is the norm

$$N : \mathbb{F}^*_{q^n} \longrightarrow \mathbb{F}^*_{q^2}.$$

If $F \in \mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ then $F(\overline{\chi}_1) = \overline{\chi}_2$ and $F(\overline{\chi}_2) = \overline{\chi}_1$ and there exists

$$\overline{\chi}_{1,2} : \mathbb{F}^*_q \longrightarrow \mathbb{C}^*$$

such that $\chi_1(z)\chi_2(z) = \overline{\chi}_{1,2}(N(z))$ for $z \in \mathbb{F}^*_{q^n}$. Also the restriction of $\overline{\chi}_1$ or $\overline{\chi}_2$ to $\mathbb{F}^*_q$ is equal to $\overline{\chi}_{1,2}$([12] p.100).

The Shintani base change in this example is given by

$$Sh(R(\chi_1, \chi_2)) = r(\overline{\chi}_1) = r(\overline{\chi}_2).$$

The second Chern class of $R(\chi_1, \chi_2)$ is given by

$$c_2(R(\chi_1, \chi_2)) = \Phi^*[\tilde{c}_{4,5}] + c_1(\mathrm{Ind}_{B_{\mathbb{F}_{q^n}}}^{GL_2\mathbb{F}_{q^n}}(1)) \bigcup \mathrm{Trace}_{B_{\mathbb{F}_{q^n}}}^{GL_2\mathbb{F}_{q^n}}(\lambda_5)$$

$$+c_2(\mathrm{Ind}_{B_{\mathbb{F}_{q^n}}}^{GL_2\mathbb{F}_{q^n}}(1))$$

where $\Phi^*[\tilde{c}_{4,5}]$ is the appropriate 4-cocycle introduced in §2 and λ_5 is the 1-dimensional cohomology class given by the character

$$\chi_1 \otimes \chi_2 : B_{\mathbb{F}_{q^n}} \longrightarrow \mathbb{F}_{q^n}^* \times \mathbb{F}_{q^n}^* \longrightarrow \mathbb{C}^*.$$

The Shintani base change of $R(\chi_1, \chi_2)$ is $r(\overline{\chi}_1) = r(\overline{\chi}_2)$, whose second Chern class is given, for $i = 1$ or $i = 2$, by

$$c_2(r(\overline{\chi}_i)) = c_2(\mathrm{Ind}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(\overline{\chi}_i \otimes \overline{\Psi})) - c_2(\mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2\mathbb{F}_q}(\overline{\chi}_i))$$

$$-c_1(r(\overline{\chi}_i)) \bigcup c_1(\mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2\mathbb{F}_q}(\overline{\chi}_i)).$$

By the discussion of §1 the first two terms in this expression are given by

$$c_2(\mathrm{Ind}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(\overline{\chi}_i \otimes \overline{\Psi})) = \Phi^*[\tilde{c}_{4,6}] + c_1(\mathrm{Ind}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(1)) \cdot \mathrm{Trace}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(\lambda_6)$$

$$+c_2(\mathrm{Ind}_{H_{\mathbb{F}_q}}^{GL_2\mathbb{F}_q}(1))$$

where $\Phi^*[\tilde{c}_{4,6}]$ is the appropriate 4-cocycle introduced in §2 and λ_6 is the 1-dimensional cohomology class given by the character $\overline{\chi}_i$ and similarly

$$c_2(\mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2\mathbb{F}_q}(\overline{\chi}_i)) = \Phi^*[\tilde{c}_{4,7}] + c_1(\mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2\mathbb{F}_q}(1)) \cdot \mathrm{Trace}_{\mathbb{F}_{q^2}^*}^{GL_2\mathbb{F}_q}(\lambda_7)$$

$$+c_2(\mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2\mathbb{F}_q}(1)).$$

Remark 5.3.

In these simple examples of Shintani base change it is plain to see how the data in the formula for the second Chern class of $Sh(\rho)$ is determined by the data in the formula for the second Chern class of ρ and vice versa.

The story is similar if one replaces the finite field by a non-Archimedean local field in the base change for admissible irreducible complex representations in the case of GL_2.

It would be interesting to see examples of a similar relationship for other GL_n base change examples.

6. Appendix: Weil representations for $GL_2\mathbb{F}_q$ and monomial resolutions

Using the notation of ([12] Chapter Three), let $r(\Theta)$ denote the Weil representation of $GL_2\mathbb{F}_q$ associated to the character $\Theta : \mathbb{F}_{q^2}^* \longrightarrow \mathbb{C}^*$ such that $F(\Theta) \neq \Theta$.

From ([12] Chapter Three) we know that the natural representation of the Borel subgroup $\mathrm{Ind}_H^B(\Theta \otimes \Psi)$ extends to a $GL_2\mathbb{F}_q$-action to give $r(\Theta)$. Hence we have

$$\mathrm{Ind}_H^{GL_2\mathbb{F}_q}(\Theta \otimes \Psi) = \mathrm{Ind}_B^{GL_2\mathbb{F}_q} r(\Theta) \longrightarrow r(\Theta)$$

by sending $g \otimes_B w$ to $g \cdot w$. This is clearly surjective.

The following result is easily proved using the conjugacy class data and character values given in ([12] Chapter Three). After verifying the character value calculation just mentioned and after working out a different proof, which proceeds to prove exactness directly, I chanced to spot this result in ([2] p.47).

The direct proof uses a careful examination of maps in the Double Coset Formula [12], which is fun but more tricky than we need here. I leave it as an exercise for the reader!

Theorem 6.1.
There is a short exact sequence of $GL_2\mathbb{F}_q$-representations

$$0 \longrightarrow \mathrm{Ind}_{\mathbb{F}^*_{q^2}}^{GL_2\mathbb{F}_q}(\Theta) \longrightarrow \mathrm{Ind}_{H}^{GL_2\mathbb{F}_q}(\Theta \otimes \Psi) \longrightarrow r(\Theta) \longrightarrow 0.$$

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