GAMES FOR AS FEW AS WILL

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A strategy, to a game theorist, is a step-by-step recipe for getting the best outcome for oneself out of the confronting situation. This situation must have a precisely described bunch of rules. These sorts of problems are fun for certain types of people, hence the "game" in Game Theory. School teachers sometimes like to entertain their students with strategical puzzles. A favourite of the classroom is the situation in which you are at a T-junction walking to London and you do not know which direction to turn. You see two people there, a few metres away, one on the easterly road and one on the westerly. You have been told about this pair. One always tells the truth and the other always lies but you do not know which is which. Also the rules allow you only one question, askable to either, to ascertain the correct route to London. What do you ask? A strategy for this "game" consists of finding a question which will always work. This is a tricky puzzle and after ten minutes of suggestions the schoolteacher delights in revealing: "If I were to ask the way to London of the other person, what would he answer?" The trick of the trade here is to come up with something involving both people and which will always give the same answer. The teachers' question in this case will result in each person pointing away from London and the strategy will then be to disregard the advice!

Another strategical example appeared during the 1990's in the newspaper column "Ask Marilyn", written by Marilyn Vos-Savant, an American journalist billed as the person with the highest IQ ever recorded (228 on the Stanford-Binet scale - beat that!). It features two goats and a car and made the headlines because of the hue and cry following Marilyn's (perfectly correct) answer. Thousands of, presumably chauvinistically male, mathematics professors protested that Ms Vos-Savant was wrong. In each of three sheds, entirely at random, have been placed one of three things - two cantankerous goats and one gleaming, brand-new Ferrari. This occurs in a TV show and you are invited by the presenter to choose a shed after which you will be given free of charge whatever is therein. This strategical problem appeared in an American newspaper so the presumption was that a new Ferrari is preferable to a malodourous goat! Once you have chosen the presenter opens a shed, different from the one you chose, to reveal a goat and gives you the option to stick with your original choice or change it to the other shed which was not opened. Ms Vos-Savant advised the counter-intuitive strategy of changing

Date: December 2010.

your choice. In North America, unbeknownst to the rest of the world, the fuss over this article raged back and forth for weeks: Were the goats indistinguishable? Were the sheds locked? Are we talking Bayesian statistics? Who cares anyway? The table below shows all the ways of distributing two goats (all goats look the same except to environmentalists) and a car.

Case	*Shed* 1	*Shed* 2	*Shed* 3
A	*Goat*	*Goat*	*Ferrari*
B	*Goat*	*Ferrari*	*Goat*
C	*Ferrari*	*Goat*	*Goat*

Each of case A,B or C is equally likely so when you are given your first choice you have one chance in three of winning the car. However once you know that the presenter is going to open a goat shed the following (different) table shows all the possibilities. From which you can see that if you change your choice you will win the Ferrari six times out of twelve so the odds of a win are one in two, which is better than one in three. The point is that the game of making one choice versus the game of choosing and later having the option to switch are two different games with different odds. The strategy of making a choice and then switching has better odds than the game with no switch option. The same is true of the strategy of making a choice and then sticking with it.

Your first choice	*Shed opened*	*Shed containing car*
Shed 1	*Shed* 2	*Shed* 1
Shed 1	*Shed* 2	*Shed* 3
Shed 1	*Shed* 3	*Shed* 1
Shed 1	*Shed* 3	*Shed* 2
Shed 2	*Shed* 1	*Shed* 2
Shed 2	*Shed* 1	*Shed* 3
Shed 2	*Shed* 3	*Shed* 2
Shed 2	*Shed* 3	*Shed* 1
Shed 3	*Shed* 2	*Shed* 3
Shed 3	*Shed* 2	*Shed* 1
Shed 3	*Shed* 1	*Shed* 3
Shed 3	*Shed* 1	*Shed* 2

Another common source of strategy problems is chess. For those who like chess problems the following diagram shows White pieces in capital letters and Black ones in lower case. The problem is to find a strategy for White, playing up the board, to checkmate Black in two moves. Those who do not like chess can skip ahead![1]

	A	B	C	D	E	F	G	H
8		*q*						
7					*p*		*B*	
6		*p*	*B*					
5	*Q*							
4						*R*	*p*	
3	*r*	*N*	*p*		*k*	*P*		*b*
2	*P*		*P*		*p*		*n*	
1			*K*		*R*		*b*	*N*

This puzzle came from the 2009-10 British Chess Problem Solving Championship. The strategy is given later in a footnote, just in case you prefer to solve it for yourself. It consists of White's first move and a choice of several second moves according to what Black does with his move.

Now let us turn our attention to strategy problems involving two or more participants.

A husband and wife, Ray and Dotty, are planning a camping trip. The mountainous region they are planning to visit is criss-crossed by four East-West trails and four North-South trails with a campsite at each junction of an East-West route with a North-South one. It is agreed that Ray will choose an East-West route and Dotty will choose a North-South one, in order to select their campsite at the intersection of their chosen routes.

Ray likes camping at high altitudes and Dotty prefers low altitudes. How are they to choose? It may help to know a little more data. In the following grid the numbers represent altitudes in thousands of feet. Each row lays out in order, from West to East corresponding to from left to right, the altitudes at the intersections on the routes which Ray has to choose from and each column does the same for Dotty's four routes, from North to South corresponding to from top to bottom (see the picture of the mountains).

[1]The solution to the checkmate in two problem:

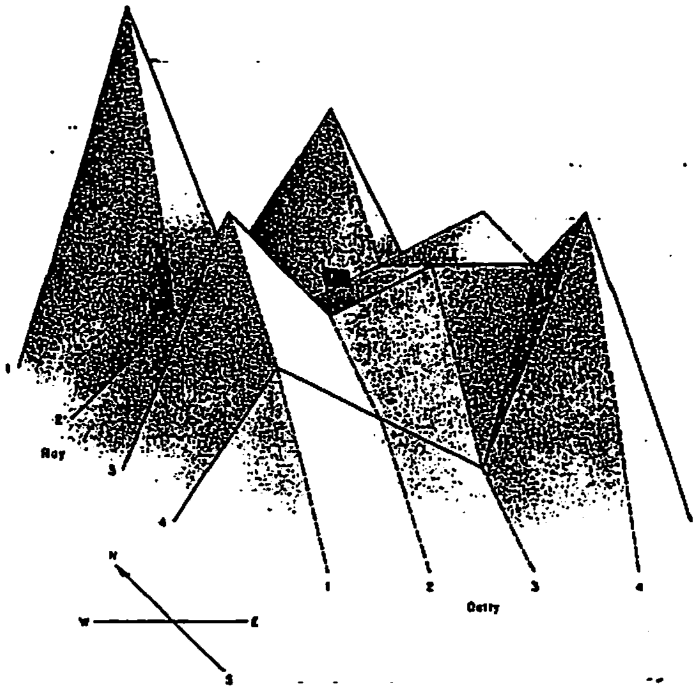
1\. N-d4 threatens 2. R - e4 ++

Refutations of Black's move:

1\. ... Q × f4 2. R × e2++ 1. ... N × f4 2. N - f5++

1\. ... K × f4 2. B - h6++ 1. ... Q - e5 2. Q × e5++

1\. ... P × f3 2. R × f3++ No other Black move can stop the threat e.g. N × e1.



THE MOUNTAINS

Dotty

Ray	1	2	3	4
1	7	2	5	1
2	2	2	3	4
3	5	3	4	4
4	3	2	1	6

Does this data help?

What is Ray thinking? At once he sees that where he would really like to camp is at seven thousand feet on his route number one. However, if he goes for route number one then Dotty would choose her route number four – if she were allowed to choose second – and they would camp at a mere one thousand feet. On the other hand, if Dotty chose first she would prefer her route four and then Ray, if allowed to go second, would choose his route number four and they would camp at six thousand feet.

This suggests, correctly, that choosing first would be a disadvantage. For this reason Ray and Dottie agree to choose simultaneously, with no cheating.

Now what? Is Ray still going to go for route one and Dotty for route four? If Ray goes for route one he realises that he is risking camping at one thousand feet, choosing route two risks two thousand, route three risks three thousand and route four risks one thousand.

So? If Ray is a risk-taker he might still go for route one, but if he is as cautious as the rest of us he will go for route three because it minimises the risk. That is, cautious Ray would like to settle for route three – because it is the route with the maximal minimum altitude for a campsite. Similarly cautious Dotty will settle for the route with the minimal maximum campsite altitude. For Dotty the maximum altitude for route one is seven thousand feet, route two is three thousand, route three is five thousand and route four is six thousand. Therefore Dotty's minimal maximum is on route two.

As it happens in this example, if Ray chooses route three and Dotty chooses route two then they will camp at three thousand feet. These choices are very fortunate because it simultaneously gives Ray's maximal minimum and Dotty's minimal maximum. This is a fluke!

When such a coincidence happens the intersection of the row and column is called a saddle point. This phenomenon gets its name because the seat of a saddle goes up at the front and back and down in both lateral directions. From the saddle point on Ray's route three the campsite altitude goes up in both directions, while from the saddle on Dotty's route two it goes down in both directions. So far then:

Moral: For happy campers search for a saddle, where the minimal column maximum and maximal row minimum coincide.

The bad news is that saddles rarely exist, so we shall have to find a more general strategy for choosing. However, before leaving the saddle, observe that Ray and Dotty would still come to the same choice of routes if we had measured altitudes as distances above one thousand feet rather than from sea level – after all the mountains are the same.

Dotty

Ray

	1	2	3	4
1	6	1	4	0
2	1	1	2	3
3	4	2	3	3
4	2	1	0	5

In fact, Ray and Dotty would come to the same choice if we added (or subtracted) the same quantity from each entry, as in the following examples.

Dotty

Ray

	1	2	3	4
1	30	25	28	24
2	25	25	26	27
3	28	26	27	27
4	26	25	24	29

Dotty

Ray

	1	2	3	4
1	4	−1	2	−2
2	−1	−1	0	1
3	2	0	1	1
4	0	−1	−2	3

Another observation is that Ray and Dotty would have made the same choice if the altitudes has been measured in hundreds of feet instead of thousands.

Dotty

Ray

	1	2	3	4
1	70	20	50	10
2	20	20	30	40
3	50	30	40	40
4	30	20	10	60

In fact the same is true if the mountains are all three times higher, seventeen times higher or, more generally, if all the numbers in the grid are multiplied by the same positive number.

Dotty

Ray

	1	2	3	4
1	21	6	15	3
2	6	6	9	12
3	15	9	12	12
4	9	6	3	18

Dotty

Ray

	1	2	3	4
1	35	10	25	5
2	10	10	15	20
3	25	15	20	20
4	15	10	5	30

Now let us return to the rarity of the existence of a saddle. A couple of slightly different grids are the following, neither has a saddle.

In the first example Dotty would like column two and Ray wants row three which intersect in 6 which is what Ray expects but not Dotty.

Dotty

Ray

	1	2	3	4
1	8	3	2	2
2	2	7	2	5
3	12	6	9	12
4	2	5	2	8

In the second example Dotty would like column two and Ray wants row two which intersect in 7 which is what Dotty expects but not Ray.

Dotty

Ray

	1	2	3	4
1	8	2	12	2
2	3	7	6	5
3	2	2	9	2
4	2	5	12	8

To proceed let us restrict ourselves to the smallest possible situation, which is still interesting. If each of Dotty and Ray have only one option the question of choosing a good strategy does not arise. Therefore we shall consider the following picture in which a, b, c, d are four numbers and Ray and Dotty have two strategies each.

Dotty

Ray

	1	2
1	a	b
2	c	d

This depicts what is called a two-by-two game. It is played as follows. Simultaneously Dotty and Ray each choose one of the numbers 1 or 2. Then Dotty gives to Ray the number of tokens (e.g. matchsticks, poker chips, dollars, pounds, euros etc.) equal to the number lying in Dotty's chosen column and Ray's chosen row. For example, if Dotty chooses 2 and Ray 1 then Dotty gives b tokens to Ray. If b is a negative number this is interpreted to mean Ray gives Dotty $(-b)$ tokens (e.g. 2 tokens from Ray to Dotty if $b = -2$).

As we shall see, the numbers may be such that no matter what strategy they choose the game is biased towards one of the players. To compensate for this we are going to work out an exchange of tokens, the same each time, before each play so as to make the game unbiased in the long run. After all, we do not want to run a crooked casino! To do the calculation now would be to get ahead of ourselves so we shall postpone it while we discuss strategies.

To aid the discussion we shall write $a \leq b$ to mean that the number a is less than or equal to the number b (e.g. $-5 \leq -4$, $-1 \leq 1$). Suppose that a saddle occurs, say, at b. We might as well assume this because we can always renumber the rows and columns to arrange the saddle to appear in the upper right (e.g. if c is the saddle switch the rows and switch the columns, if a is the saddle switch the columns and if it is d switch the rows). Since b is the minimal column maximum we must have $d \leq b$ and since it is the maximal row minimum we have $b \leq a$ giving $d \leq b \leq a$. How big is c? No one knows! The possibilities are $d \leq b \leq a \leq c$, $d \leq b \leq c \leq a$, $d \leq c \leq b \leq a$ and $c \leq d \leq b \leq a$, which all give a saddle at b. If b is a saddle and both players play the saddle strategy then Dotty gives b tokens to Ray at every play. To make it fair Ray should give b tokens to Dotty before each play.

Therefore if both players follow the saddle strategy the fair game results in no one wining anything. What happens if they abandon the saddle strategy? If Dotty sticks to it but Ray does not they choose column two and row two which intersect at d so Ray gives Dotty b tokens beforehand and then wins back only d tokens, which for Ray is a net loss on the play. If Ray sticks to it but Dotty does not they choose column one and row one which intersect at a so Ray gives Dotty b tokens beforehand and then wins back a tokens, which for Dotty is a net loss on the play. If they both abandon the saddle strategy then Ray gives Dotty b tokens beforehand and then wins back c tokens, for which the profit depends on the value of c.

Let us consider a couple of numerical examples, remembering that Ray likes the row in which the minimum is maximal and Dotty likes the column where the maximum is minimal. For example in the following game

Dotty

Ray

	1	2
1	6	5
2	5	4

Ray is happy with the top row (where the minimum is 5) and Dotty is happy with the right-hand column (where the maximum is 5). Therefore we have found a saddle. If Ray and Dotty choose it each time they play the game then Dotty contentedly pays Ray five pounds. As it stands this "game" is unfair to Dotty but we can fix that if we arrange that before each play Ray gives Dotty five pounds. It is now a fair game. If each player goes for the saddle strategy they will not lose but if they want to gamble they can choose differently on some of the plays.

The next example does not have a saddle. Here Ray would like to choose

Dotty

Ray

	1	2
1	3	6
2	5	4

the bottom row (where the minimum is 4) and Dotty prefers the left-hand column (where the maximum is 5), which is not a saddle since the intersection of their preferences does not give the payouts which each expects. The intersection of these is 5, which more than satisfies Ray but after a couple of plays Dotty would notice that Ray's payout was greater than he expected and she would switch to choosing the right-hand column. Then Ray would see an opportunity and switch rows etc etc. The nett result would be chaotic strategies or, equivalently, no strategy at all.

What are we to do when there is no saddle? An obvious suggestion would be for each player sometimes to make one choice and sometimes the other in a random manner. For example, according to the outcome of tossing a coin. This leads to the notion of a mixed strategy, which was mathematicised in the pioneering book on game theory by Morgenstern and von Neumann [1]. Let us return to the general two person game.

Dotty

Ray

	1	2
1	a	b
2	c	d

Suppose that Ray has a gadget which randomly tells him which row to choose in such a way that over a long run of N games the number of choices is on average αN top rows and $(1-\alpha)N$ bottom rows for some number α between 0 and 1. For example, an unbiased coin toss would give $\alpha = 1/2$. If Dotty always chooses the left-hand column Ray's pay off would be $\alpha Na + (1-\alpha)Nc$ while if she chooses the right-hand it would be $\alpha Nb + (1-\alpha)Nd$. From Ray's point of view these numbers had better be equal because if not, Dotty could play consistly in the column giving the smaller value. This means that Ray would like

$$\alpha a + (1-\alpha)c = \alpha b + (1-\alpha)d$$

or equivalently

$$\alpha(a-b) = (1-\alpha)(d-c).$$

This means that Ray can calculate what he would like α to be from the equalities of ratios

$$\text{top row : bottom row} = \alpha : (1-\alpha) = (d-c) : (a-b).$$

Since both α and $1-\alpha$ are positive numbers less than or equal to one we can only solve this ratio condition if either $c \leq d$ and $b \leq a$ or $d \leq c$ and $a \leq b$. That is, we must have $d-c$ and $a-b$ either both positive or both negative - because the ratios $x : y$ and $-x : -y$ are equal.

If Dotty has a similarly random gadget we obtain the following condition. If Ray always chooses the top row Dotty's payoff would be $N\beta a + N(1-\beta)b$ and if he chooses the lower row if would be $N\beta c + N(1-\beta)d$ so Dotty would like to find a number β between 0 and 1 satisfying

$$\beta(a-c) = (1-\beta)(d-b).$$

This leads to the ratio condition

$$\text{left column : right column} = \beta : (1-\beta) = (d-b) : (a-c).$$

We can only solve this for β between 0 and 1 providing that either $b \leq d$ and $c \leq a$ or $d \leq b$ and $a \leq c$.

In the above calculations the condition that we can solve for both numbers $0 \leq \alpha, \beta \leq 1$ is equivalent to one of the following four cases:

I: $a \leq b, d \leq c, a \leq c, d \leq b$
II: $a \leq b, d \leq c, c \leq a, b \leq d$
III: $b \leq a, c \leq d, a \leq c, d \leq b$
IV: $b \leq a, c \leq d, c \leq a, b \leq d$.

The recipe for α and β appears without the mathematical explanation in ([3] pp.40-41) where a quick way to decide whether we are in one of cases I-IV or not is given. One subtracts the numbers in the top row from the numbers below in the bottom row to give a pair of numbers, called the row oddment in [3][2]

row oddment

$c-a$	$d-b$

and one subtracts the numbers in column 1 from those in column 2 to give the column oddment

column oddment

$b-a$
$d-c$

.

One of cases I-IV occurs precisely when the row and column oddments each consist of one positive and one negative number. By the way, one can subtract the rows in the opposite order to give

$a-c$	$b-d$

and the test will still work. (The same is true for columns). Incidentally, let us agree that if either oddment consists of two zeroes then the game fails the test. We shall also agree that an oddment consisting of zero and a non-zero number like either

1	0

or

0
−3

passes the test.

Let us suppose that the game passes the oddment test. The game as it stands, with Dotty paying Ray after each play, is clearly biased. We would like to modify the play to make it into a fair game.

What can we possibly mean by a fair game? If Ray finds α so that $\alpha a + (1-\alpha)c = \alpha b + (1-\alpha)d$, he expects in N games to receive $N\alpha a + N(1-\alpha)c$ from Dotty. On the other hand, having found β such that $\beta a + (1-\beta)b = \beta c + (1-\beta)d$, Dotty expects to pay $N\beta a + N(1-\beta)b$ to Ray. If $\alpha a + (1-\alpha)c$ is not equal to $\beta a + (1-\beta)b$ then Ray and Dotty will not be able to agree

[2]The bottom line of ([3] p.40) is not correct. It states that every two by two game will pass the oddment test. As we shall see, this is false. What is true is that only when the oddment test works can the game be made into a fair game by the mixed strategy method.

about how to make a fair game. But if $\alpha a + (1-\alpha)c = \beta a + (1-\beta)b$ then they can make the game fair by Ray giving $\alpha a + (1-\alpha)c$ to Dotty before each play.

A little algebra shows that $\alpha(a-b-c+d) = d-c$ and $\beta(a-b-c+d) = d-b$. If the game has passed the oddment test then $a-b-c+d$ is not zero and one finds that Ray's and Dotty's calculations of the expected payoff are both equal to $\frac{ad-bc}{a-b-c+d}$. In this case, each play of the fair game consists of Ray giving $\frac{ad-bc}{a-b-c+d}$ to Dotty and then each one simultaneously choosing one of his/her two strategies and calculating the payoff, which Dotty pays to Ray. If each player follows their calculated mixed strategy then, in the long run, they will be quits. That's fair!

In addition, the payoff will not be zero except in the case where the top row is proportional to the bottom row (in which case the first column will be proportional to the second and vice versa).

For convenience, let us record the mixed strategy programme for the game

Dotty

Ray

	1	2
1	a	b
2	c	d

Step 1: Check that the oddments

$c-a$	$d-b$

and

$b-a$
$d-c$

each consist of one negative and one positive number.

Step 2: Calculate

$$\alpha = \frac{d-c}{a-b-c+d} \quad \text{and} \quad \beta = \frac{d-b}{a-b-c+d}.$$

Step 3: Calculate the expected average payoff per play from Dotty to Ray

$$P = \frac{ad-bc}{a-b-c+d}.$$

Step 4: The fair game: at each play Ray pays Dotty P then they simultaneously choose a strategy and Dotty pays Ray the amount (a, b, c or d) where their chosen strategies intersect.

Let us consider a couple of numerical examples to which the mixed strategy method does not apply. For example, in the game

Dotty

Ray	1	2
1	6	5
2	5	4

the row oddment equals

$$\begin{array}{||c|c||} \hline 5-6 & 4-5 \\ \hline \end{array} = \begin{array}{||c|c||} \hline -1 & -1 \\ \hline \end{array}$$

which fails and, not surprisingly, we cannot find α or β satisfying $\alpha(a-b-c+d) = d-c$ and $\beta(a-b-c+d) = d-b$ because $a-b-c+d = 6-5-5+4 = 0$.

The next numerical example is

Dotty

Ray	1	2
1	8	6
2	5	4

.

Here $a-b-c+d = 8-6-5+4 = 1$, which is non-zero but the row oddment is

$$\begin{array}{||c|c||} \hline 5-8 & 4-6 \\ \hline \end{array} = \begin{array}{||c|c||} \hline -3 & -2 \\ \hline \end{array}$$

which fails. On the other hand, the 6 in row 1, column 2 is where Ray finds the maximal row minimum and Dotty finds the minimum column maximum, which means it is a saddle. As a labour saving plan, Ray and Dotty will usually search first for a saddle strategy, since it is easier to apply. For example, in this example, to make the game fair Ray pays Dotty 6 before each play.

Unless some of the numbers a, b, c, d are equal there is no game for which both the saddle strategy and the mixed strategy apply. For example, suppose that there is a saddle at b in the game

Dotty

Ray	1	2
1	a	b
2	c	d

According to Ray b is the maximal row minimum so that $b \le a$ and $\min(c, d) \le b$. According to Dotty b is the minimal column maximum so that $d \le b$ and $b \le \max(a, c)$. This means that $d \le b \le a$ and we have one of the cases:

(i) $c \le d \le b \le a$
(ii) $d \le c \le b \le a$

(iii) $d \leq b \leq c \leq a$
(iv) $d \leq b \leq a \leq c$.

However the row oddment fails for (i)-(iii) and the column oddment fails for (iv), unless some of the numbers are equal.

Before closing with a "real life" (!) example of a two by two game in action let us briefly consider the general two person game. This may be depicted by an array of numbers

Dotty

Ray

	1	2	3	⋯	⋯	⋯	⋯	s
1	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	⋯	⋯	⋯	⋯	$a_{1,s}$
2	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	⋯	⋯	⋯	⋯	$a_{2,s}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$t-1$	$a_{t-1,1}$	$a_{t-1,2}$	$a_{t-1,3}$	⋯	⋯	⋯	⋯	$a_{t-1,s}$
t	$a_{t,1}$	$a_{t,2}$	$a_{t,3}$	⋯	⋯	⋯	⋯	$a_{t,s}$

In this game Ray has the choice of t strategies, one for each row, and Dotty has the choice of s strategies, one for each column. The play consists of each simultaneously choosing a strategy and then Dotty pays Ray the amount at the intersection of Dotty's column and Ray's row. One may attempt to calculate how to turn this into a fair game using mixed strategies. For example, Dotty would try to find numbers $x_1, x_2, \ldots, x_s$ lying between 0 and 1 and adding up to 1 to be the frequencies with which some random machine chooses columns $1, 2, \ldots, s$ respectively. She would like each of the sums

$$x_1 a_{1,1} + x_2 a_{1,2} + \ldots + x_s a_{1,s},$$

$$x_1 a_{2,1} + x_2 a_{2,2} + \ldots + x_s a_{2,s},$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$x_1 a_{t,1} + x_2 a_{t,2} + \ldots + x_s a_{t,s},$$

to be equal. This amounts to what is known as a set of simultaneous linear equations. There is a similar set for Ray's mixed strategy. To make the game fair we would have to solve Dotty's and Ray's equations in a compatible manner. It is hard enough just to solve the two sets of simultaneous linear equations if s and t are large. With the aid of a computer this may be accomplished using the Simplex Method in linear programming. Linear programming was a technique discovered by Leonid Kantorovich, a Russian mathematician in 1939. During World War II linear programming was further developed as a method to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy. It was also used in order optimally to schedule submarine movements. The techniques were kept secret until 1947 when George B. Dantzig, its inventor, published the simplex method. The simplex method, with embellishments by many mathematicians including John von Neumann, remained the main technique for solving sets of simultaneous linear equations or inequalities until 1984 when Narendra Karmarkar introduced a new interior point method for solving linear programming problems.

Coming from a typical immigrant family of eastern european intellectuals, in college George Dantzig was a very clever and industrious student. One day, after arriving late to a lecture of Jerzy Neyman (famous for the Neyman-Pearson test in statistics) George mistakenly took down, as the homework for the day, two unsolved and long-standing problems in statistics. Then he went home and solved the homework!

We shall conclude with two examples of which the first is taken from ([3] pp. 47-48) and is depicted as follows

Red

Blue

	1	2
1	60	100
2	100	80

The context concerns a Blue bombing mission involving Bomber 1 and Bomber 2. The mission is apt to be attacked by a Red fighter, which makes a one-pass attempt at shooting down one of the bombers. These bombers fly in such a way that Bomber 1 derives considerably more protection from the guns of Bomber 2 than the latter does from those of Bomber 1. The two bombers are to carry only one bomb. Which plane should carry it?

Seeing the bombers in formation the pilot of the Red fighter can determine which plane is better protected. He then has to choose between two strategies

Red 1 : attack the less well protected bomber

Red 2 : attack the more well protected bomber.

The two strategies of Blue concern where to put the bomb

Blue 1 : bomb in the less well protected bomber

Blue 2 : bomb in the more well protected bomber.

The numbers indicate, as a percentage, the probability that the bomb will escape unscathed from the attack. If Red adopts strategy 1 it attacks Bomber 2 and if Blue adopts strategy 1 the bomb is in Bomber 2 and will escape 60 percent of the time. If Red adopts strategy 1 it attacks Bomber 2 and if Blue adopts strategy 2 the bomb is in Bomber 1 and will escape 100 percent of the time. Conversely, if Red adopts strategy 2 it attacks Bomber 1 and if Blue adopts strategy 1 the bomb is in Bomber 2 and will escape 100 percent of the time. If Red adopts strategy 2 it attacks Bomber 1 and if Blue adopts strategy 2 the bomb is in Bomber 1 and will escape 80 percent of the time. These figures have a certain amount of realism in that when the bomb is in the plane which is attacked it fares better when it is in the better protected Bomber 1.

Now let us return to the question of where to put the bomb. Group Captain Mike "Buffo" Hawkeye of RAF Bomber Command immediately knew the answer: "By jove! Go for the better protection, what!" By which we assume he means put the bomb in Bomber 1. This will ensure that, at worst, the bomb escapes 80 per cent of the time.

However, the backroom team in Bletchley Park had read an (encrypted, top secret) account of the method of George B. Dantzig. They notice that the oddments of this "game" are

40	−20

and

40
−20

which both pass the oddment test. Applying the mixed strategy formula we obtain

$$\alpha = \frac{80 - 100}{60 - 100 - 100 + 80} = \frac{1}{3}.$$

This means that, by adopting the mixed strategy, Blue expects the bomb to escape attack

$$(1/3) \times 60 + (2/3) \times 100 = (1/3) \times 100 + (2/3) \times 80 = 86.66666..... \%$$

of the time. This represents an improvement over the "Go for the better protection" strategy by about 8 per cent.

The second concluding example is James Lovelock's Daisyworld, which features what is essentially a two person game where the "players" are dark and light daisies. Lovelock is the British inventor and scientist renowned for the invention of the microwave oven and for the discovery of polar thinning of the ozone layer. First published in [2] Daisyworld is a computer simulation. What it is doing amounts to playing the strategical two daisy game over and over again. White petalled daisies reflect light, while black petalled daisies absorb light. The simulation tracks the two daisy populations and the surface temperature of Daisyworld as the sun's rays grow more powerful. The surface temperature of Daisyworld remains almost constant over a broad range of solar output. The importance of this example is to show how, from our two-person game point of view, the mixed strategy of the daisies inadvertently works to the benefit of the entire environment. Some of the arguments based upon this model have been highly contentious, engaging the industry of many popular science writers in a very entertaining way - for details see Wikipedia.

References

[1] John von Neumann and Oskar Morgenstern: *Theory of Games and Economic Behaviour*; Princeton University Press (1944).

[2] A.J. Watson and J.E. Lovelock: Biological homeostasis of the global environment: the parable of Daisyworld; Tellus B (Int. Met. Inst.) 35 (4) 286-289 (1983).

[3] J.D. Williams: *The Compleat Strategyst*; Dover Publications (1986).