HOMOLOGICAL DETECTION

VICTOR P. SNAITH

Contents

1. The 2-cocycle

Let $\mathbb{F} = \mathbb{F}_{2^d}$ and consider the group extension

$$M_n^0 \longrightarrow SL_n\mathbb{F}[t]/(t^3) \stackrel{\pi}{\longrightarrow} SL_n\mathbb{F}[t]/(t^2)$$

where M_n^0 is the subgroup of $\mathrm{Ker}(\pi)$ given by

$$M_n^0 = \{1 + Xt^2 \mid \det(1 + Xt^2) = 1\}.$$

Since

$$\det(\begin{pmatrix} 1 + X_{11}t^2 & X_{12}t^2 & \ldots & \ldots & X_{1n}t^2 \\ X_{21}t^2 & 1 + X_{22}t^2 & \ldots & \ldots & X_{2n}t^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{n1}t^2 & X_{n2}t^2 & \ldots & \ldots & 1 + X_{nn}t^2 \end{pmatrix})$$

$$\equiv 1 + \mathrm{Trace}(X)t^2 \text{ (modulo } t^3)$$

so that M_n^0 is isomorphic to the additive abelian group given by the trace zero $n \times n$ matrices with entries in in $\mathbb{F}$.

Choose a map of sets, which is a right inverse to π,

$$h : SL_n\mathbb{F}[t]/(t^2) \longrightarrow SL_n\mathbb{F}[t]/(t^3).$$

If $X \in SL_n\mathbb{F}[t]/(t^2)$ we may write X as $X = X_0 + X_1 t$ where $X_0 \in SL_n\mathbb{F}$ and X_1 is a trace zero $n \times n$ matrix with entries in in $\mathbb{F}$. The formula for the determinant shows that there is an $n \times n$ matrix X_2 with entries in $\mathbb{F}$ such that $h(X) = X_0 + X_1 t + X_2 t^2 \in SL_n\mathbb{F}[t]/(t^3)$.

Example 1.1. Consider the 2×2 matrix

$$x = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \in SL_2\mathbb{F}[t]/(t^2).$$

We have $x^2 = 1 \in SL_2\mathbb{F}[t]/(t^2)$. In $GL_2\mathbb{F}[t]/(t^3)$ we have

$$y = \begin{pmatrix} 1 + t^2 & t \\ t & 1 \end{pmatrix}$$

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which satisfies $\pi(y) = x$ and

$$\det(y) = 1 + t^2 + t^2 \equiv 1$$

so that we may choose $h(x) = y$. However

$$h(x)^2 = y^2 = \begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix} \begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1+t^2 & 0 \\ 0 & 1+t^2 \end{pmatrix} \neq 1.$$

Therefore the restricted group extension

$$M_n^0 \longrightarrow \pi^{-1}(\langle x \rangle) \xrightarrow{\pi} \langle x \rangle$$

is not split.

Next we are going to write down explicitly the 2-cycle

$$\Delta : SL_n\mathbb{F}[t]/(t^2) \times SL_n\mathbb{F}[t]/(t^2) \longrightarrow M_n^0,$$

in the convention of the inhomogeneous bar resolution ([**?**] p. 41). In terms of a section h of π the formula is

$$\Delta([X_1|X_2]) = h(X_1)h(X_2)h(X_1X_2)^{-1} \in M_n^0.$$

We need to show that Δ is a 2-cycle, which is the condition

$$\Delta(\delta(X_0|X_1|X_2]) = 1.$$

Explicitly

$$X_0(\Delta([X_1|X_2]))(\Delta([X_0X_1|X_2]))^{-1}\Delta([X_0|X_1X_2])(\Delta([X_0|X_1]))^{-1} = 1.$$

In the first of these four terms the action of X_0 on $\Delta([X_1|X_2])$ is by reduction $X_0 \mapsto \overline{X}_0 \in SL_n\mathbb{F}$ following by conjugation.

Write

$$A = X_0(\Delta([X_1|X_2])) = \overline{X}_0(h(X_1)h(X_2)h(X_1X_2)^{-1})\overline{X}_0^{-1},$$

$$B^{-1} = \Delta([X_0X_1|X_2]) = h(X_0X_1)h(X_2)h(X_0X_1X_2)^{-1},$$

$$B = h(X_0X_1X_2)h(X_2)^{-1}h(X_0X_1)^{-1},$$

$$C = \Delta([X_0|X_1X_2]) = h(X_0)h(X_1X_2)h(X_0X_1X_2)^{-1},$$

$$D^{-1} = \Delta([X_0|X_1]) = h(X_0)h(X_1)h(X_0X_1)^{-1},$$

$$D = h(X_0X_1)h(X_1)^{-1}h(X_0)^{-1}.$$

Now we have

$$\Delta(\delta(X_0|X_1|X_2]) = ABCD = ACBD.$$

Therefore

$$
\begin{aligned}
&\Delta(\delta(X_0|X_1|X_2])\\
&= \overline{X}_0(h(X_1)h(X_2)h(X_1X_2)^{-1})\overline{X}_0^{-1}\times\\
&\quad h(X_0)h(X_1X_2)h(X_0X_1X_2)^{-1}\times h(X_0X_1X_2)h(X_2)^{-1}h(X_0X_1)^{-1}\\
&\qquad \times h(X_0X_1)h(X_1)^{-1}h(X_0)^{-1}\\
&= \overline{X}_0(h(X_1)h(X_2)h(X_1X_2)^{-1})\overline{X}_0^{-1}\times\\
&\quad h(X_0)h(X_1X_2)\times h(X_2)^{-1}h(X_1)^{-1}h(X_0)^{-1}\\
&= \overline{X}_0(h(X_1)h(X_2)h(X_1X_2)^{-1})\overline{X}_0^{-1}\times\\
&\quad \overline{X}_0h(X_1X_2)\times h(X_2)^{-1}h(X_1)^{-1}\overline{X}_0^{-1}
\end{aligned}
$$

because $h(X_1X_2)\times h(X_2)^{-1}h(X_1)^{-1}\in M_n^0$ and $h(X_0)$ conjugates M_n^0 by first reducing to $\overline{X}_0$. Therefore

$$
\begin{aligned}
&\Delta(\delta(X_0|X_1|X_2])\\
&= \overline{X}_0h(X_1)h(X_2)h(X_1X_2)^{-1}h(X_1X_2)\times h(X_2)^{-1}h(X_1)^{-1}\overline{X}_0^{-1}\\
&= 1,
\end{aligned}
$$

as required.

This shows that there is a cohomology class

$$[\Delta]\in H^2(SL_n\mathbb{F}[t]/(t^2);M_n^0)$$

which must be non-zero since the restriction to $H^2(\langle x\rangle;M_n^0)$ represents the non-split extension of Example **??**.

Now consider the composition

$$H_2(SL_n\mathbb{F}[t]/(t^2); M_n^0) \otimes H^2(SL_n\mathbb{F}[t]/(t^2); M_n^0)$$

$$\downarrow \text{ evaluation}$$

$$H_0(SL_n\mathbb{F}[t]/(t^2); M_n^0 \otimes M_n^0)$$

$$\downarrow \cong$$

$$(M_n^0 \otimes M_n^0)_{SL_n\mathbb{F}[t]/(t^2)}$$

$$\downarrow T$$

$$\mathbb{F}$$

where $T(A \otimes B) = \text{Trace}(AB)$.

Next we shall attempt to calculate the image under this map, but without the final map T.

Theorem 1.2. *(Charlap and Vasquez)*

The above composite map, but without the final map T, is the differential

$$d_2 : E^2_{2,1} = H_2(SL_2\mathbb{F}[t]/(t^2); M_2^0) \cong \mathbb{F} \oplus \mathbb{F} \longrightarrow E^2_{0,2} = (M_n^0 \otimes M_n^0)_{SL_n\mathbb{F}[t]/(t^2)}.$$

Firstly let us make the pairing explicit on the chain level. If $(\underline{B}_* G, d)$ is the inhomogeneous bar resolution with left free G-action then $H^2(SL_n\mathbb{F}[t]/(t^2); M_n^0)$ is the 2-dimensional homology of the cochain complex

$$\text{Hom}_{SL_n\mathbb{F}[t]/(t^2)}(\underline{B}_* SL_n\mathbb{F}[t]/(t^2), M_n^0)$$

with differential $d^* = (- \cdot d)$ and $SL_n\mathbb{F}[t]/(t^2)$ acting on the left of M_n^0 by reduction to $SL_n\mathbb{F}$) following by conjugation $X(A) = \overline{X} A \overline{X}^{-1}$. The homology $H_2(SL_n\mathbb{F}[t]/(t^2); M_n^0)$ is the 2-dimensional homology of the chain complex

$$M_n^0 \otimes_{SL_n\mathbb{F}[t]/(t^2)} \underline{B}_* SL_n\mathbb{F}[t]/(t^2)$$

with differential $1 \otimes d$. This time $SL_n\mathbb{F}[t]/(t^2)$ acts on the right of M_n^0 by reduction to $SL_n\mathbb{F}$) following by conjugation $(A)X = \overline{X}^{-1} A \overline{X}$.

With these actions we have

$$A \otimes_{SL_n\mathbb{F}[t]/(t^2)} X(b) = \overline{X}^{-1} A \overline{X} \otimes_{SL_n\mathbb{F}[t]/(t^2)} b \in M_n^0 \otimes_{SL_n\mathbb{F}[t]/(t^2)} \underline{B}_* SL_n\mathbb{F}[t]/(t^2)$$

so that if

$$f \in \text{Hom}_{SL_n\mathbb{F}[t]/(t^2)}(\underline{B}_* SL_n\mathbb{F}[t]/(t^2), M_n^0)$$

then

$$A \otimes \overline{X} f(b) \overline{X}^{-1} = \overline{X}^{-1} A \overline{X} \otimes f(b) \in (M_n^0 \otimes M_n^0)_{SL_n\mathbb{F}[t]/(t^2)}$$

so that $1 \otimes f$ gives a well-defined map which lands in the $SL_n\mathbb{F}[t]/(t^2)$-coinvariant quotient of $M_n^0 \otimes M_n^0$ where the action is the diagonal action on the left of each M_n^0-factor. These invariants are mapped in a well-defined manner to $\mathbb{F}$ by the trace of the product.

Consider matrices of the form, $\lambda, \mu \in \mathbb{F}^*$

$$x_{\lambda,\mu} = \begin{pmatrix} 1 & \lambda t \\ \mu t & 1 \end{pmatrix} \in SL_2\mathbb{F}[t]/(t^2).$$

We have

$$\begin{pmatrix} 1 & \lambda t \\ \mu t & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda t \\ \mu t & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $x_{\lambda,\mu}$ commutes with $x_{\lambda',\mu'}$. In fact

$$\begin{pmatrix} 1 & \lambda t \\ \mu t & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda' t \\ \mu' t & 1 \end{pmatrix} = \begin{pmatrix} 1 & (\lambda+\lambda')t \\ (\mu+\mu')t & 1 \end{pmatrix}.$$

Also a lift of $x_{\lambda,\mu}$ to $SL_2\mathbb{F}[t]/(t^3)$ is given by

$$\begin{pmatrix} 1+\lambda\mu t^2 & \lambda t \\ \mu t & 1 \end{pmatrix}.$$

The square of this lift is given by

$$\begin{pmatrix} 1+\lambda\mu t^2 & \lambda t \\ \mu t & 1 \end{pmatrix}\begin{pmatrix} 1+\lambda\mu t^2 & \lambda t \\ \mu t & 1 \end{pmatrix} = \begin{pmatrix} 1+\lambda\mu t^2 & 0 \\ 0 & 1+\lambda\mu t^2 \end{pmatrix}.$$

Therefore for each X, λ, μ we have a cycle

$$X \otimes_{SL_n\mathbb{F}[t]/(t^2)} ([x_{\lambda,\mu}|x_{\lambda.\mu}] - [1|1]) \in M_n^0 \otimes_{SL_n\mathbb{F}[t]/(t^2)} \underline{B}_2 SL_n\mathbb{F}[t]/(t^2)$$

since $x_{\lambda,\mu}$ maps to the identity in $SL_2\mathbb{F}$.

The image of

$$[X \otimes_{SL_n\mathbb{F}[t]/(t^2)} ([x_{\lambda,\mu}|x_{\lambda.\mu}] - [1|1])] \otimes [\Delta]$$

in

$$(M_n^0 \otimes M_n^0)_{SL_n\mathbb{F}[t]/(t^2)}$$

is the class of

$$d_2(X \otimes_{SL_n\mathbb{F}[t]/(t^2)} [x_{\lambda,\mu}|x_{\lambda,\mu}]) = X \otimes \begin{pmatrix} \lambda\mu t^2 & 0 \\ 0 & \lambda\mu t^2 \end{pmatrix}.$$

Choosing X to equal the 2×2 identity matrix gives a map to a copy of $\mathbb{F}$ in $(M_n^0 \otimes M_n^0)_{SL_n\mathbb{F}[t]/(t^2)}$.

Next consider the matrix

$$z_\lambda = \begin{pmatrix} 1+\lambda\mu t & 0 \\ 0 & 1+\lambda t \end{pmatrix} \in SL_2\mathbb{F}[t]/(t^2).$$

We have

$$z_\lambda^2 = \begin{pmatrix} 1+\lambda^2 t^2 & 0 \\ 0 & 1+\lambda^2 t^2 \end{pmatrix} = 1.$$

We may choose

$$h(z_\lambda) = \begin{pmatrix} 1+\lambda t+\lambda^2 t^2 & 0 \\ 0 & 1+\lambda t \end{pmatrix}$$

since $\det(h(z_\lambda) \equiv 1+\lambda^2t^2+\lambda^2t^2 \equiv 1$ (modulo t^3) so that $h(z_\lambda) \in SL_2\mathbb{F}[t]/(t^3)$. In addition

$$h(z_\lambda) = \begin{pmatrix} 1+\lambda t+\lambda^2t^2 & 0 \\ 0 & 1+\lambda t \end{pmatrix} \begin{pmatrix} 1+\lambda t+\lambda^2t^2 & 0 \\ 0 & 1+\lambda t \end{pmatrix} = (1+\lambda^2t^2)I_2.$$

Therefore $X \otimes_{SL_2\mathbb{F}[t]/(t^2)} ([z_\lambda|z_\lambda] - [1|1])$ is a 2-cycle and

$$d_2(X \otimes_{SL_2\mathbb{F}[t]/(t^2)} ([z_\lambda|z_\lambda] - [1|1])) = X \otimes \begin{pmatrix} \lambda^2t^2 & 0 \\ 0 & \lambda^2t^2 \end{pmatrix}.$$

Therefore

$$d_2(X \otimes_{SL_2\mathbb{F}[t]/(t^2)} ([z_\lambda|z_\lambda] - [x_{\lambda,\lambda}|x_{\lambda,\lambda}]) = 0.$$

Question 1.3. Is

$$X \otimes_{SL_2\mathbb{F}[t]/(t^2)} ([z_\lambda|z_\lambda] - [x_{\lambda,\lambda}|x_{\lambda,\lambda}])$$

non-zero in $E^2_{2,1}$? In ([**?**] p.519 Prop 5.19) it is shown that $E^2_{2,1} \cong \mathbb{F} \oplus \mathbb{F}$ by an elaborate series of calculations. Perhaps the details of these would answer this question?

This question will be resolved in Section Three.

Example 1.4. Consider $SL_2\mathbb{F}_2 = GL_2\mathbb{F}_2 \cong \Sigma_3$ generated by

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

satisfying $\tau^2 = 1 = \sigma^3, \tau\sigma\tau = \sigma^2$.

$SL_2\mathbb{F}_2[t]/(t^2) = \Sigma_3 \propto M_2^0$ where

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \ W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $\tau(U) = U = \sigma(U), \tau(V) = W, \sigma(V) = W, \sigma(W) = U + V + W$.

However there is another Σ_3-action on M_2^0 given by $\tau(U) = U = \sigma(U), \tau(V) = W, \sigma(V) = W, \sigma(W) = V + W$.

If $\alpha : M_2^0 \longrightarrow M_2^0$ transport the first action to the second, assume that the matrix of α with respect to the ordered basis is

$$\alpha = \begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}.$$

Therefore $\alpha(\tau(U)) = \tau(\alpha(U))$ implies that

$$aU + bV + cW = aU + cV + bW$$

so that $b = c$. Also $\alpha(\sigma(U)) = \sigma(\alpha(U))$ implies that

$$aU + bV + bW = aU + bW + b(V + W) = aU + bV$$

so that $b = 0$ and $a = 1$. Next consider $\alpha(\tau(V)) = \tau(\alpha(V))$ which implies

$$a''U + b''V + c''W = \tau(a'U + b'V + c'W) = a'U + b'W + c'V$$

so that $a' = a''$, $b'' = c'$ and $c'' = b'$. The relation $\alpha(\sigma(V)) = \sigma(\alpha(V))$ which implies

$$a'U + c'V + b'W = \sigma(a'U + b'V + c'W) = a'U + b'W + c'(V + W)$$

so that $c' = 0$. The relation $\alpha(\sigma(W)) = \sigma(\alpha(W))$ implies

$$U + a'U + b'V + a'U + b'W = \sigma(a'U + b'W) = a'U + b'(V + W)$$

so that

$$\alpha = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix satisfies $\alpha^2 = 1$.

2. The construction of the spectral sequence

The homology spectral sequence used in [?]

$$E^2_{r,s} = H_r(SL_n\mathbb{F}[t]/(t^2); H_s(M^0_n)) \Longrightarrow H_{r+s}(SL_n\mathbb{F}[t]/(t^3); \mathbb{Z}),$$

whose differentials have the form

$$d_t : E^t_{r,s} \longrightarrow E^t_{r-t,s+t-1},$$

is an example of the first Grothendieck spectral sequence ([?] p.297) of a bicomplex $(A_{*,*}, d_I + d_{II})$.

Let us partially recall its construction. We have two group extensions

$$G^3_n \longrightarrow SL_n\mathbb{F}[t]/(t^3) \longrightarrow SL_n\mathbb{F}$$

and

$$G^2_n \longrightarrow SL_n\mathbb{F}[t]/(t^2) \longrightarrow SL_n\mathbb{F}.$$

Recall that the inhomogeneous bar resolution of G has a differential

$$\partial([x_1|\dots|x_m])$$

$$= x_1[x_2|\dots|x_m] + \sum_{i=1}^{m-1} (-1)^i[x_1|\dots|x_ix_{i+1}|\dots|x_m]$$

$$+(-1)^m[x_1|\dots|x_{m-1}].$$

Set

$$A_{r,s} = \oplus^r_{a=0}\, C_{r-a,a,s}$$

and

$$C_{i,j,k} = \mathbb{Z} \otimes_{SL_n\mathbb{F}[t]/(t^3)} \underline{B}_k G^3_n \otimes (\underline{B}_j G^2_n \otimes \underline{B}_i SL_n\mathbb{F}).$$

Here $g \in SL_n\mathbb{F}$ acts by conjugation $g(y[y_1|\dots\ y_u]) = g(y)[g(y_1)|\dots\ g(y_u)]$ on $\underline{B}_* G^3_n$ and $\underline{B}_* G^2_n$. G^3_n acts as usual on its inhomogeneous bar resolution and on that of G^2_n via reduction and then the usual G^2_n-action on $\underline{B}_* G^2_n$. Putting these actions together - diagonally - gives an $SL_n\mathbb{F}[t]/(t^2)$-action making

$$\underline{B}_j G^2_n \otimes \underline{B}_i SL_n\mathbb{F}$$

into a free resolution of the semi-direct product $SL_n\mathbb{F}[t]/(t^2)$. Therefore $A_{*,*}$ is a free resolution of $SL_n\mathbb{F}[t]/(t^3)$ with the differential $d_I + d_{II}$ given by

$$d_I(x) = (-1)^k 1 \otimes \partial \otimes 1 + (-1)^{k+j} 1 \otimes 1 \otimes \partial)(x), \quad x \in C_{i,j,k}$$

and

$$d_{II}(x) = (\partial \otimes 1 \otimes 1)(x), \quad x \in C_{i,j,k}.$$

Note that

$$d_I : A_{r,s} \longrightarrow A_{r-1,s} \text{ and } d_{II} : A_{r,s} \longrightarrow A_{r,s-1}.$$

The filtration which gives rise to the first spectral sequence is

$$F^p(\oplus_{r+s=m}\ A_{r,s}) = \oplus_{r+s=m, s\leq p}\ A_{r,s}$$

so that $F^{p-1} \subseteq F^p$ and $d_{II}(F^p) \subseteq F^{p-1}$. In addition

$$F^p(\oplus_{r+s=m}\ A_{r,s})/F^{p-1}(\oplus_{r+s=m}\ A_{r,s}) \cong A_{m-p,p}$$

and $E^1_{m-p,p}$ is the homology at $A_{m-p,p}$ of the complex

$$\ldots \xrightarrow{d_{II}} A_{m-p,p} \xrightarrow{d_{II}} A_{m-p,p-1} \xrightarrow{d_{II}} \ldots.$$

The differential d_I induces a chain complex

$$\ldots \xrightarrow{d_I} E^1_{m-p,p} \xrightarrow{d_I} E^1_{m-p-1,p} \xrightarrow{d_I} \ldots$$

whose homology at $E^1_{m-p,p}$ is $E^2_{m-p,p} = H_I H_{II}$.

We have

$$A_{m-p,p} = \oplus_{a=0}^{m-p}\ \mathbb{Z} \otimes_{SL_n\mathbb{F}[t]/(t^3)}\ \underline{B}_p G_n^3 \otimes \underline{B}_a G_n^2 \otimes \underline{B}_{m-p-a} SL_n\mathbb{F}$$

$$\downarrow\ d_{II} = \partial \otimes 1 \otimes 1$$

$$A_{m-p,p-1} = \oplus_{a=0}^{m-p}\ \mathbb{Z} \otimes_{SL_n\mathbb{F}[t]/(t^3)}\ \underline{B}_{p-1} G_n^3 \otimes \underline{B}_a G_n^2 \otimes \underline{B}_{m-p-a} SL_n\mathbb{F}$$

so that

$$E^1_{m-p,p} = H_p(G_n^3) \otimes_{SL_n\mathbb{F}[t]/(t^2)} \tilde{B}_{m-p}\mathbb{Z}\otimes_{SL_n\mathbb{F}[t]/(t^2)}$$

where $\tilde{B}_*\mathbb{Z}\otimes_{SL_n\mathbb{F}[t]/(t^2)}$ is a free $SL_n\mathbb{F}[t]/(t^2)$-resolution of $\mathbb{Z}$ so that

$$(H_I H_{II})_{r,s} = E^2_{r,s} \cong H_r(SL_n\mathbb{F}[t]/(t^2); H_s(M_n^0)),$$

as required.

The successive differentials on the successive $E^t_{*,*}$'s eventually gives a "terminal" answer $E^\infty_{*,*}$ which is isomorphic to the associated graded of the filtration on $H_{r+s}(SL_n\mathbb{F}[t]/(t^3); \mathbb{Z})$ corresponding to the images of the homology the $[F^p A_{*,*})$'s.

It is show in [?] that, in the spectral sequence we are considering,

$$F^0 = 0, F^1/F^0 \cong \mathbb{F} \oplus \mathbb{F}, F^2/F^1 \cong \mathbb{F}, F^3/F^2 \otimes \mathbb{Z}_2 = 0.$$

3. Addressing Question ??

In this section I shall compute the injective transfer map

$$i^* : H_2(SL_2\mathbb{F}_2[t]/(t^2); M_2^0\mathbb{F}_2) \longrightarrow H_2(\langle\tau\rangle \propto M_2^0\mathbb{F}_2; M_2^0\mathbb{F}_2).$$

This map is injective because

$$SL_2\mathbb{F}_2 = GL_2\mathbb{F}_2 \cong \Sigma_3 = \{\tau, \sigma \mid \tau^2 = 1 = \sigma^3, \tau\sigma\tau = \sigma^2\}$$

and

$$SL_2\mathbb{F}_2[t]/(t^2) \cong \Sigma_3 \propto M_2^0\mathbb{F}_2.$$

If z generates a cyclic group of order n write $P_*(g) \stackrel{\epsilon}{\longrightarrow} \mathbb{Z}$ for the free resolution of the trivial module $\mathbb{Z}$ of the form

$$\ldots \longrightarrow \mathbb{Z}[\langle g\rangle]e_2(g) \longrightarrow \mathbb{Z}[\langle g\rangle]e_1(g) \longrightarrow \mathbb{Z}[\langle g\rangle]e_0(g) \longrightarrow \mathbb{Z}$$

where $d(e_{2m}(g)) = (1+g+\ldots+g^{n-1})e_{2m-1}(g)$, $d(e_{2m-1}(g)) = (1-g)e_{2m-2}(g)$ and $\epsilon(e_0(g)) = 1$.

Since $M_2^0 = \langle U\rangle \oplus \langle V\rangle \oplus \langle W\rangle$ a free $\mathbb{Z}[M_2^0]$-resolution is given by

$$P_*(U) \otimes P_*(V) \otimes P_*(W)$$

with differential on $P_a(U) \otimes P_b(V) \otimes P_c(W)$

$$d = d \otimes 1 \otimes 1 + (-1)^a d \otimes 1 \otimes 1 + (-1)^{a+b} d \otimes 1 \otimes 1.$$

Now to construct a free $\Sigma_3 = \langle\tau\rangle \propto \langle\sigma\rangle$-resolution. First we must decide how to let τ act on the left of $P_*(\sigma)$. The action on the inhomogeneous bar resolution of $\langle\sigma\rangle$ would be

$$\tau(g[g_1|g_2|\ldots|g_t]) = \tau(g)[\tau(g_1)|\tau(g_2)|\ldots|\tau(g_t)$$

and the map ([**?**] p.17)

$$\phi : P_*(\sigma) \longrightarrow \underline{B}_*\langle\sigma\rangle$$

is given by

$$\phi(e_i(\sigma)) = \begin{cases} \sum_I \ [\sigma^{i_1}|\sigma|\sigma^{i_2}|\sigma|\ldots|\sigma^{i_s}|\sigma] & \text{if } i = 2s \\ \\ \sum_I \ [\sigma|\sigma^{i_1}|\sigma|\sigma^{i_2}|\sigma|\ldots|\sigma^{i_s}|\sigma] & \text{if } i = 2s+1. \end{cases}$$

It seems difficult to define the involution τ directly on the small resolution so form the tensor product

$$C_s = \oplus_{a+b=s} \ \underline{B}_a\langle\sigma\rangle \ \otimes P_b(\tau)$$

with differential $d = d\otimes 1 + (-1)^a 1\otimes d$ and Σ_3-action given by $\sigma(x\otimes y) = \sigma\cdot x\otimes y$ and $\tau(x\otimes y) = \tau(x)\otimes\tau\cdot y$. This is a chain complex of free Σ_3 modules because

$$\begin{aligned} \sigma(d(x\otimes y)) &= \sigma\cdot d(x)\otimes y + (-1)^a\sigma\cdot x\otimes d(y) \\ &= d(\sigma\cdot x)\otimes y + (-1)^a\sigma\cdot x\otimes d(y) \end{aligned}$$

and

$$\begin{aligned}\tau(d(x\otimes y)) &= \tau(d(x))\otimes \tau\cdot y + (-1)^a\tau(x)\otimes\tau\cdot d(y)\\ &= d(\tau(x))\otimes\tau\cdot y + (-1)^a\tau(x)\otimes\tau\cdot d(y).\end{aligned}$$

Furthermore

$$\tau(\sigma(x\otimes y)) = \tau(\sigma\cdot x\otimes y) = \tau(\sigma\cdot x)\otimes\tau\cdot y = \tau(\sigma)\cdot\tau(x)\otimes\tau\cdot y$$

while

$$\sigma^2(\tau(x\otimes y)) = \sigma^2(\tau(x)\otimes\tau\cdot y) = \sigma^2\cdot\tau(x)\otimes\tau\cdot y) = \tau(\sigma)\cdot\tau(x)\otimes\tau\cdot y$$

which shows that the actions provide a Σ_3-module structure.

Next we need a Σ_3-action on the resolution for $M_2^0\mathbb{F}_2$. This does not seem to work on the resolution $P_*(U)\otimes P_*(V)\otimes P_*(W)$. Recall that the action on U, V, W in the semi-direct product is given by

$$\tau(U) = U = \sigma(U), \tau(V) = W, \sigma(V) = W, \sigma(W) = U + V + W.$$

This suggests the action

$$\tau(e_s(U)) = e_s(U) = \sigma(e_s(U)),\ \tau(e_s(V)) = e_s(W),\ \tau(e_s(W)) = e_s(V)$$

and

$$\sigma(e_s(V)) = e_s(W),\ \sigma(e_s(W)) = e_s(U) + e_s(V) + e_s(W).$$

This is indeed a $\mathbb{F}_2[\Sigma_3]$-action because

$$\begin{aligned}\sigma^3(e_s(V)) &= \sigma^2(e_s(W)) = \sigma(e_s(U) + e_s(V) + e_s(W))\\ &= e_s(U) + e_s(W)) + \sigma(e_s(U) + e_s(V) + e_s(W))\\ &\equiv e_s(V)\ (\text{modulo } 2)\end{aligned}$$

which will almost suffice for our purposes since we are eventually going to work in cohomology with coefficients in M_2^0.

However, the problem is that $P_*(U)\otimes P_*(V)\otimes P_*(W)$ is not preserved by this action, which spreads out onto something more like the symmetric algebra.

Therefore we had better use the inhomogeneous bar resolution $\underline{B}M_2^0$. Since $g\in\Sigma_3$ acts on the group M_2^0 as described above it acts on the bar resolution by

$$g(z_0[z_1|\ldots\ z_m]) = g(z_0)[g(z_1)|\ldots\ g(z_m)].$$

Form the $\mathbb{Z}[SL_2\mathbb{F}_2[t]/(t^2)]$-resolution of $\mathbb{Z}$ of the form

$$\underline{B}M_2^0\otimes C_*$$

with $g\in\Sigma_3$ acting via

$$g(a\otimes c) = g(a)\otimes g\cdot c$$

and $m\in M_2^0$ acting via

$$m(a\otimes c) = m\cdot a\otimes c.$$

We have to verify that both sides of

$$(g,1)(1,g^{-1}(m)) = (g,m) = (1,m)(g,1) = (g,m) \in \Sigma_3 \propto M_2^0$$

which follows since

$$(g,1)(1,g^{-1}(m))(a\otimes c) = (g,1)(g^{-1}(m)\cdot a\otimes c) = m\cdot g(a)\otimes g\cdot c$$

while

$$(1,m)(g,1)(a\otimes c) = (1,m)(g(a)\otimes g\cdot c) = m\cdot g(a)\otimes g\cdot c.$$

Now the first of the two 2-cycle representatives which we wish to map is, in terms of the inhomogeneous bar resolution,

$$I_2\otimes([(1+t)I_2|(1+t)I_2]-[1|1]) \in M_2^0\otimes_{SL_2\mathbb{F}_2[t]/(t^2)} \underline{B}_* SL_2\mathbb{F}_2[t]/(t^2).$$

We know that this represents a non-zero homology class, because d_2 is non-zero on it, and we can see that it originates in

$$M_2^0\otimes_{\langle U\rangle} \underline{B}_*\langle U\rangle$$

so that in terms of the other resolution it is represented by

$$I_2\otimes([(1+t)I_2|(1+t)I_2]-[1|1])\otimes 1 \in M_2^0\otimes_{SL_2\mathbb{F}_2[t]/(t^2)} \underline{B}M_2^0\otimes C_*$$

where $1\in C_0$ on the right and the other 1 is the neutral element of the group.

The transfer is induced by the chain map from

$$M_2^0\otimes_{SL_2\mathbb{F}_2[t]/(t^2)} \underline{B}M_2^0\otimes C_*$$

to

$$M_2^0\otimes_{\langle\tau\rangle\propto M_2^0} \underline{B}M_2^0\otimes C_*$$

which sends $X\otimes_{SL_2\mathbb{F}_2[t]/(t^2)} a\otimes c$ to

$$\sum_{g=1,\sigma,\sigma^2} (X)g^{-1}\otimes_{\langle\tau\rangle\propto M_2^0} g(a)\otimes g\cdot c$$

so that the transfer of

$$I_2\otimes([(1+t)I_2|(1+t)I_2]-[1|1])\otimes 1 = I_2\otimes([U|U]-[1|1])\otimes 1]$$

is represented by

$$I_2\otimes([U|U]-[1|1])\otimes(1+\sigma+\sigma^2)\cdot 1.$$

Next we want to calculate the image under the transfer of

$$I_2\otimes_{SL_2\mathbb{F}[t]/(t^2)}([x_{1,1}|x_{1,1}]-[1|1])$$

$$= I_2\otimes_{SL_2\mathbb{F}[t]/(t^2)}([V+W|V+W])-[1|1])\otimes 1$$

which equals

$$\sum_{g=1,\sigma,\sigma^2} I_2 \otimes_{\langle\tau\rangle\propto M_2^0} ([g(V)+g(W)|g(V)+g(W)]-[1|1])\otimes g\cdot 1$$

$$= I_2 \otimes_{\langle\tau\rangle\propto M_2^0} ([V+W|V+W]-[1|1])\otimes 1$$

$$+I_2 \otimes_{\langle\tau\rangle\propto M_2^0} ([W+U+V+W|W+U+V+W]-[1|1])\otimes \sigma\cdot 1$$

$$+I_2 \otimes_{\langle\tau\rangle\propto M_2^0} ([U+V+W+V)|U+V+W+V]-[1|1])\otimes \sigma^2\cdot 1$$

$$= I_2 \otimes_{\langle\tau\rangle\propto M_2^0} ([V+W|V+W]-[1|1])\otimes 1$$

$$+I_2 \otimes_{\langle\tau\rangle\propto M_2^0} ([U+V|U+V]-[1|1])\otimes \sigma\cdot 1$$

$$+I_2 \otimes_{\langle\tau\rangle\propto M_2^0} ([U+W)|U+W]-[1|1])\otimes \sigma^2\cdot 1.$$

The M_2^0 on the left of

$$M_2^0 \otimes_{\langle\tau\rangle\propto M_2^0} \underline{B}M_2^0 \otimes C_*$$

is the direct sum of the trivial $\mathbb{Z}[\langle\tau\rangle \propto M_2^0]$-module $\mathbb{F}\langle U\rangle$ and the module induced from the trivial $\mathbb{Z}[M_2^0]$-module. Both our 2-cycles are in the former summand because of the $I_2 \otimes \ldots$ tensor factor.

This summand is $H_*(\langle\tau\rangle \propto M_2^0; \mathbb{Z}/2)$. However there is a group isomorphism

$$\langle\tau\rangle \propto M_2^0 \cong \langle U\rangle \times (\langle\tau\rangle \propto \langle V, W\rangle) \cong C_2 \times D_8.$$

Therefore by the Kunneth formula the first summand is

$$H_*(C_2; \mathbb{Z}/2)) \otimes H_*(D_8; \mathbb{Z}/2).$$

The mod 2 cohomology of these groups is computed in detail in ([**?**] p.16 and p.24).

REWRITE FROM HERE ON

Form the $\mathbb{F}_2[SL_2\mathbb{F}_2[t]/(t^2)]$-resolution of $\mathbb{F}_2$ of the form

$$P_*(U) \otimes P_*(V) \otimes P_*(W) \otimes C_*$$

with $g \in \Sigma_3$ acting via

$$g(a \otimes b \otimes c \otimes d) = g(a) \otimes g(b) \otimes g(c) \otimes g \cdot d$$

and $m \in M_2^0$ acting via

$$m(a \otimes b \otimes c \otimes d) = a \otimes b \otimes c \otimes m \cdot d.$$

Then, if $\mathbb{F}_2[SL_2\mathbb{F}_2[t]/(t^2)]$ acts on M_2^0 on the right by mapping to $SL_2\mathbb{F}_2$ and then conjugating on the right $(X)Y = Y^{-1}XY$ so that

$$((X)Y_1)Y_2) = (Y_1^{-1}XY_1)Y_2 = Y_2^{-1}Y_1^{-1}XY_1Y_2 = (X)(Y_1Y_2).$$

Our model for $H_2(SL_2\mathbb{F}_2[t]/(t^2); M_n^0)$ is the 2-dimensional homology of

$$M_2^0 \otimes_{SL_2\mathbb{F}_2[t]/(t^2)} P_*(U) \otimes P_*(V) \otimes P_*(W) \otimes C_*.$$

The transfer is induced by the chain map from the above complex to

$$M_2^0 \otimes_{\langle\tau\rangle \propto M_2^0} P_*(U) \otimes P_*(V) \otimes P_*(W) \otimes C_*$$

which sends $X \otimes_{SL_2\mathbb{F}_2[t]/(t^2)} a \otimes b \otimes c \otimes d$ to

$$\sum_{g=1,\sigma,\sigma^2} (X)g^{-1} \otimes_{\langle\tau\rangle \propto M_2^0} g(a \otimes b \otimes c \otimes d).$$

Now the first of the two 2-cycle representatives which we wish to map is, in terms of the inhomogeneous bar resolution,

$$I_2 \otimes [(1+t)I_2|(1+t)I_2] \in M_2^0 \otimes_{SL_2\mathbb{F}_2[t]/(t^2)} \underline{B}_* SL_2\mathbb{F}_2[t]/(t^2).$$

We know that this represents a non-zero homology class, because d_2 is non-zero on it, and we can see that it originates in

$$M_2^0 \otimes_{\langle U\rangle} \underline{B}_* \langle U\rangle$$

so that in terms of the other resolution it must be represented by

$$I_2 \otimes_{\langle\tau\rangle \propto M_2^0} e_2(U) \otimes e_0(V) \otimes e_0(W) \otimes 1 \in M_2^0 \otimes_{\langle\tau\rangle \propto M_2^0} P_*(U) \otimes P_*(V) \otimes P_*(W) \otimes C_*.$$

Since I_2 is central the image under the transfer of this element is

$$\begin{aligned}
&I_2 \otimes_{\langle\tau\rangle \propto M_2^0} e_2(U) \otimes e_0(V) \otimes e_0(W) \otimes 1 \\
&+I_2 \otimes_{\langle\tau\rangle \propto M_2^0} e_2(U) \otimes e_0(W) \otimes (e_0(U) + e_0(V) + e_0(W)) \otimes \sigma \cdot 1 \\
&+I_2 \otimes_{\langle\tau\rangle \propto M_2^0} e_2(U) \otimes (e_0(U) + e_0(V) \\
&\qquad +e_0(W)) \otimes (e_0(U) + e_0(W) + (e_0(U) + e_0(V) + e_0(W)) \otimes \sigma^2 \cdot 1 \\
&= I_2 \otimes_{\langle\tau\rangle \propto M_2^0} e_2(U) \otimes (1 + \sigma + \sigma^2) \cdot 1.
\end{aligned}$$

The second of the two 2-cycle representatives which we wish to map is, in terms of the inhomogeneous bar resolution,

$$I_2 \otimes [V+W|V+W] \in M_2^0 \otimes_{SL_2\mathbb{F}_2[t]/(t^2)} \underline{B}_* SL_2\mathbb{F}_2[t]/(t^2).$$

4. Another group

I want to examine in detail the group of matrices in $GL_2\mathbb{F}_2[t]/(t^3)$ which reduce to matrices in $SL_2\mathbb{F}_2[t]/(t^2)$. Firstly I shall list all the elements of

$SL_2\mathbb{F}_2[t]/(t^2)$.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tau\sigma^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1+t & 0 \\ 0 & 1+t \end{pmatrix},\ V = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\ W = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

$$UV = \begin{pmatrix} 1+t & t \\ 0 & 1+t \end{pmatrix},\ UW = \begin{pmatrix} 1+t & 0 \\ t & 1+t \end{pmatrix},\ VW = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$$

$$UVW = \begin{pmatrix} 1+t & t \\ t & 1+t \end{pmatrix}, U^{e(u)}V^{e(v)}W^{e(w)}\tau^{e(\tau)}\sigma^{e(\sigma)}$$

with $0 \leq e(u), e(v), e(w), e(\tau) \leq 1$ and $0 \leq e(\sigma) \leq 2$. In $GL_2\mathbb{F}_2[t]/(t^3$ we have

$$\begin{pmatrix} 1+at^2 & bt^2 \\ ct^2 & 1+dt^2 \end{pmatrix} U^{e(u)}V^{e(v)}W^{e(w)}\tau^{e(\tau)}\sigma^{e(\sigma)}$$

with $0 \leq a, b, c, d \leq 1$.

Next let us examine conjugation by the base on the fibre. We have

$$\begin{pmatrix} 1+at^2 & bt^2 \\ ct^2 & 1+dt^2 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+at^2 & t+bt^2 \\ ct^2 & 1+dt^2 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+at^2 & bt^2 \\ ct^2 & 1+dt^2 \end{pmatrix} = \begin{pmatrix} 1+at^2 & bt^2+t \\ ct^2 & 1+dt^2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1+at^2 & bt^2 \\ ct^2 & 1+dt^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1+at^2 & bt^2 \\ ct^2+t & 1+dt^2 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} 1+at^2 & bt^2 \\ ct^2 & 1+dt^2 \end{pmatrix} = \begin{pmatrix} 1+at^2 & bt^2 \\ t+ct^2 & 1+dt^2 \end{pmatrix}$$

so that U, V, W all centralise the fibre subgroup.

Can one find 3 matrices in the fibre which transform in the second manner under conjugation. We would need

$$u = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

fixed by σ-conjugation. Therefore

$$\begin{aligned}\sigma u \sigma^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} b & a \\ a+b & a+b \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a+b & b \\ 0 & a+b \end{pmatrix}.\end{aligned}$$

Therefore

$$u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ aka } \begin{pmatrix} 1+t^2 & 0 \\ 0 & 1+t^2 \end{pmatrix}.$$

Next we would like $\tau(v) = w = \sigma(v)$. If

$$v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } w = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

the τ-action implies

$$w = \begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

In addition

$$\begin{aligned}\sigma v \sigma^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} c & d \\ a+c & b+d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} c+d & c \\ a+b+c+d & a+c \end{pmatrix}.\end{aligned}$$

so that $c = 0, a = d = 1$. Hence

$$v = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ and } w = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

Next we conjugate w by σ

$$\begin{aligned}\sigma w \sigma^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} b & 1 \\ 1+b & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1+b & b \\ b & 1+b \end{pmatrix}.\end{aligned}$$

so that if $b = 1$

$$\sigma w \sigma^2 = \sigma^2 v \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = v + w.$$

5. A third group

Consider the homomorphism

$$SL_n\mathbb{Z}/4[t]/(t^2) \xrightarrow{\pi} SL_n\mathbb{Z}/2[t]/(t^2).$$

Let

$$X = \begin{pmatrix} a + bt & c + dt \\ e + ft & g + ht \end{pmatrix} \in SL_2\mathbb{Z}/4[t]/(t^2).$$

In order to lie in $SL_n\mathbb{Z}/4[t]/(t^2)$ we must have

$$\det(X) = (a+bt)(g+ht) - (c+dt)(e+ft) = ag + bgt + aht - ce - det - cft \equiv 1$$

which means that

$$ag - ce \equiv 1, \; bg + ah \equiv de + cf \text{ (modulo 4)}.$$

Now consider $\ker(\pi)$ when $n = 2$. A matrix in this kernel has the form

$$X = \begin{pmatrix} 1 + 2a + 2bt & 2c + 2dt \\ 2e + 2ft & 1 + 2g + 2ht \end{pmatrix}$$

in addition to satisfying the congruences

$$(1 + 2a)(1 + 2g) \equiv 1, \; 2b(1 + 2g) + (1 + 2a)2h \equiv 0 \text{ (modulo 4)}$$

which is equivalent to

$$2a + 2g \equiv 0, \; 2b + 2h \equiv 0 \text{ (modulo 4)}.$$

Hence $X \in \ker(\pi) \bigcap SL_2\mathbb{Z}/4[t]/(t^2)$ has the form

$$X = \begin{pmatrix} 1 + 2a + 2bt & 2c + 2dt \\ 2e + 2ft & 1 + 2a + 2bt \end{pmatrix}.$$

Next we observe that

$$\begin{pmatrix} 1+2a+2bt & 2c+2dt \\ 2e+2ft & 1+2a+2bt \end{pmatrix} \cdot \begin{pmatrix} 1+2a'+2b't & 2c'+2d't \\ 2e'+2f't & 1+2a'+2b't \end{pmatrix}$$

$$= \begin{pmatrix} (1+2a+2bt)(1+2a'+2b't) & 2c+2dt+2c'+2d't \\ 2e+2ft+2e'+2f't & (1+2a+2bt)(1+2a'+2b't) \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt+2a'+2b't & 2c+2dt+2c'+2d't \\ 2e+2ft+2e'+2f't & 1+2a+2bt+2a'+2b't) \end{pmatrix}$$

so that $\ker(\pi)\bigcap SL_2\mathbb{Z}/4[t]/(t^2)$ is an abelian group isomorphic to $M_2^0\mathbb{F}_2 \times M_2^0\mathbb{F}_2$.

Next we examine how $M_2^0\mathbb{F}_2 \subset \Sigma_3 \propto M_2^0\mathbb{F}_2 \cong SL_2\mathbb{Z}/2[t]/(t^2)$ acts on $\ker(\pi)\bigcap SL_2\mathbb{Z}/4[t]/(t^2)$. $M_2^0\mathbb{F}_2$ is generated by the three matrices U, V, W given by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \ W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence U corresponds to the matrix

$$U = \begin{pmatrix} 1+t & 0 \\ 0 & 1+t \end{pmatrix}$$

which lifts to

$$\hat{U} = \begin{pmatrix} 1+3t & 0 \\ 0 & 1+t \end{pmatrix}, \ \hat{U}^{-1} = \begin{pmatrix} 1+t & 0 \\ 0 & 1+3t \end{pmatrix}.$$

We have

$$\begin{pmatrix} 1+3t & 0 \\ 0 & 1+t \end{pmatrix} \begin{pmatrix} 1+2a+2bt & 2c+2dt \\ 2e+2ft & 1+2a+2bt \end{pmatrix} \begin{pmatrix} 1+t & 0 \\ 0 & 1+3t \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt+3t+2at & 2c+2dt+2ct \\ 2e+2ft+2et & 1+2a+2bt+t+2at \end{pmatrix} \begin{pmatrix} 1+t & 0 \\ 0 & 1+3t \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt+3t+2at+t+2at & 2c+2dt+2ct+2ct \\ 2e+2ft+2et+2et & 1+2a+2bt+t+2at+3t+2at \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt & 2c+2dt \\ 2e+2ft & 1+2a+2bt \end{pmatrix}$$

so that U acts trivially.

Similarly V corresponds to the matrix

$$V = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

which lifts to

$$\hat{V} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \; \hat{V}^{-1} = \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+2a+2bt & 2c+2dt \\ 2e+2ft & 1+2a+2bt \end{pmatrix} \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt+2et & 2c+2dt+t+2at \\ 2e+2ft & 1+2a+2bt \end{pmatrix} \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt+2et & 2c+2dt+t+2at+3t+2at \\ 2e+2ft & 1+2a+2bt+2et \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt+2et & 2c+2dt \\ 2e+2ft & 1+2a+2bt+2et \end{pmatrix}.$$

Therefore V acts trivially on the fibre group if $2e \equiv 0$ (modulo 4).

Similarly W corresponds to the matrix

$$W = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

which lifts to

$$\hat{W} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \; \hat{W}^{-1} = \begin{pmatrix} 1 & 0 \\ 3t & 1 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 1+2a+2bt & 2c+2dt \\ 2e+2ft & 1+2a+2bt \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3t & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt & 2c+2dt \\ 2e+2ft+t+2at & 1+2a+2bt+2ct \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3t & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt+2ct & 2c+2dt \\ 2e+2ft+t+2at+3t+2at & 1+2a+2bt+2ct \end{pmatrix}$$

$$= \begin{pmatrix} 1+2a+2bt+2ct & 2c+2dt \\ 2e+2ft & 1+2a+2bt+2ct \end{pmatrix}.$$

Therefore W acts trivially on the fibre group if $2c \equiv 0$ (modulo 4).

Now consider the 2-cycles made from the matrices in $SL_2\mathbb{F}_2[t]/(t^2)$

$$z = \begin{pmatrix} 1+t & 0 \\ 0 & 1+t \end{pmatrix}, \text{ and } x = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$$

We lift x to

$$X = \begin{pmatrix} 1+2a+2bt & 2c+(1+2d)t \\ 2e+(1+2f)t & 1+2a+2ht \end{pmatrix} \in SL_2\mathbb{Z}/4[t]/(t^2)$$

The conditions that $X \in SL_2\mathbb{Z}/4[t]/(t^2)$ are

$$(1+2a)(1+2a) - 2c \cdot 2e \equiv 1,$$

$$2b(1+2a) + (1+2a)2h \equiv (1+2d)2e + 2c(1+2f) \text{ (modulo 4)}$$

of which the first condition is satisfied automatically by X and the second is equivalent to

$$2b + 2h \equiv 2e + 2c \text{ (modulo 4)}$$

so we may write

$$X = \begin{pmatrix} 1+2a+2bt & 2c+(1+2d)t \\ 2e+(1+2f)t & 1+2a+(2c+2b+2e)t \end{pmatrix} \in SL_2\mathbb{Z}/4[t]/(t^2)$$

Next we must compute

$$
\begin{aligned}
& X \cdot X \\
&= \begin{pmatrix} 1+2a+2bt & 2c+(1+2d)t \\ 2e+(1+2f)t & 1+2a+(2c+2b+2e)t \end{pmatrix} \cdot \\
&\qquad \begin{pmatrix} 1+2a+2bt & 2c+(1+2d)t \\ 2e+(1+2f)t & 1+2a+(2c+2b+2e)t \end{pmatrix} \\
&= \begin{pmatrix} 1+2ct+2et & 2+(2c+2e)t \\ 2+(2c+2e)t & 1+2et+2ct \end{pmatrix}
\end{aligned}
$$

We lift

$$
z = \begin{pmatrix} 1+t & 0 \\ 0 & 1+t \end{pmatrix}
$$

to

$$
Z = \begin{pmatrix} 1+2a+(1+2b)t & 2c+2dt \\ 2e+2ft & 1+2a+(3+2b)t \end{pmatrix} \in SL_2\mathbb{Z}/4[t]/(t^2)
$$

whose determinant is equal to

$$
\begin{aligned}
& (1+2a)(1+2a+3t+2bt)+(1+2b)t(1+2a+3t+2bt) \\
&= 1+2a+3t+2bt+2a+2at+t+2at+2bt \\
&= 1+4t
\end{aligned}
$$

so that $Z \in SL_2\mathbb{Z}/4[t]/(t^2)$.

Next we compute

$$Z \cdot Z$$

$$= \begin{pmatrix} 1+2a+(1+2b)t & 2c+2dt \\ 2e+2ft & 1+2a+(3+2b)t \end{pmatrix}$$

$$\begin{pmatrix} 1+2a+(1+2b)t & 2c+2dt \\ 2e+2ft & 1+2a+(3+2b)t \end{pmatrix}$$

$$= \begin{pmatrix} 1+2t & 0 \\ 0 & 1+2t \end{pmatrix}$$

Therefore choosing $e = c = 0$ in the lift X we find that pairing with the difference of our two 2-cycles yields

$$I_2 \otimes \begin{pmatrix} 1 & 2t \\ 2t & 1 \end{pmatrix} - I_2 \otimes \begin{pmatrix} 1+2t & 0 \\ 0 & 1+2t \end{pmatrix} \in (M_2^0 \otimes M_2^0)_{SL_2\mathbb{F}_2[t]/(t^2)}$$

which is non-zero!

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