

**DERIVED LANGLANDS II: HYPER-HECKE ALGEBRAS,
MONOCENTRIC RELATIONS AND
 $\mathcal{M}_{cmc,\phi}(G)$ -ADMISSIBILITY**

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ABSTRACT. In the book “Derived Langlands” I claimed to give a construction which enabled one to consider admissible representations of a locally profinite Lie group as chain complexes of monomial representations - i.e. representations given by compactly supported induction from one-dimensional representations of compact, open modulo the centre subgroups. In the jargon, this process amounts to embedding the smooth representation category into a derived category of the monomial category. The book went on to illustrate some aspects of the construction as they pertained to different parts of the Langlands Programme.

The generality described above may appear rather daunting. However, the constructions are potentially interesting and useful even in the simple case of finite-dimensional representations of finite groups or Galois groups - particularly over fields of positive characteristic. Therefore I shall spend the first two lectures introducing the monomial category in this case, where it was developed by Robert Boltje. The morphisms form a ring, which I call the hyperHecke algebra. I shall introduce the notion of exactness in this category which is necessary to define monomial resolution of a representation. I shall spend some time on the case of complex representations of general linear groups over finite fields which illustrates several features - invariants of representations such as Kondo-Gauss sums, bijective correspondences such as Shintani base change - which are simple analogues of important ingredients of the Langlands Programme.

Later lectures will describe (i) how the monomial resolution construction and the hyperHecke algebra generalises to the case of admissible representations of compact, open modulo the centre subgroups and (ii) how algebraic topology extends the resolutions to admissible representations of an arbitrary locally p -adic Lie group. The circumstances under which this works is that of “monomial admissibility” which bears the same relation to the hyperHecke algebra as does the (weaker) usual notion of admissibility to the usual Hecke algebra.

There are several other bits and pieces: (i) how the famous Bernstein centre can be approached via the hyperHecke algebra (Deligne’s essay on the subject uses the Hecke algebra but rather differently) (ii) how the morphisms in the monomial category may be described by means of the convolution product in the classical Hecke algebra (iii) monomial resolutions of automorphic representations and the way spaces of classical modular forms appear in the notion of monomial exactness.

Since starting this investigation a long while ago, I have dug up more questions than answers. I am interested to understand, for example, whether some of this material impinges on modularity of Galois representations. Hope to see you there!!

1. INTRODUCTION

This essay is a very intuitive development of the ideas in [15]. The book was little more than a curiosity for the experts in the Langlands Programme¹.

¹For example the Lecture Notes Series of the L.M.Soc. took several years to realise that no referee could be found for it! Expert comments were not very encouraging (from

In this context “intuitive” is intended to indicate that this essay has the sort of tangential rigour-level which is only to be expected in the author’s context (i.e. age, retirement, mathematical isolation which is so customary in the UK, health [12], dilettante ignorance!).

This is a provisional version, up loaded to my home-page in order to be there in time for my Sheffield lectures. Later I shall replace this apology with a complete introductory description of the contents.

2. THE HYPERHECKE ALGEBRA

Let G be a locally profinite group and let k be an algebraically closed field. Suppose that $\underline{\phi} : Z(G) \rightarrow k^*$ is a fixed k -valued, continuous character on the centre $Z(G)$ of G . Let $\mathcal{M}_{cmc,\underline{\phi}}(G)$ be the poset of pairs (H, ϕ) where H is a subgroup of G , containing $Z(G)$, which is compact, open modulo the centre of G and $\phi : H \rightarrow k^*$ is a k -valued, continuous character whose restriction to $Z(G)$ is $\underline{\phi}$.

We define the hyperHecke algebra, $\mathcal{H}_{cmc}(G)$, to be the k -algebra given by the following generators and relations. For $(H, \phi), (K, \psi) \in \mathcal{M}_{cmc,\underline{\phi}}(G)$, write $[(K, \psi), g, (H, \phi)]$ for any triple consisting of $g \in G$, characters ϕ, ψ on subgroups $H, K \leq G$, respectively such that

$$(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

which means that $K \leq g^{-1}Hg$ and that $\psi(k) = \phi(h)$ where $k = g^{-1}hg$ for $h \in H, k \in K$.

Let \mathcal{H} denote the k -vector space with basis given by these triples. Define a product on these triples by the formula

$$[(H, \phi), g_1, (J, \mu)] \cdot [(K, \psi), g_2, (H, \phi)] = [(K, \psi), g_1g_2, (J, \mu)]$$

and zero otherwise. This product makes sense because

- (i) if $K \leq g_2^{-1}Hg_2$ and $H \leq g_1^{-1}Jg_1$ then $K \leq g_2^{-1}Hg_2 \leq g_2^{-1}g_1^{-1}Jg_1g_2$ and
- (ii) if $\psi(k) = \phi(h) = \mu(j)$, where $k = g_2^{-1}hg_2, h = g_1^{-1}jg_1$ then $k = g_2^{-1}g_1^{-1}jg_1g_2$.

This product is clearly associative and we define an algebra $\mathcal{H}_{cmc}(G)$ to be \mathcal{H} modulo the relations ([15]²)

$$[(K, \psi), gk, (H, \phi)] = \psi(k^{-1})[(K, \psi), g, (H, \phi)]$$

and

$$[(K, \psi), hg, (H, \phi)] = \phi(h^{-1})[(K, \psi), g, (H, \phi)].$$

P.S. “Why are you doing this?”; from G.H. a cheerful joke about New Year resolutions). Eventually I was approached by World Scientific, who were so helpful that I was delighted to take up their offer of publication.

²For the purposes of this essay I am using throughout a convention in which g is replaced by g^{-1} to make the composition coincide with the conventions of induced representations which I shall use here.

3. THE MONOCENTRE OF A GROUP

Suppose that we are in the situation of §2.

To recapitulate: Let G be a locally profinite group and let k be an algebraically closed field. Suppose that $\underline{\phi} : Z(G) \longrightarrow k^*$ is a fixed k -valued, continuous character on the centre $Z(G)$ of G . Let $\mathcal{M}_{cmc,\underline{\phi}}(G)$, as in §2, be the poset of pairs $(H, \underline{\phi})$ where H is a subgroup of G , containing $Z(G)$, which is compact, open modulo the centre of G and $\underline{\phi} : H \longrightarrow k^*$ is a k -valued, continuous character whose restriction to $Z(G)$ is $\underline{\phi}$.

As (K, ψ) varies over $\mathcal{M}_{cmc,\underline{\phi}}(G)$ suppose that we have a family of elements of G , $\{x_{(K,\psi)} \in \text{stab}_G(K, \psi)\}$ indexed by pairs (K, ψ) where $\text{stab}_G(K, \psi)$ denotes the stabiliser of (K, ψ)

$$\text{stab}_G(K, \psi) = \{z \in G \mid zKz^{-1} = K, \psi(zkz^{-1}) = \psi(k) \text{ for all } k \in K\}.$$

This is equivalent to $K \leq x_{(K,\psi)}^{-1} K x_{(K,\psi)}$ and, for all $k \in K$,

$$\psi(x_{(K,\psi)}^{-1} k x_{(K,\psi)}) = \psi(k) = x_{(K,\psi)}^*(\psi)(x_{(K,\psi)}^{-1} k x_{(K,\psi)})$$

so that $[(K, \psi), x_{(K,\psi)}, (K, \psi)]$ is one of the basis vectors for \mathcal{H} of §2.

Next suppose that $(H, \phi) \in \mathcal{M}_{cmc,\underline{\phi}}(G)$ and $x_{(H,\phi)}$ are similar data for another pair and that $[(K, \psi), g, (H, \phi)]$ is another basis element of \mathcal{H} in §2.

The **monocentre condition** relating these elements is defined by

(i) $g x_{(K,\psi)} g^{-1} \in \text{stab}_G(H, \phi)$

and

(ii) $g x_{(K,\psi)} g^{-1} = x_{(H,\phi)} \in \text{stab}_G(H, \phi) / \text{Ker}(\phi)$.

Observe that $\text{Ker}(\phi)$ is a normal subgroup of $\text{stab}_G(H, \phi)$. Therefore if $[(K, \psi), g, (H, \phi)]$, $x_{(K,\psi)}$ and $x_{(H,\phi)}$ satisfy the monocentre condition then so do $[(K, \psi), g, (H, \phi)]$, $x_{(K,\psi)}^{-1}$ and $x_{(H,\phi)}^{-1}$.

Furthermore, if $[(K, \psi), g, (H, \phi)]$, $x_{(K,\psi)}$ and $x_{(H,\phi)}$ satisfy the monocentre condition and $w \in \text{Ker}(\psi) \leq K$ then $[(K, \psi), g, (H, \phi)]$, $x_{(K,\psi)} w$ and $x_{(H,\phi)} g w g^{-1}$ also satisfy the condition and $g w g^{-1} \in \text{Ker}(\phi) \leq H$.

Proposition 3.1.

The monocentre condition implies that the two compositions, defined in §2,

$$[(K, \psi), g, (H, \phi)] \cdot [(K, \psi), x_{(K,\psi)}, (K, \psi)]$$

and

$$[(H, \phi), x_{(H,\phi)}, (H, \phi)] \cdot [(K, \psi), g, (H, \phi)]$$

are equal in the algebra $\mathcal{H}_{cmc}(G)$ of §2.

Proof: By definition we have

$$[(K, \psi), g, (H, \phi)] \cdot [(K, \psi), x_{(K,\psi)}, (K, \psi)] = [(K, \psi), g x_{(K,\psi)}, (H, \phi)]$$

and

$$[(H, \phi), x_{(H,\phi)}, (H, \phi)] \cdot [(K, \psi), g, (H, \phi)] = [(K, \psi), x_{(H,\phi)} g, (H, \phi)].$$

The monomial condition implies that there exists $z \in \text{Ker}(\phi) \subseteq H$ such that

$$g^{-1}x_{(H,\phi)}^{-1} = x_{(K,\psi)}^{-1}g^{-1}z$$

or, equivalently, that

$$zx_{(H,\phi)}g = gx_{(K,\psi)}.$$

By the relation defining $\mathcal{H}_{cmc}(G)$ at the end of §2

$$\begin{aligned} [(K, \psi), zx_{(H,\phi)}g, (H, \phi)] &= \phi(z)^{-1}[(K, \psi), zx_{(H,\phi)}g, (H, \phi)] \\ &= [(K, \psi), zx_{(H,\phi)}g, (H, \phi)]. \end{aligned}$$

□

Definition 3.2. The monocentre of G , denoted by $Z_{\mathcal{M}}(G)$, is the set of families $\{x_{(K,\psi)} \in \text{stab}_G(K, \psi)/\text{Ker}(\psi)\}$ such that for every $x_{(K,\psi)}$, $x_{(H,\phi)}$ and g such that $(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi))$ the monocentre condition holds, as introduced above.

As we shall see in more detail, because the monocentre condition includes a central character which is common to the pairs (K, ψ) and (H, ϕ) , $Z_{\mathcal{M}}(G)$ is the product of subgroups

Remark 3.3. *The maximal pairs*

The elements in a family corresponding to an element of $Z_{\mathcal{M}}(G)$ are cosets such as $x_{(K,\psi)} \in \text{stab}_G(K, \psi)/\text{Ker}(\psi)$ as (K, ψ) varies over $\mathcal{M}_{cmc,\underline{\phi}}(G)$.

If (K, ψ) is maximal in the partially ordered set $\mathcal{M}_{cmc,\underline{\phi}}(G)$ and we have

$$(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

we must have $K = g^{-1}Hg$ and $\phi(gkg^{-1}) = \psi(k)$ for all $k \in K$. Then any pair

$$x_{(K,\psi)} \in \text{stab}_G(K, \psi)/\text{Ker}(\psi), \quad x_{(H,\phi)} \in \text{stab}_G(H, \phi)/\text{Ker}(\phi)$$

in the family must satisfy $gx_{(K,\psi)}g^{-1} \in \text{stab}_G(H, \phi)$ automatically and also that

$$gx_{(K,\psi)}g^{-1} = x_{(H,\phi)} \in \text{stab}_G(H, \phi)/\text{Ker}(\phi).$$

This means that the elements in a monocentre family corresponding to maximal pairs are determined by any one maximal pair - the other entries being conjugates as shown above. - which depend only on g 's modulo the stabiliser, which is precisely the indeterminacy which the conditions impose on g .

Notwithstanding the above remark, it remains to decide whether or not $Z_{\mathcal{M}}(G)$ can be non-trivial.

Example 3.4. $G = D_8$

The dihedral group of order eight is given by

$$D_8 = \langle x, y \mid x^4 = 1 = y^2, \quad yxy = x^3 \rangle.$$

The subgroups and the one-dimensional complex characters on them are

$$\begin{aligned}
D_8 &: 1, \chi_1, \chi_2, \chi_1\chi_2 \\
&\quad \chi_1(x) = -1, \chi_1(y) = 1, \chi_2(x) = 1, \chi_2(y) = -1 \\
\langle x \rangle &: 1, \phi, \phi^2, \phi^3 \\
&\quad \phi(x) = i = \sqrt{-1} \\
\langle x^2, y \rangle &: 1, \tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_1\tilde{\chi}_2 \\
&\quad \tilde{\chi}_1(x^2) = -1, \tilde{\chi}_1(y) = 1, \tilde{\chi}_2(x^2) = 1, \tilde{\chi}_2(y) = -1 \\
\langle x^2, xy \rangle &: 1, \hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_1\hat{\chi}_2 \\
&\quad \hat{\chi}_1(x^2) = -1, \hat{\chi}_1(xy) = 1, \hat{\chi}_2(x^2) = 1, \hat{\chi}_2(xy) = -1 \\
\langle x^2 \rangle &: 1, \chi \\
\langle 1 \rangle &: 1
\end{aligned}$$

Therefore the elements of $\mathcal{M}_{cmc,1}(D_8)$ are

$$\begin{aligned}
D_8 &: 1, \chi_1, \chi_2, \chi_1\chi_2 \\
\langle x \rangle &: 1, \phi^2, \\
\langle x^2, y \rangle &: 1, \tilde{\chi}_2 \\
\langle x^2, xy \rangle &: 1, \hat{\chi}_2 \\
\langle x^2 \rangle &: 1
\end{aligned}$$

Now consider

$$x_{(\langle x^2 \rangle, 1)} \in \text{stab}_{D_8}(\langle x^2 \rangle, 1) / \text{Ker}(1) = D_8 / \langle x^2 \rangle.$$

We have inclusions

$$\begin{aligned}
\langle \langle x^2 \rangle, 1 \rangle &\leq (D_8, 1) \\
\langle \langle x^2 \rangle, 1 \rangle &\leq (D_8, \chi_1) \\
\langle \langle x^2 \rangle, 1 \rangle &\leq (D_8, \chi_2) \\
\langle \langle x^2 \rangle, 1 \rangle &\leq (D_8, \chi_1\chi_2) \\
\langle \langle x^2 \rangle, 1 \rangle &\leq \langle x \rangle, 1) \\
\langle \langle x^2 \rangle, 1 \rangle &\leq (\langle x \rangle, \phi^2) \\
\langle \langle x^2 \rangle, 1 \rangle &\leq (\langle x^2, y \rangle, 1) \\
\langle \langle x^2 \rangle, 1 \rangle &\leq (\langle x^2, y \rangle, \tilde{\chi}_2) \\
\langle \langle x^2 \rangle, 1 \rangle &\leq (\langle x^2, xy \rangle, 1) \\
\langle \langle x^2 \rangle, 1 \rangle &\leq (\langle x^2, xy \rangle, \hat{\chi}_2)
\end{aligned}$$

which require monocentre conditions which are depicted as follows:

$$\begin{aligned}
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(D_8, 1)}, D_8 / \langle x^2 \rangle \longrightarrow \{1\}, \\
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(D_8, \chi_1)}, D_8 / \langle x^2 \rangle \longrightarrow D_8 / \langle x^2, y \rangle, \\
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(D_8, \chi_2)}, D_8 / \langle x^2 \rangle \longrightarrow D_8 / \langle x \rangle, \\
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(D_8, \chi_1\chi_2)}, D_8 / \langle x^2 \rangle \longrightarrow D_8 / \langle x^2, xy \rangle, \\
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(\langle x \rangle, 1)}, D_8 / \langle x^2 \rangle \longrightarrow D_8 / \langle x \rangle, \\
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(\langle x \rangle, \phi^2)}, D_8 / \langle x^2 \rangle \longrightarrow D_8 / \langle x^2 \rangle, \\
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(\langle x^2, y \rangle, 1)}, D_8 / \langle x^2 \rangle \longrightarrow D_8 / \langle x^2, y \rangle, \\
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(\langle x^2, y \rangle, \tilde{\chi}_2)}, D_8 / \langle x^2 \rangle \longrightarrow D_8 / \langle x^2 \rangle, \\
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(\langle x^2, xy \rangle, 1)}, D_8 / \langle x^2 \rangle \longrightarrow D_8 / \langle x^2, xy \rangle, \\
x_{(\langle x^2 \rangle, 1)} &\mapsto x_{(\langle x^2, xy \rangle, \hat{\chi}_2)}, D_8 / \langle x^2 \rangle \longrightarrow D_8 / \langle x^2 \rangle.
\end{aligned}$$

Therefore $Z_{\mathcal{M}_{cmc,1}}(D_8) \cong D_8/\langle x^2 \rangle$.

Next consider

$$x_{(\langle x^2 \rangle, \chi)} \in \text{stab}_{D_8}(\langle x^2 \rangle, \chi) / \text{Ker}(\chi) = D_8.$$

The elements of $\mathcal{M}_{cmc,\chi}(D_8)$ are

$$\begin{aligned} \langle x \rangle &: \phi, \phi^3 \\ \langle x^2, y \rangle &: \tilde{\chi}_1, \tilde{\chi}_1 \tilde{\chi}_2 \\ \langle x^2, xy \rangle &: \hat{\chi}_1, \hat{\chi}_1 \hat{\chi}_2 \\ \langle x^2 \rangle &: \chi \end{aligned}$$

so we have inclusions

$$\begin{aligned} (\langle x^2 \rangle, \chi) &\leq (\langle x \rangle, \phi), \\ (\langle x^2 \rangle, \chi) &\leq (\langle x \rangle, \phi^3), \\ (\langle x^2 \rangle, \chi) &\leq (\langle x^2, y \rangle, \tilde{\chi}_1), \\ (\langle x^2 \rangle, \chi) &\leq (\langle x^2, y \rangle, \tilde{\chi}_1 \tilde{\chi}_2), \\ (\langle x^2 \rangle, \chi) &\leq (\langle x^2, xy \rangle, \hat{\chi}_1), \\ (\langle x^2 \rangle, \chi) &\leq (\langle x^2, xy \rangle, \hat{\chi}_1 \hat{\chi}_2). \end{aligned}$$

which require monocentre conditions which are depicted as follows:

$$\begin{aligned} x_{(\langle x^2 \rangle, \chi)} &\longrightarrow x_{(\langle x \rangle, \phi)}, \quad D_8 \longrightarrow \langle x \rangle \\ x_{(\langle x^2 \rangle, \chi)} &\longrightarrow x_{(\langle x \rangle, \phi^3)}, \quad D_8 \longrightarrow \langle x \rangle \\ x_{(\langle x^2 \rangle, \chi)} &\longrightarrow x_{(\langle x^2, y \rangle, \tilde{\chi}_1)}, \quad D_8 \longrightarrow \langle x^2, y \rangle / \langle y \rangle \\ x_{(\langle x^2 \rangle, \chi)} &\longrightarrow x_{(\langle x^2, y \rangle, \tilde{\chi}_1 \tilde{\chi}_2)}, \quad D_8 \longrightarrow \langle x^2, y \rangle / \langle x^2 y \rangle \\ x_{(\langle x^2 \rangle, \chi)} &\longrightarrow x_{(\langle x^2, xy \rangle, \hat{\chi}_1)}, \quad D_8 \longrightarrow \langle x^2, xy \rangle / \langle xy \rangle \\ x_{(\langle x^2 \rangle, \chi)} &\longrightarrow x_{(\langle x^2, xy \rangle, \hat{\chi}_1 \hat{\chi}_2)}, \quad D_8 \longrightarrow \langle x^2, xy \rangle / \langle x^3 y \rangle. \end{aligned}$$

In the case of the first two conditions in the list we see that

$$x_{(\langle x^2 \rangle, \chi)} \in \langle x \rangle$$

is necessary. However, in order to satisfy the monocentre condition in relation to the last four entries in the list we require that

$$x_{(\langle x^2 \rangle, \chi)} \in \langle x^2 \rangle.$$

Therefore we obtain

$$Z_{\mathcal{M}}(D_8) = Z_{\mathcal{M}_{cmc,1}}(D_8) \times Z_{\mathcal{M}_{cmc,\chi}}(D_8) \cong D_8/\langle x^2 \rangle \times \langle x^2 \rangle.$$

This dihedral example reveals the following result.

Theorem 3.5.

The monocentre group, $Z_{\mathcal{M}}(G)$, is the product of the subgroups $Z_{\mathcal{M}_{cmc,\underline{\phi}}}(G)$ as $\underline{\phi}$ varies over the central characters. Also the set of elements in a family $\{x_{(K,\psi)} \in \text{stab}_G(K, \psi) / \text{Ker}(\psi)\}$ representing an element of $Z_{\mathcal{M}_{cmc,\underline{\phi}}}(G)$ are determined by the

$$x_{(Z(G), \underline{\phi})} \in G / \text{Ker}(\underline{\phi})$$

such that the image of $x_{(Z(G), \underline{\phi})}$ represents an element $x_{(K,\psi)} \in \text{stab}_G(K, \psi) / \text{Ker}(\psi)$ for every $(K, \phi) \in \mathcal{M}_{cmc,\underline{\phi}}$.

Remark 3.6. (i) The monocentre group is an entertaining construction, but it will turn out to be too restrictive for our purposes. Although it might be less trivial - even useful! - in the case of modular representations.

(ii) More important is the following situation (see §11.3 and §11.4). Fix a central character $\underline{\phi}$ as usual.

Suppose, for $i = 1, 2$, that we are given

$$[(K_i, \psi_i), g_i, (H_i, \phi_i)] \text{ and}$$

$$\{x_{(K_i, \psi_i)} \in \text{stab}_G(K_i, \psi_i)/\text{Ker}(\psi_i)\} \text{ and}$$

$$\{x_{(H_i, \phi_i)} \in \text{stab}_G(H_i, \phi_i)/\text{Ker}(\phi_i)\}$$

which satisfy both

$$\begin{aligned} & [(H_1, \phi_1), x_{(H_1, \phi_1)}, (H_1, \phi_1)] \cdot [(K_1, \psi_1), g_1, (H_1, \phi_1)] \\ &= [(K_1, \psi_1), g_1, (H_1, \phi_1)] \cdot [(K_1, \psi_1), x_{(K_1, \psi_1)}, (K_1, \psi_1)] \end{aligned}$$

and

$$\begin{aligned} & [(H_2, \phi_2), x_{(H_2, \phi_2)}, (H_2, \phi_2)] \cdot [(K_2, \psi_2), g_2, (H_2, \phi_2)] \\ &= [(K_2, \psi_2), g_2, (H_2, \phi_2)] \cdot [(K_2, \psi_2), x_{(K_2, \psi_2)}, (K_2, \psi_2)]. \end{aligned}$$

Under these conditions we require that for all

$$[(H_1, \phi_1), g_3, (H_2, \phi_2)] \text{ and } [(K_1, \psi_1), g_4, (K_2, \psi_2)]$$

such that

$$\begin{aligned} & [(H_1, \phi_1), g_3, (H_2, \phi_2)] \cdot [(K_1, \psi_1), g_1, (H_1, \phi_1)] \\ &= [(K_2, \psi_2), g_2, (H_2, \phi_2)] \cdot [(K_1, \psi_1), g_4, (K_2, \psi_2)] \end{aligned}$$

the $\{x_{(K_i, \psi_i)}, x_{(H_i, \phi_i)}\}$ satisfy

$$\begin{aligned} & [(H_2, \phi_2), x_{(H_2, \phi_2)}, (H_2, \phi_2)] \cdot [(H_1, \phi_1), g_3, (H_2, \phi_2)] \\ &= [(H_1, \phi_1), g_3, (H_2, \phi_2)] \cdot [(H_1, \phi_1), x_{(H_1, \phi_1)}, (H_1, \phi_1)] \end{aligned}$$

and also that

$$\begin{aligned} & [(K_2, \psi_2), x_{(K_2, \psi_2)}, (K_2, \psi_2)] \cdot [(K_1, \psi_1), g_4, (K_2, \psi_2)] \\ &= [(K_1, \psi_1), g_4, (K_2, \psi_2)] \cdot [(K_1, \psi_1), x_{(K_1, \psi_1)}, (K_1, \psi_1)]. \end{aligned}$$

4. EXTENDING THE DEFINITION OF ADMISSIBILITY

If G is a locally profinite group and k is an algebraically closed field then a k -representation of G is a vector space V with a left, k -linear G -action. Let $\underline{\phi} : Z(G) \longrightarrow k^*$ be a continuous character on the centre of G . Let $\mathcal{M}_{cmc, \underline{\phi}}(G)$, as in §2, denote the poset of pairs (H, ϕ) where H is a subgroup of G , such

that $Z(G) \subseteq H$, which is compact open modulo the centre and $\phi : H \longrightarrow k^*$ is a continuous character which extends $\underline{\phi}$.

Suppose that V is acted upon by $g \in Z(G)$ via multiplication by $\underline{\phi}(g)$. The representation V is called smooth if

$$V = \bigcup_{K \subset G, K \text{ compact, open}} V^K.$$

V is called admissible if $\dim_k(V^K) < \infty$ for all compact open subgroups K . Define a subspace of V , denoted by $V^{(H, \phi)}$, for $(H, \phi) \in \mathcal{M}_{cmc, \underline{\phi}}(G)$ by

$$V^{(H, \phi)} = \{v \in V \mid g \cdot v = \phi(g)v \text{ for all } g \in H\}.$$

Hence $V^K = V^{(Z(G) \cdot K, \phi)}$ if ϕ is a continuous character which is trivial on K .

We shall say that V is $\mathcal{M}_{cmc, \underline{\phi}}(G)$ -smooth if

$$V = \bigcup_{(H, \phi) \in \mathcal{M}_{cmc, \underline{\phi}}(G)} V^{(H, \phi)}.$$

In addition we shall say that V is $\mathcal{M}_{cmc, \underline{\phi}}(G)$ -admissible if $\dim_k V^{(H, \phi)} < \infty$ for all $(H, \phi) \in \mathcal{M}_{cmc, \underline{\phi}}(G)$.

Proposition 4.1.

Let G be a locally profinite group and let k be an algebraically closed field. Let V be a k -representation of G with central character ϕ . Suppose that every continuous, k -valued character of a compact open subgroup of G has finite image. Then V is $\mathcal{M}_{cmc, \underline{\phi}}(G)$ -admissible if and only if it is admissible.

Proof:

If K is compact open then $K \cap Z(G)$ is also compact open. It is certainly compact, being a closed subset of a compact subspace. For $G = GL_n F$ with F a p -adic local field the assumption is true. More generally, it holds if the quotient of $Z(G)$ by its maximal compact subgroup is discrete³.

Suppose that V is admissible. If H is a subgroup of G which is compact open modulo the centre then $H = Z(G) \cdot K$ for some compact open subgroup. In this case suppose that ϕ is a character of H extending the central character. Then $V^{(H, \phi)} = V^{(K, \mu)}$ where $\mu = \text{Res}_K^H(\phi)$. Since the image of μ is finite the kernel of μ is compact open and $V^{(K, \mu)} \subseteq V^{\text{Ker}(\mu)}$, which is finite-dimensional.

Next suppose that $0 \neq v \in V$. There exists a compact open subgroup K such that $v \in V^K$. Set $H = Z(G) \cdot K$, which is compact open modulo $Z(G) \subset H$. If $g \in Z(G) \cap K$ then $v = g \cdot v = \underline{\phi}(g) \cdot v$ so that the central character is trivial on $Z(G) \cap K$. Hence the central character induces a character λ on H which factors through $K/Z(G) \cap K \cong Z(G) \cdot K/K$ and so $v \in V^{(H, \lambda)}$, which completes the proof of $\mathcal{M}_{cmc, \underline{\phi}}(G)$ -admissibility.

³If this condition is not true in general it is true in the main cases of interest. Therefore let us treat it as an unimportant assumption for the time being!

Assume that V is $\mathcal{M}_{cmc,\underline{\phi}}(G)$ -admissible. If $0 \neq v \in V$ belongs to $V^{(H,\phi)}$ where H is compact open modulo the centre then $H = Z(G) \cdot K$ where K is compact open. Hence $v \in V^J$ where J is the compact open subgroup given by $J = \text{Ker}(\text{Res}_K^H(\phi))$.

Next suppose that K is a compact open subgroup. If V^K is non-trivial then $V^K \subseteq V^{(Z(G) \cdot K, \lambda)}$ where $\lambda : H = Z(G) \cdot K \rightarrow k^*$ is the character which was constructed in the first half of the proof. Since $V^{(Z(G) \cdot K, \lambda)}$ is assumed to be finite-dimensional this concludes the proof of admissibility. \square

Question 4.2. *Di- p -adic Langlands*

In the last 20 years I believe that several authors have studied the “ p -adic Langlands programme”. This is the situation where, for example, one studies “admissible” representations of a locally p -adic Lie group on vector spaces over the algebraic closure of a p -adic local field (or its residue field).

I intend to call this the di- p -adic situation since it is no more complicated to say and indicates the involvement of p -adic fields twice. In addition to [1] there are lots of papers on this subject⁴ and a useful source for these (brought to my attention by Rob Kurinczuk) is the bibliography of [7].

The question arises: Are the sort of representations considered by the di- p -adic professionals $\mathcal{M}_{cmc,\underline{\phi}}(G)$ -admissible?

5. INDUCED REPRESENTATIONS AND LOCALLY PROFINITE GROUPS

Let G be a locally profinite group. In this section we are going to study admissible representations of G and its subgroups in relation to induction. These representations will be given by left-actions of the groups on vector spaces over k , which is an algebraically closed field of arbitrary characteristic.

Let us begin by recalling, from ([15] Chapter Two §1), induced and compactly induced smooth representations.

Definition 5.1.

Let G be a locally profinite group and $H \subseteq G$ a closed subgroup. Thus H is also locally profinite. Let

$$\sigma : H \rightarrow \text{Aut}_k(W)$$

be a smooth representation of H . Set X equal to the space of functions $f : G \rightarrow W$ such that (writing simply $h \cdot w$ for $\sigma(h)(w)$ if $h \in H, w \in W$)

- (i) $f(hg) = h \cdot f(g)$ for all $h \in H, g \in G$,
- (ii) there is a compact open subgroup $K_f \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K_f$.

The (left) action of G on X is given by $(g \cdot f)(x) = f(xg)$ and

$$\Sigma : G \rightarrow \text{Aut}_k(X)$$

gives a smooth representation of G .

⁴Regrettably I have not got round to reading any of them!

The representation Σ is called the representation of G smoothly induced from σ and is usually denoted by $\Sigma = \text{Ind}_H^G(\sigma)$.

5.2.

$$(g \cdot f)(hg_1) = f(hg_1g) = hf(g_1g) = h(g \cdot f)(g_1)$$

so that $(g \cdot f)$ satisfies condition (i) of Definition 5.1.

Also, by the same discussion as in the finite group case (see Appendix §4), the formula will give a left G -representation, providing that $g \cdot f \in X$ when $f \in X$. However, condition (ii) asserts that there exists a compact open subgroup K_f such that $k \cdot f = f$ for all $k \in K_f$. The subgroup gK_fg^{-1} is also a compact open subgroup and, if $k \in K_f$, we have

$$(gkg^{-1}) \cdot (g \cdot f) = (gkg^{-1}g) \cdot f = (gk) \cdot f = (g \cdot (k \cdot f)) = (g \cdot f)$$

so that $g \cdot f \in X$, as required.

The smooth representations of G form an abelian category $\text{Rep}(G)$.

Proposition 5.3.

The functor

$$\text{Ind}_H^G : \text{Rep}(H) \longrightarrow \text{Rep}(G)$$

is additive and exact.

Proposition 5.4. (*Frobenius Reciprocity*)

There is an isomorphism

$$\text{Hom}_G(\pi, \text{Ind}_H^G(\sigma)) \xrightarrow{\cong} \text{Hom}_H(\pi, \sigma)$$

given by $\phi \mapsto \alpha \cdot \phi$ where α is the H -map

$$\text{Ind}_H^G(\sigma) \longrightarrow \sigma$$

given by $\alpha(f) = f(1)$.

5.5. In general, if $H \subseteq Q$ are two closed subgroups there is a Q -map

$$\text{Ind}_H^G(\sigma) \longrightarrow \text{Ind}_H^Q(\sigma)$$

given by restriction of functions. Note that α in Proposition 5.4 is the special case where $H = Q$.

5.6. *The c -Ind variation*

Inside X let X_c denote the set of functions which are compactly supported modulo H . This means that the image of the support

$$\text{supp}(f) = \{g \in G \mid f(g) \neq 0\}$$

has compact image in $H \backslash G$. Alternatively there is a compact subset $C \subseteq G$ such that $\text{supp}(f) \subseteq H \cdot C$.

The Σ -action on X preserves X_c , since $\text{supp}(g \cdot f) = \text{supp}(f)g^{-1} \subseteq HCg^{-1}$, and we obtain $X_c = c - \text{Ind}_H^G(W)$, the compact induction of W from H to G .

This construction is of particular interest when H is open. There is a canonical left H -map (see the Appendix in induction in the case of finite groups)

$$f : W \longrightarrow c - \text{Ind}_H^G(W)$$

given by $w \mapsto f_w$ where f_w is supported in H and $f_w(h) = h \cdot w$ (so $f_w(g) = 0$ if $g \notin H$).

For $g \in G$ we have

$$\begin{aligned} (g \cdot f_w)(x) = f_w(xg) &= \begin{cases} 0 & \text{if } xg \notin H, \\ (xg^{-1}) \cdot w & \text{if } xg \in H, \end{cases} \\ &= \begin{cases} 0 & \text{if } x \notin Hg^{-1}, \\ (xg^{-1}) \cdot w & \text{if } x \in Hg^{-1}. \end{cases} \end{aligned}$$

We shall be particularly interested in the case when $\dim_k(W) = 1$. In this case we write $W = k_\phi$ where $\phi : H \longrightarrow k^*$ is a continuous/smooth character and, as a vector space with a left H -action $W = k$ on which $h \in H$ acts by multiplication by $\phi(h)$. In this case α_c is an injective left $k[H]$ -module homomorphism of the form

$$f : k_\phi \longrightarrow c - \text{Ind}_H^G(k_\phi).$$

Lemma 5.7.

Let H be an open subgroup of G . Then

(i) $f : w \mapsto f_w$ is an H -isomorphism onto the space of functions $f \in c - \text{Ind}_H^G(W)$ such that $\text{supp}(f) \subseteq H$.

(ii) If $w \in W$ and $h \in H$ then $h \cdot f_w = f_{h \cdot w}$.

(iii) If \mathcal{W} is a k -basis of W and \mathcal{G} is a set of coset representatives for $H \backslash G$ then

$$\{g \cdot f_w \mid w \in \mathcal{W}, g \in \mathcal{G}\}$$

is a k -basis of $c - \text{Ind}_H^G(W)$.

Proof

If $\text{supp}(f)$ is compact modulo H there exists a compact subset C such that

$$\text{supp}(f) \subseteq HC = \bigcup_{c \in C} Hc.$$

Each Hc is open so the open covering of C by the Hc 's refines to a finite covering and so

$$C = Hc_1 \bigcup \dots \bigcup Hc_n$$

and so

$$\text{supp}(f) \subseteq HC = Hc_1 \bigcup \dots \bigcup Hc_n.$$

For part (i), the map f is an H -homomorphism to the space of functions supported in H with inverse map $f \mapsto f(1)$.

For part (ii), from §?? we have

$$(h \cdot f_w)(x) = f_w(xh) = \begin{cases} 0 & \text{if } x \notin H, \\ xh \cdot w & \text{if } x \in H. \end{cases}$$

so that, for all $x \in G$, $(h \cdot f_w)(x) = f_{h \cdot w}(x)$, as required.

For part (iii), the support of any $f \in c - \text{Ind}_H^G(W)$ is a finite union of cosets Hg where the g 's are chosen from the set of coset representatives \mathcal{G} of $H \backslash G$. The restriction of f to any one of these Hg 's also lies in $c - \text{Ind}_H^G(W)$. If $\text{supp}(f) \subseteq Hg$ then $(g \cdot f)(z) \neq 0$ implies that $zg \in Hg$ so that $g \cdot f$ has support contained in H . Hence $g \cdot f$ on H is a finite linear combination of the functions f_w with $w \in \mathcal{W}$. Therefore f is a finite linear combination of $g \cdot f_w$'s where $w \in \mathcal{W}, g \in \mathcal{G}$. Clearly the set of functions $g \cdot f_w$ with $g \in \mathcal{G}$ and $w \in \mathcal{W}$ is linearly independent. \square

Example 5.8. Let K be a p -adic local field with valuation ring \mathcal{O}_K and π_K a generator of the maximal ideal of \mathcal{O}_K . Suppose that $G = GL_n K$ and that H is a subgroup containing the centre of G (that is, the scalar matrices K^*). If H is compact, open modulo K^* then there is a subgroup H' of finite index in H such that $H' = K^* H_1$ with H_1 compact, open in $SL_n K$. This can be established by studying the simplicial action of $GL_n K$ on a suitable barycentric subdivision of the Bruhat-Tits building of $SL_n K$ (see [15] Chapter Four §1).

To show that H is both open and closed it suffices to verify this for H' . Firstly H' is open, since it is $H' = \bigcup_{z \in K^*} z H_1 = \bigcup_{s \in \mathbb{Z}} \pi_K^s H_1$.

Also $H' = K^* H_1$ is closed. Suppose that $X' \notin K^* H_1$. $K^* H_1$ is closed under multiplication by the multiplicative group generated by π_K so that $\pi_K^m X' \notin K^* H_1$ for all m . By conjugation we may assume that H_1 is a subgroup of $SL_n \mathcal{O}_K$, which is the maximal compact open subgroup of $SL_n K$, unique up to conjugacy. Choose the smallest non-negative integer m such that every entry of $X = \pi_K^m X'$ lies in \mathcal{O}_K . Therefore we may write $0 \neq \det(X) = \pi_K^s u$ where $u \in \mathcal{O}_K^*$ and $1 \leq s$. Now suppose that V is an $n \times n$ matrix with entries in \mathcal{O}_K such that $X + \pi_K^t V \in K^* H_1$. Then

$$\det(X + \pi_K^t V) \equiv \pi_K^s u \pmod{\pi_K^t}.$$

So that if $t > s$ then s must have the form $s = nw$ for some integer w and $\pi_K^{-w}(X + \pi_K^t V) \in GL_n \mathcal{O}_K \cap K^* H_1 = H_1$. Therefore all the entries in $\pi_K^{-w} X$ lie in \mathcal{O}_K and $\pi_K^{-w} X \in GL_n \mathcal{O}_K$. Enlarging t , if necessary, we can ensure that $\pi_K^{-w} X \in H_1$, since H_1 is closed (being compact), and therefore $X' \in K^* H_1$, which is a contradiction.

Since H is both closed and open in $GL_n K$ we may form the admissible representation $c - \text{Ind}_H^{GL_n K}(k_\phi)$ for any continuous character $\phi : H \rightarrow k^*$ and apply Lemma ??.

If $g \in GL_n K, h \in H$ then $(g \cdot f_1)(x) = \phi(xg)$ if $xg \in H$ and zero otherwise. On the other hand, $(gh \cdot f_1)(x) = \phi(xgh) = \phi(h)\phi(xg)$ if $xg \in H$ and

zero otherwise. Therefore as a left GL_nK -representation $c - \text{Ind}_H^{GL_nK}(k_\phi)$ is isomorphic to

$$k[GL_nK]/(\phi(h)g - gh \mid g \in GL_nK, h \in H)$$

with left action induced by $g_1 \cdot g = g_1g$.

This vector space is isomorphic to the k -vector space whose basis is given by k -bilinear tensors over H of the form $g \otimes_{k[H]} 1$ as in the case of finite groups. The basis vector $g \cdot f_1$ corresponds to $g \otimes_H 1$ and GL_nK acts on the tensors by left multiplication, as usual (see Appendix §4 in the finite group case).

Proposition 5.9.

The functor

$$c - \text{Ind}_H^G : \text{Rep}(H) \longrightarrow \text{Rep}(G)$$

is additive and exact.

Proposition 5.10.

Let $H \subseteq G$ be an open subgroup and (σ, W) smooth. Then there is a functorial isomorphism

$$\mathrm{Hom}_G(c - \mathrm{Ind}_H^G(W), \pi) \xrightarrow{\cong} \mathrm{Hom}_H(W, \pi)$$

given by $F \mapsto F \cdot f$, the composition with the H -map f of Lemma 5.7.

Example 5.11. $c - \mathrm{Ind}_H^G(\phi)$

Suppose that $\phi : H \rightarrow k^*$ is a continuous character (i.e. a one-dimensional smooth representation of H).

Suppose that we are in a situation analogous to that of Example 5.8. Namely suppose that H is open and closed, contains $Z(G)$, the centre of G , and is compact open modulo $Z(G)$. A basis for k is given by $1 \in k^*$ and we have the function $f_1 \in X_c$ given by $f_1(h) = \phi(h)$ if $h \in H$ and $f_1(g) = 0$ if $g \notin H$.

If, following Lemma 5.7, \mathcal{G} is a set of coset representatives for $H \backslash G$ then a k -basis for $c - \mathrm{Ind}_H^G(\phi)$ is given by

$$\{g \cdot f_1 \mid g \in \mathcal{G}\}.$$

For $g \in G$ we have

$$\begin{aligned} (g \cdot f_1)(x) = f_1(xg) &= \begin{cases} 0 & \text{if } xg \notin H, \\ \phi(xg) & \text{if } xg \in H, \end{cases} \\ &= \begin{cases} 0 & \text{if } x \notin Hg^{-1}, \\ \phi(xg) & \text{if } x \in Hg^{-1}. \end{cases} \end{aligned}$$

Before going further let us introduce the presence of (H, ϕ) into the notation.

Definition 5.12. Let H be a closed subgroup of G containing the centre, $Z(G)$, which is compact open modulo $Z(G)$. Let $\phi : H \rightarrow k^*$ be a continuous character of H . Denote by $X_c(H, \phi)$ the k -vector space of functions $f : G \rightarrow k$ such that

- (i) $f(hg) = \phi(h)f(g)$ for all $h \in H, g \in G$,
- (ii) there is a compact open subgroup $K_f \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K_f$,
- (iii) f is compactly supported modulo H .

As in §5.6, the left action of G on $X_c(H, \phi)$ is given by $(g \cdot f)(x) = f(xg)$ and therefore

$$\Sigma : G \rightarrow \mathrm{Aut}_k(X_c(H, \phi))$$

gives a smooth representation of G - denoted by $\Sigma = c - \mathrm{Ind}_H^G(\phi)$.

Henceforth we shall denote the map written as f_1 in Example 5.11 by $f_{(H, \phi)} \in X_c(H, \phi)$.

Therefore, for $g \in G$, we have

$$\begin{aligned} (g \cdot f_{(H,\phi)})(x) = f_{(H,\phi)}(xg) &= \begin{cases} 0 & \text{if } xg \notin H, \\ \phi(xg) & \text{if } xg \in H, \end{cases} \\ &= \begin{cases} 0 & \text{if } x \notin Hg^{-1}, \\ \phi(xg) & \text{if } x \in Hg^{-1}. \end{cases} \end{aligned}$$

Definition 5.13. For (H, ϕ) and (K, ψ) as in Definition 5.12, write $[(K, \psi), g, (H, \phi)]$ for any triple consisting of $g \in G$, characters ϕ, ψ on subgroups $H, K \leq G$, respectively such that

$$(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

which means that $K \leq g^{-1}Hg$ and that $\psi(k) = \phi(h)$ where $k = g^{-1}hg$ for $h \in H, k \in K$.

Let \mathcal{H} denote the k -vector space with basis given by these triples. Define a product on these triples by the formula

$$[(H, \phi), g_1, (J, \mu)] \cdot [(K, \psi), g_2, (H, \phi)] = [(K, \psi), g_1g_2, (J, \mu)]$$

and zero otherwise. This product makes sense because

- (i) if $K \leq g_2^{-1}Hg_2$ and $H \leq g_1^{-1}Jg_1$ then $K \leq g_2^{-1}Hg_2 \leq g_2^{-1}g_1^{-1}Jg_1g_2$ and
- (ii) if $\psi(k) = \phi(h) = \mu(j)$, where $k = g_2^{-1}hg_2, h = g_1^{-1}jg_1$ then $k = g_2^{-1}g_1^{-1}jg_1g_2$.

This product is clearly associative and we define an algebra $\mathcal{H}_{cmc}(G)$ to be \mathcal{H} modulo the relations (c.f. Appendix §4)

$$[(K, \psi), gk, (H, \phi)] = \psi(k^{-1})[(K, \psi), g, (H, \phi)]$$

and

$$[(K, \psi), hg, (H, \phi)] = \phi(h^{-1})[(K, \psi), g, (H, \phi)].$$

We observe that

$$[(K, \psi), g, (H, \phi)] = [(g^{-1}Hg, g^*\phi), g, (H, \phi)] \cdot [(K, \psi), 1, (g^{-1}Hg, g^*\phi)]$$

We shall refer to this algebra as the compactly supported modulo the centre (CSMC-algebra) of G .

Lemma 5.14.

Let $[(K, \psi), g, (H, \phi)]$ be a triple as in Definition 5.13. Associated to this triple define a left $k[G]$ -homomorphism

$$[(K, \psi), g, (H, \phi)] : X_c(K, \psi) \longrightarrow X_c(H, \phi)$$

by the formula $g_1 \cdot f_{(K,\psi)} \mapsto (g_1g^{-1}) \cdot f_{(H,\phi)}$.

For a proof, which is the same as in the case when G is finite, can be found in (the Appendix on induction in the case of finite groups).

Theorem 5.15.

Let $\mathcal{M}_c(G)$ denote the partially order set of pairs (H, ϕ) as in Definitions 5.12 and 5.13 (so that $X_c(H, \phi) = c - \text{Ind}_H^G(\phi)$). Then

$$M_c(\underline{n}, G) = \bigoplus_{\alpha \in \mathcal{A}, (H, \phi) \in \mathcal{M}_c(G)} \underline{n}_\alpha X_c(H, \phi)$$

is a left $k[G] \times \mathcal{H}_{cmc}(G)$ -module.

Proof

We have only to verify associativity of the module multiplication, which is obvious. \square

Definition 5.16. ${}_{k[G]}\text{mon}$, the monomial category of G

The monomial category of G is the additive category (non-abelian) whose objects are the k -vector spaces given by direct sums of $X_c(H, \phi)$'s of §5.15 and whose morphisms are elements of the hyperHecke algebra $\mathcal{H}_{cmc}(G)$. In other words the subcategory of the category of $k[G] \times \mathcal{H}_{cmc}(G)$ -modules of which one example is $M_c(\underline{n}, G)$ in §5.15.

6. IDEMPOTENTED ALGEBRAS ([6] P.309)

Definition 6.1. Let k be a field and H a k -algebra. Let \mathcal{E} denote a set of idempotents of H . Assume that if $e_1, e_2 \in \mathcal{E}$ then there exists $e_0 \in \mathcal{E}$ such that $e_0 e_1 = e_1 e_0 = e_1$ and $e_0 e_2 = e_2 e_0 = e_2$. In addition assume for every $\phi \in H$ that there exists $e \in \mathcal{E}$ such that $e\phi = \phi e = \phi$.

With these assumptions H is called an idempotent k -algebra.

Write $f \leq e$ if $ef = fe = f$. This gives \mathcal{E} the structure of a partially ordered set (i.e. a poset).

If R is a ring and e an idempotent denote eRe by $R[e]$. If M is a left R -module write $M[e]$ for the $R[e]$ -module eM . If H is an idempotent algebra then $H[e]$ is a k -algebra with unit e and $M[e]$ is an $H[e]$ -module.

M is smooth if $M = \bigcup_{e \in \mathcal{E}} M[e]$ and is admissible if it is smooth and for each $e \in \mathcal{E}$ we have $\dim_k(M[e]) < \infty$.

If (H_i, \mathcal{E}_i) are idempotent algebras for $i = 1, 2$ then so is $H_1 \otimes H_2$ with idempotents $e_1 \otimes e_2$ for $e_i \in \mathcal{E}_i$.

6.2. The idempotent algebra $\mathcal{H}_{cmc}(G)$

Let \mathcal{E} be the collection of finite additive combinations in $\mathcal{H}_{cmc}(G)$, the algebra of Definition 5.13, of the form

$$e = \sum_{i=1}^n [(H_i, \phi_i), 1, (H_i, \phi_i)]$$

in which $(H_i, \phi_i) = (H_j, \phi_j)$ if and only if $i = j$. Then $e \cdot e = e$ and all the idempotents in $\mathcal{H}_{cmc}(G)$ have this form.

We shall write $e_{(H, \phi)}$ for the idempotent $[(H, \phi), 1, (H, \phi)]$.

Define the homomorphism

$$[(K', \psi'), g, (H', \phi')] : X_c(K, \psi) \longrightarrow X_c(H, \phi)$$

to be zero unless $(K', \psi') = (K, \psi)$ and $(H, \phi) = (H', \phi')$. The following result is clear.

Theorem 6.3.

(i) In §6.2 $(\mathcal{H}_{cmc}(G), \mathcal{E})$ is an idempotented algebra and $M_c(G)$ is an $\mathcal{H}_{cmc}(G)$ -module in the category of smooth $k[G]$ -modules.

(ii) In this idempotented algebra $e = \sum_{i=1}^n [(H_i, \phi_i), 1, (H_i, \phi_i)]$ and f satisfy $ef = fe = f$ in and only if the idempotent f is a subsum of e , which fits very nicely with the $f \leq e$ notation.

(iii) If $M_c(\underline{n}, G)$ is the module of Theorem 5.15 then $M_c(\underline{n}, G)[e]$ is the direct sum of the $\underline{n}_\alpha X_c(H, \phi)$'s for which $e_{(H, \phi)}$ appears in the sum for e .

6.4. Hecke algebras

The Hecke algebra of a locally compact, totally disconnected group is a related idempotented algebra.

Let G be a locally compact, totally disconnected group. Assume that G is unimodular - that is, the left invariant Haar measure equals the right-invariant Haar measure of G ([6] p.137).

The Hecke algebra of G , denoted by \mathcal{H}_G is the space $C_c^\infty(G)$ of locally constant, compactly supported k -valued functions on G with the convolution product ([6] p.140 and p.255)

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(gh)\phi_2(h^{-1})dh = \int_G \phi_1(h)\phi_2(h^{-1}g)dh.$$

This integral requires only one of ϕ_1, ϕ_2 to be compactly supported in order to land in \mathcal{H}_G .

Suppose that $K_0 \subseteq G$ is a compact, open subgroup. Define an idempotent

$$e_{K_0} = \frac{1}{\text{vol}(K_0)} \cdot \chi_{K_0}$$

where χ_{K_0} is the characteristic function of K_0 . If $K_0 \subseteq K_1$ then $e_{K_0} * e_{K_1} = e_{K_1}$.

This is seen using left invariance of the Haar measure

$$\int_G \frac{\chi_K(zh)}{\text{vol}(K)} \frac{\chi_H(h^{-1})}{\text{vol}(H)} = \int_G \frac{\chi_K(h)}{\text{vol}(K)} \frac{\chi_H(h^{-1}z)}{\text{vol}(H)}.$$

The integrand is zero unless $h \in K$ and then it is zero unless $z \in H$. When $z \in H$ we are integrating

$$\int_G \frac{\chi_K(h)}{\text{vol}(K)} \frac{1}{\text{vol}(H)} = \frac{\chi_H(z)}{\text{vol}(H)},$$

as required.

\mathcal{H}_G is an idempotented algebra because G has a base of neighbourhoods consisting of compact open subgroups.

A function $f \in \mathcal{H}_G$ is called K -finite if the subspace spanned by all its (left) translates by K is finite-dimensional ([6] p.299).

7. MONOMIAL MORPHISMS AS CONVOLUTION PRODUCTS

It is my belief and eventual intention that the material of this section will remain true for the general G as in §2 provided that all continuous k -valued characters on compact, open subgroups have finite image.

However, throughout this section I shall assume that G is a locally profinite group whose centre $Z(G)$ is compact. Let H be a subgroup which is compact, open modulo the centre. Let k be an algebraically closed field for which all continuous characters $\phi : H \rightarrow k^*$ have finite image when H is compact, open.

The following two results give some examples of G for which $Z(G)$ is compact.

Lemma 7.1.

Let K be a p -adic local field. Then $Z(SL_n K)$ is finite. In particular it is compact.

Proof

Consider the relation

$$\begin{pmatrix} x_1 & 0 & \cdot & \cdots & 0 \\ 0 & x_2 & \cdot & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & x_n \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & \cdot & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix} \\ = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdot & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 & 0 & \cdot & \cdots & 0 \\ 0 & x_2 & \cdot & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & x_n \end{pmatrix}$$

In the (i, j) entry we find $x_i a_{i,j} = a_{i,j} x_j$ and since we may suppose $a_{i,j} \neq 0$ we see that $x_1 = x_2 = \dots = x_n$ and $x_1^n = 1$. Therefore $Z(SL_n K) = \mu_n(K)$, the group of n -th roots of unity in K . \square

Lemma 7.2.

Let K be a p -adic local field with ring of integers \mathcal{O}_K and prime π_K . Then $Z(GL_n K / \langle \pi_K \rangle) \cong \mathcal{O}_K^*$. In particular it is compact. Here $\langle \pi_K \rangle$ denotes the centre subgroup generated by π_K times the identity matrix.

Proof

The relation used in the proof of §7.1 implies that for each (i, j) we have $\pi_K^\alpha x_i a_{i,j} = a_{i,j} x_j \pi_K^\beta$ for some pair α, β . Therefore we may suppose that $x_1 \in \mathcal{O}_K^*$ and that $x_j = x_1 \pi_K^{e_j}$. Now taking a matrix with $a_{1,j} a_{j,1} \neq 0$ for $j =$

$2, 3, \dots, n$ we find that $\pi_K^\alpha x_1 = x_j \pi_K^\beta = x_1 \pi_K^{\beta+e_j}$ for $j = 2, 3, \dots, n$. This implies that $x_1 \pi_K^e = x_2 = x_3 = \dots = x_n$ which implies that $e = 0$. \square

The next two results ensure that we are free to use convolution products in our context.

Lemma 7.3.

Let G be a locally profinite group whose centre $Z(G)$ is compact. If H is a subgroup of G , containing $Z(G)$, which is compact, open modulo the centre then H is compact, open.

Proof

There is a compact open subset C of G such that $H = Z(G) \cdot C$. Multiplication is a continuous map from the compact space $Z(G) \times C$ onto H so that H is compact. Furthermore any point of H may be written as $h = z \cdot c$ with $z \in Z(G)$ and $c \in C$. Therefore $z \cdot N \subseteq H$ for any open neighbourhood of c in C is an open neighbourhood of h in H , which is therefore open. \square

Lemma 7.4.

Let G be a locally profinite group whose centre $Z(G)$ is compact and let H be a subgroup which is compact, open modulo the centre. Let k be an algebraically closed field for which all continuous characters $\phi : H \rightarrow k^*$ have finite image when H is compact, open. Then the vector space, X_c , of §5.6 on which $c - \text{Ind}_H^G(k_\phi)$ is defined is a subspace of the Hecke algebra of G , \mathcal{H}_G , the space of locally constant, compactly supported k -valued functions on G .

Proof:

By §5.7 it suffices to verify that the function f_w of §5.6 is locally constant, compactly supported for $w = 1 \in k^*$. This function is given by the formula

$$f_1(x) = \begin{cases} 0 & \text{if } x \notin H, \\ \phi(x) & \text{if } x \in H, \end{cases}$$

By §7.3 H , the support of f_1 , is compact. Since the image of ϕ is finite the function f_1 is locally constant. \square

Recall from §§5.13-5.14 that we have defined

$$[(K, \psi), g, (H, \phi)] : X_c(K, \psi) \rightarrow X_c(H, \phi)$$

by the formula $g_1 \cdot f_{(K, \psi)} \mapsto (g_1 g_1^{-1}) \cdot f_{(H, \phi)}$.

If χ_W is the characteristic function of $W \subseteq G$ we may define $g_1 \cdot f_{(K, \psi)}$ using characteristic functions in the following manner. By definition

$$g_1 \cdot f_{(K, \psi)}(x) = \begin{cases} \psi(xg_1^{-1}) & \text{if } xg_1^{-1} \in K, \\ 0 & \text{if } xg_1^{-1} \notin K. \end{cases}$$

Suppose that v_1, \dots, v_t are coset representatives for $K/\text{Ker}(\psi)$. Then, if $xg_1^{-1} \in K$ we must have $xg_1^{-1} \in \text{Ker}(\psi)v_{j(xg_1^{-1})}$ for some $1 \leq j(xg_1^{-1}) \leq t$

and therefore $\psi(xg_1^{-1}) = \psi(v_{j(xg_1^{-1})})$. Hence we have the fomula

$$g_1 \cdot f_{(K,\psi)} = \sum_{j=1}^t \psi(v_j) \chi_{\text{Ker}(\psi)v_j g_1}$$

because $\bigcup \text{Ker}(\psi)v_j g_1 = Kg_1$ so that the right hand side is zero unless $xg_1^{-1} \in K$ and is $\psi(v_{j_0})$ precisely when $j_0 = j(xg_1^{-1})$.

Next, from Definition 5.13

$$(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

implies that $\psi(k) = \phi(h)$ where $k = g^{-1}hg$ for $h \in H, k \in K$. Therefore if $k \in \text{Ker}(\psi)$ then $h \in \text{Ker}(\phi)$ and so $\text{Ker}(\psi) \leq g^{-1}\text{Ker}(\phi)g$.

Consider the convolution product

$$\chi_{g_1 \text{Ker}(\psi)} * \chi_{g^{-1}\text{Ker}(\phi)}(z) = \int_G \chi_{g_1 \text{Ker}(\psi)}(h) \chi_{g^{-1}\text{Ker}(\phi)}(h^{-1}z) dh.$$

The integrand is zero unless $h \in g_1 \text{Ker}(\psi)$ in addition to the condition

$$z \in hg^{-1}\text{Ker}(\phi) \subseteq g_1 \text{Ker}(\psi)g^{-1}\text{Ker}(\phi) = g_1 g^{-1}g \text{Ker}(\psi)g^{-1}\text{Ker}(\phi) \subseteq g_1 g^{-1}\text{Ker}(\phi)$$

and conversely. Therefore

$$\chi_{g_1 \text{Ker}(\psi)} * \chi_{g^{-1}\text{Ker}(\phi)} = \text{vol}(g_1 \text{Ker}(\psi)) \chi_{g_1 g^{-1}\text{Ker}(\phi)}.$$

Similarly, if $v \in K$ and $u \in H$, we have a convolution product

$$\chi_{g_1 \text{Ker}(\psi)v} * \chi_{g^{-1}\text{Ker}(\phi)u}(z) = \int_G \chi_{g_1 \text{Ker}(\psi)v}(h) \chi_{g^{-1}\text{Ker}(\phi)u}(h^{-1}z) dh.$$

The integrand is zero unless $h \in g_1 \text{Ker}(\psi)v$ in addition to the condition

$$z \in hg^{-1}\text{Ker}(\phi)u \subseteq g_1 \text{Ker}(\psi)v g^{-1}\text{Ker}(\phi)u \subseteq g_1 g^{-1}\text{Ker}(\phi)(gvg^{-1}) \cdot u$$

and conversely. Therefore

$$\chi_{g_1 \text{Ker}(\psi)v} * \chi_{g^{-1}\text{Ker}(\phi)u} = \text{vol}(g_1 \text{Ker}(\psi)v) \chi_{g_1 g^{-1}\text{Ker}(\phi)gvg^{-1}u}.$$

Lemma 7.5.

Suppose that $v_1, \dots, v_t \in K$ is a set of coset representatives for $K/\text{Ker}(\psi)$. Then

$$g_1 \cdot f_{(K,\psi)} = \sum_{j=1}^t \psi(v_j) \cdot \chi_{\text{Ker}(\psi)v_j g_1^{-1}}.$$

Proof:

Consider the functions in the equation applied to $x \in G$. The left hand side is zero if $xg_1 \notin K$ which is equivalent to there being no j such that $xg_1 \in \text{Ker}(\psi)v_j$ or $x \in \text{Ker}(\psi)v_j g_1^{-1}$. Under these conditions every characteristic function on the right hand side also vanishes on x . On the other hand if $xg_1 \in K$ there exists a unique j_0 such that $x \in \text{Ker}(\psi)v_{j_0} g_1^{-1}$ and so, evaluated at xg_1 , there is one and only one term on the right hand side which contributes. It yields $\psi(v_{j_0})$ which is the value of $g_1 \cdot f_{(K,\psi)}$ at x , as required. \square

7.6. The image $\phi(H)$ is a finite cyclic group, being a finite subgroup of k^* , which contains $\phi(gKg^{-1}) = \psi(K)$. Therefore there exist v_1, \dots, v_t which are coset representatives for $K/\text{Ker}(\psi)$ and u_1, \dots, u_s which give distinct cosets in $H/\text{Ker}(\phi)$ such that the set $\{(gv_i g^{-1})u_j \mid 1 \leq i \leq t, 1 \leq j \leq s\}$ is a set of coset representatives for $H/\text{Ker}(\phi)$.

Definition 7.7. Define an involution $T : C_c^\infty(G) \longrightarrow C_c^\infty(G)$ by $T(F)(x) = F(x^{-1})$. For example $T(\chi_{\text{Ker}(\psi)v_j g_1^{-1}}) = \chi_{g_1 \text{Ker}(\psi)v_j^{-1}}$.

In the notation of §7.6 set

$$\Phi_{[(K,\psi),g,(H,\phi)]} = \sum_{j=1}^s \phi(u_j) \cdot \chi_{g^{-1}\text{Ker}(\phi)u_j}.$$

Theorem 7.8.

In the notation of Definition 7.7

$$\begin{aligned} [(K, \psi), g, (H, \phi)](g_1 \cdot f_{(K,\psi)}) &= g_1 g^{-1} \cdot f_{(H,\phi)} \\ &= \frac{1}{\text{vol}(\text{Ker}(\psi))} T(T(g_1 \cdot f_{(K,\psi)}) * \Phi_{[(K,\psi),g,(H,\phi)]}). \end{aligned}$$

Proof:

We observe that $\psi(v_i)(\chi_{\text{Ker}(\psi)v_i g_1^{-1}})(x) = \psi(v_i) = \psi(xg_1)$ if $x \in \text{Ker}(\psi)v_i g_1^{-1} = v_i \text{Ker}(\psi)g_1^{-1}$ and zero otherwise. Therefore

$$T(\psi(v_i)(\chi_{\text{Ker}(\psi)v_i g_1^{-1}}))(x) = \psi(v_i)(\chi_{\text{Ker}(\psi)v_i g_1^{-1}})(x^{-1}) = \psi(v_i)$$

if $x^{-1} \in \text{Ker}(\psi)v_i g_1^{-1}$ and zero otherwise. In the non-zero case $x \in g_1 \text{Ker}(\psi)v_i^{-1}$ and $\psi(v_i) = \psi(g_1^{-1}x)^{-1}$ so that

$$T(\psi(v_i)(\chi_{\text{Ker}(\psi)v_i g_1^{-1}})) = \psi(v_i)^{-1} \chi_{g_1 \text{Ker}(\psi)v_i^{-1}}.$$

From Lemma 7.1 we have

$$T(g_1 \cdot f_{(K,\psi)}) = \sum_{i=1}^t \psi(v_i)^{-1} \cdot \chi_{g_1 \text{Ker}(\psi)v_i^{-1}}.$$

Therefore

$$\begin{aligned} &T(g_1 \cdot f_{(K,\psi)}) * \Phi_{[(K,\psi),g,(H,\phi)]} \\ &= \sum_{i=1}^t \sum_{j=1}^s \psi(v_i)^{-1} \phi(u_j) (\chi_{g_1 \text{Ker}(\psi)v_i^{-1}} * \chi_{g^{-1}\text{Ker}(\phi)u_j}) \\ &= \sum_{i=1}^t \sum_{j=1}^s \psi(v_i)^{-1} \phi(u_j) \text{vol}(\text{Ker}(\psi)) \chi_{g_1 g^{-1} \text{Ker}(\phi) (g v_i^{-1} g^{-1}) u_j}. \end{aligned}$$

Hence

$$\begin{aligned}
& T(T(g_1 \cdot f_{(K,\psi)}) * \Phi_{[(K,\psi),g,(H,\phi)]}) \\
&= T(\sum_{i=1}^t \sum_{j=1}^s \psi(v_i)^{-1} \phi(u_j) \text{vol}(\text{Ker}(\psi)) \chi_{g_1 g^{-1} \text{Ker}(\phi)(g v_i^{-1} g^{-1}) u_j}) \\
&= T(\sum_{i=1}^t \sum_{j=1}^s \phi(g v_i g^{-1})^{-1} \phi(u_j) \text{vol}(\text{Ker}(\psi)) \chi_{g_1 g^{-1} \text{Ker}(\phi)(g v_i^{-1} g^{-1}) u_j}) \\
&= \text{vol}(\text{Ker}(\psi)) \sum_{i=1}^t \sum_{j=1}^s T(\phi((g v_i^{-1} g^{-1}) u_j) \chi_{g_1 g^{-1} \text{Ker}(\phi)(g v_i^{-1} g^{-1}) u_j}) \\
&= \text{vol}(\text{Ker}(\psi)) \sum_{i=1}^t \sum_{j=1}^s \phi((g v_i^{-1} g^{-1}) u_j)^{-1} \chi_{u_j^{-1} (g v_i g^{-1}) \text{Ker}(\phi) g g_1^{-1}} \\
&= \text{vol}(\text{Ker}(\psi)) g_1 g^{-1} \cdot f_{(H,\phi)},
\end{aligned}$$

by Lemma 7.1. \square

Remark 7.9. (i) Theorem 7.8 has shown that, under the special conditions which were stated at the start of this section, the morphisms of the monomial category ${}_{k[G]}\mathbf{mon}$ of Definition 5.16 are given in terms of the convolution product of §6.4 of the Hecke algebra \mathcal{H}_G .

(ii) My belief is that Theorem 7.8 remains true in general, in some sense, providing that all continuous characters $\phi : H \rightarrow k^*$ have finite image when H is compact, open. This belief is based on the following: [15] claims to construct for each admissible representation V of G a monomial resolution in the derived category ${}_{k[G]}\mathbf{mon}$ ⁵ and (see §9; also [8] pp.2-3) such V are intimately related to Hecke modules. Therefore one should expect a connection between the morphisms in that resolution and convolutions products.

The difficulty, in the case of a general locally profinite group G , with the treatment of this section is that $X_c(H, \phi)$'s are spaces of locally constant functions which are compactly supported modulo H , rather than actually being compactly supported.

It might be that I can get away with using the Schwartz space of locally constant, compactly supported functions of $G/Z(G)$, but I have not yet had time to examine this generalisation⁶.

8. THE BAR-MONOMIAL RESOLUTION: I. FINITE MODULO THE CENTRE CASE

8.1. Let G is a finitely generated group with centre $Z(G)$ and finite quotient group $G/Z(G)$ with other notation as in §2 and other earlier sections. In this section I summarise ([15] Chapter One, §2).

⁵In a later section I shall give a self-contained construction of these resolutions based on the hyperHecke algebra and which applies to any V is $\mathcal{M}_{cmc,\phi}(G)$ -admissible V .

⁶To that end, as a novice, I should re-read §9, several sections of [6] on Hecke modules and the material of ([13] §1.11 p.63).

Denote by \hat{G} the group of character homomorphisms $\text{Hom}(G, k^*)$ from G to k^* . For H be a subgroup of G containing $Z(G)$ denote by \hat{H}_ϕ the finite subset of \hat{H} of characters which are equal to ϕ when restricted to $Z(G)$.

For an arbitrary k -algebra A we write ${}_A\mathbf{mod}$ (resp. \mathbf{mod}_A) for the category of left (resp. right) A -modules. We denote by ${}_A\mathbf{lat}$ (resp. \mathbf{lat}_A) the category of left (resp. right) A -lattices i.e. the subcategory of ${}_A\mathbf{mod}$ consisting of those A -modules which are finitely generated and A -projective. The rank of a free k -module M will be denoted by $\text{rk}_k(M)$. When $A = k[G]$ we have subcategories ${}_{k[G], \phi}\mathbf{mod} \subset {}_{k[G]}\mathbf{mod}$ and ${}_{k[G], \phi}\mathbf{lat} \subset {}_{k[G]}\mathbf{lat}$ whose objects are those on which $Z(G)$ acts via ϕ .

Recall that the poset of $\mathcal{M}_\phi(G)$ of pairs (H, ϕ) admits a left G -action by conjugation for which the G -orbit of (H, ϕ) will be denoted by $(H, \phi)^G$.

Definition 8.2. A finite (G, ϕ) -lineable left $k[G]$ -module M^7 is a left $k[G]$ -module together with a fixed finite direct sum decomposition

$$M = M_1 \oplus \cdots \oplus M_m$$

where each of the M_i is a free k -module of rank one on which $Z(G)$ acts via ϕ and the G -action permutes the M_i . The M_i 's are called the lines of M . For $1 \leq i \leq m$ let H_i denote the subgroup of G with stabilises the line M_i . Then there exists a unique $\phi_i \in \hat{H}_{i, \phi}$ such that $h \cdot v = \phi_i(h)v$ for all $v \in M_i, h \in H_i$. The pair $(H_i, \phi_i) \in \mathcal{M}_\phi(G)$ is called the stabilising pair of M_i .

The k -submodule of M given by

$$M^{((H, \phi))} = \bigoplus_{1 \leq i \leq m, (H, \phi) \leq (H_i, \phi_i)} M_i$$

is called the (H, ϕ) -fixed points of M .

A morphism between (G, ϕ) -lineable modules from M to $N = N_1 \oplus \cdots \oplus N_n$ is defined to be a $k[G]$ -module homomorphism $f : M \longrightarrow N$ such that

$$f(M^{((H, \phi))}) \subseteq N^{((H, \phi))}$$

for all $(H, \phi) \in \mathcal{M}_\phi(G)$.

The (left) finite (G, ϕ) -lineable modules and their morphisms define an additive category denoted by ${}_{k[G], \phi}\mathbf{mon}$.

By definition each (G, ϕ) -Line Bundle is a k -free $k[G]$ -module so there is a forgetful functor

$$\mathcal{V} : {}_{k[G], \phi}\mathbf{mon} \longrightarrow {}_{k[G], \phi}\mathbf{mod}.$$

Remark 8.3. *Some natural operations and constructions*

There are several operations which are obvious lifts to ${}_{k[G], \phi}\mathbf{mon}$ of well-known operations in ${}_{k[G], \phi}\mathbf{mod}$. This means that the resulting functors commute with the forgetful functor $\mathcal{V} : {}_{k[G], \phi}\mathbf{mon} \longrightarrow {}_{k[G], \phi}\mathbf{mod}$.

⁷Here I have taken my own terminological advice given in the footnote to ([15] Chapter One, Definition 1.2).

Direct sum and tensor product, denoted by $M \oplus N$ and $M \otimes N$, are lineable in the obvious way as is the $k[G]$ -module $\text{Hom}_k(M, N)$, giving an object of $k[G], \phi_M \cdot \phi_N^{-1}$ **mon**. A homomorphism, preserving centres, $f : G' \rightarrow G$ induces a restriction map Res_f from $k[G], \phi$ **mon** to $k[G'], \phi \cdot f$ **mon**.

Suppose that $Z(G) \subseteq H \subseteq G$ and the index of H in G is finite and $P \in k[H], \phi$ **mon** then the usual induced $k[G]$ -module construction (see, Appendix: Comparison of Inductions) $\text{Ind}_H^G(P)$ gives a lineable module $\underline{\text{Ind}}_H^G(P)$ in such a way that, if the stabilising pair for P_i is (H_i, ϕ_i) , then the stabilising pair of the Line $g \otimes_{k[H]} P_i$ is $g(H_i, \phi_i)$.

Analogues of the usual distributivity isomorphism of direct sums over tensor products, the Frobenius reciprocity isomorphism and the Mackey decomposition isomorphism all hold in $k[G], \phi$ **mon**.

We call a M in $k[G], \phi$ **mon** indecomposable if it is not isomorphic to a non-trivial direct sum $N \oplus P$ in $k[G], \phi$ **mon**.

Proposition 8.4. The set of (G, ϕ) -lineable modules given by

$$\{\underline{\text{Ind}}_H^G(k_\phi) \mid (H, \phi) \in G \setminus \mathcal{M}_\phi(G)\}$$

is a full set of pairwise non-isomorphic representatives for the isomorphism classes of indecomposable objects in $k[G], \phi$ **mon**. Moreover any object in $k[G], \phi$ **mon** is canonically isomorphic to the direct sum of objects in this set.

8.5. Let $[(K, \psi), g, (H, \phi)]$ be one of the basic generators of the hyperHecke algebra $\mathcal{H}_{cmc}(G)$ of §2 then we have a morphism

$$[(K, \psi), g, (H, \phi)] \in \text{Hom}_{k[G], \phi \text{ mon}}(\underline{\text{Ind}}_K^G(k_\psi), \underline{\text{Ind}}_H^G(k_\phi))$$

defined by the same formula as in the case of induced modules (see, Appendix: Comparison of Inductions). In addition the composition of morphisms in $k[G], \phi$ **mon** coincides with the product in the hyperHecke algebra.

Let $(K, \psi) \in \mathcal{M}_\phi(G)$ and let N be an object of $k[G], \phi$ **mon**. Then there is a k -linear isomorphism

$$\text{Hom}_{k[G], \phi \text{ mon}}(\underline{\text{Ind}}_K^G(k_\psi), N) \xrightarrow{\cong} N^{((K, \psi))}$$

given by $f \mapsto f(1 \otimes_K 1)$. The inverse isomorphism is given by

$$n \mapsto ((g \otimes_K v \mapsto vg \cdot n)).$$

Lemma 8.6. *Projectivity in $k[G], \phi$ **mon***

Consider the diagram

$$M \xrightarrow{h} N \xleftarrow{f} P$$

in which $M, P \in k[G], \phi$ **mon** and $N \in k[G], \phi$ **mod** with h, f being morphisms in $k[G], \phi$ **mod**. Assume, for all $(H, \phi) \in \mathcal{M}_\phi(G)$, that

$$f(P^{((H, \phi))}) \subseteq h(M^{((H, \phi))}).$$

Then there exists $j \in \text{Hom}_{k[G], \phi \text{ mon}}(P, M)$ such that $h \cdot j = f$.

In particular we include the situation where $N' \in {}_{k[G], \underline{\phi}} \mathbf{mon}$ with h, f being morphisms to N' in ${}_{k[G], \underline{\phi}} \mathbf{mon}$ and the diagram above being the result of applying the forgetful functor \mathcal{V} with $N = \mathcal{V}(N')$.

8.7. Let V be a finitely generated $k[G]$ -module on which $Z(G)$ acts via the central character $\underline{\phi}$. That is, V is an object of ${}_{k[G], \underline{\phi}} \mathbf{mod}$. I shall define the notion of a ${}_{k[G], \underline{\phi}} \mathbf{mon}$ -resolution of V . This is a chain complex of morphisms in ${}_{k[G], \underline{\phi}} \mathbf{mon}$ with certain properties which will ensure that it exists and is unique up to chain homotopy in ${}_{k[G], \underline{\phi}} \mathbf{mon}$.

For $V \in {}_{k[G], \underline{\phi}} \mathbf{mod}$ and $(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)$ define the (H, ϕ) -fixed points of V by

$$V^{(H, \phi)} = \{v \in V \mid h \cdot v = \phi(h)v \text{ for all } h \in H\}.$$

Clearly $g(V^{(H, \phi)}) = V^{g(H, \phi)}$, $V^{(Z(G), \underline{\phi})} = V$ and $(K, \psi) \leq (H, \phi)$ implies that $V^{(H, \phi)} \subseteq V^{(K, \psi)}$. Note that $f \in \text{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mod}}(V, W)$ satisfies $f(V^{(H, \phi)}) \subseteq W^{(H, \phi)}$ for all $(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)$. In addition, if $M \in {}_{k[G], \underline{\phi}} \mathbf{mod}$ then $M^{((H, \phi))} \subseteq M^{(H, \phi)}$ so that

Definition 8.8. ([15] Chapter One §2)

Let $V \in {}_{k[G], \underline{\phi}} \mathbf{mod}$. A ${}_{k[G], \underline{\phi}} \mathbf{mon}$ -resolution of V is a chain complex

$$M_* : \quad \dots \xrightarrow{\partial_{i+1}} M_{i+1} \xrightarrow{\partial_i} M_i \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0$$

with $M_i \in {}_{k[G], \underline{\phi}} \mathbf{mon}$ and $\partial_i \in \text{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mon}}(M_{i+1}, M_i)$ for all $i \geq 0$ together with $\epsilon \in \text{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mod}}(\mathcal{V}(M_0), V)$ such that

$$\dots \xrightarrow{\partial_i} M_i^{((H, \phi))} \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_1} M_1^{((H, \phi))} \xrightarrow{\partial_0} M_0^{((H, \phi))} \xrightarrow{\epsilon} V^{(H, \phi)} \longrightarrow 0$$

is an exact sequence of k -modules for each $(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)$. In particular, when $(H, \phi) = (Z(G), \underline{\phi})$ we see that

$$\dots \xrightarrow{\partial_i} M_i \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0 \xrightarrow{\epsilon} V \longrightarrow 0$$

is an exact sequence in ${}_{k[G], \underline{\phi}} \mathbf{mod}$.

Proposition 8.9.

Let $V \in {}_{k[G], \underline{\phi}} \mathbf{mod}$ and let

$$\dots \longrightarrow M_n \xrightarrow{\partial_{n-1}} M_{n-1} \xrightarrow{\partial_{n-2}} \dots \xrightarrow{\partial_0} M_0 \xrightarrow{\epsilon} V \longrightarrow 0$$

be a ${}_{k[G], \underline{\phi}} \mathbf{mon}$ -resolution of V . Suppose that

$$\dots \longrightarrow C_n \xrightarrow{\partial'_{n-1}} C_{n-1} \xrightarrow{\partial'_{n-2}} \dots \xrightarrow{\partial'_0} C_0 \xrightarrow{\epsilon'} V \longrightarrow 0$$

a chain complex where each ∂'_i and C_i belong to ${}_{k[G], \underline{\phi}} \mathbf{mon}$ and ϵ' is a ${}_{k[G], \underline{\phi}} \mathbf{mod}$ homomorphism such that $\epsilon'(C_0^{((H, \phi))}) \subseteq V^{(H, \phi)}$ for each $(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)$.

Then there exists a chain map of ${}_{k[G], \underline{\phi}} \mathbf{mon}$ -morphisms $\{f_i : C_i \longrightarrow M_i, i \geq 0\}$ such that

$$\epsilon \cdot f_0 = \epsilon', \quad f_{i-1} \cdot \partial'_i = \partial_i \cdot f_i \text{ for all } i \geq 1.$$

In addition, if $\{f'_i : C_i \longrightarrow M_i, i \geq 0\}$ is another chain map of ${}_{k[G], \underline{\phi}}\mathbf{mon}$ -morphisms such that $\epsilon \cdot f_0 = \epsilon \cdot f'_0$ then there exists a ${}_{k[G], \underline{\phi}}\mathbf{mon}$ -chain homotopy $\{s_i : C_i \longrightarrow M_{i+1}, \text{ for all } i \geq 0\}$ such that $\partial_i \cdot s_i + s_{i-1} \cdot \partial'_i = f_i - f'_i$ for all $i \geq 1$ and $f_0 - f'_0 = \partial_0 \cdot s_0$.

Proof

This is the usual homological algebra argument using Lemma 8.6. \square

Remark 8.10. (i) Needless to say, Proposition 8.9 has an analogue to the effect that every ${}_{k[G], \underline{\phi}}\mathbf{mod}$ -homomorphism $V \longrightarrow V'$ extends to a ${}_{k[G], \underline{\phi}}\mathbf{mon}$ -morphism between the monomial resolutions of V and V' , if they exist, and the extension is unique up to ${}_{k[G], \underline{\phi}}\mathbf{mon}$ -chain homotopy.

(ii) The category ${}_{k[G], \underline{\phi}}\mathbf{mon}$ is additive but not abelian. Homological algebra (e.g. a projective resolution) is more conveniently accomplished in an abelian category. To overcome this difficulty we shall embed ${}_{k[G], \underline{\phi}}\mathbf{mon}$ into more convenient abelian categories. This is reminiscent of the Freyd-Mitchell Theorem which embeds every abelian category into a category of modules.

8.11. *The functor category $\mathit{funct}_k^o({}_{k[G], \underline{\phi}}\mathbf{mon}, {}_k\mathbf{mod})$*

Let $\mathit{funct}_k^o({}_{k[G], \underline{\phi}}\mathbf{mon}, {}_k\mathbf{mod})$ denote the category of contravariant functors, \mathcal{F}, \mathcal{G} etc, from ${}_{k[G], \underline{\phi}}\mathbf{mon}$ to the category of finitely generated k -modules whose morphisms are k -linear natural transformations $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ etc.

Let ${}_{k[G], \underline{\phi}}\mathbf{mod}$ denote the category of finite rank $k[G]$ -modules with central character $\underline{\phi}$ (see §8.1). Consider the functor

$$\mathcal{I} : {}_{k[G], \underline{\phi}}\mathbf{mod} \longrightarrow \mathit{funct}_k^o({}_{k[G], \underline{\phi}}\mathbf{mon}, {}_k\mathbf{mod})$$

given on objects by

$$\mathcal{I}(V) = \mathit{Hom}_{{}_{k[G], \underline{\phi}}\mathbf{mod}}(\mathcal{V}(-), V)$$

with $\mathcal{I}(\beta : V \longrightarrow W) = (f \mapsto \beta \cdot f)$.

The relation of \mathcal{I} to morphisms, particularly to the morphisms given by the $[(K, \psi), g, (H, \phi)]$'s of the hyperHecke algebra, is analysed in detail in [15] Chapter One §3).

Proposition 8.12. *The functor \mathcal{I} on morphisms*

Let $V, W \in {}_{k[G], \underline{\phi}}\mathbf{mod}$ and set $\mathcal{F}(-) = \mathcal{I}(V)(-)$, $\mathcal{G}(-) = \mathcal{I}(W)(-)$. Given a natural transformation $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ for each $M \in {}_{k[G], \underline{\phi}}\mathbf{mon}$ we have

$$\alpha(M) : \mathcal{F}(M) \longrightarrow \mathcal{G}(M)$$

such that if $\beta : M \longrightarrow M'$ is a morphism in ${}_{k[G], \underline{\phi}}\mathbf{mon}$ we have a commutative diagram

If we have a homomorphism $\gamma : V \longrightarrow W$ in ${}_{k[G], \underline{\phi}}\mathbf{mod}$ we obtain a natural transformation

$$\gamma_* : \mathcal{F} \longrightarrow \mathcal{G}$$

given by $\gamma_*(M)(f) = \gamma \cdot f \in \mathcal{G}(M)$ for all $f : M \longrightarrow V$ in $\mathcal{F}(M)$.

Any natural transformation

$$\alpha : \mathcal{F} \longrightarrow \mathcal{G}$$

there is a unique homomorphism $\gamma : V \longrightarrow W$ in ${}_{k[G], \underline{\phi}} \mathbf{mod}$ such that $\alpha = \gamma_*$, which completely determines α .

Proposition 8.13. Let \mathcal{I} denote the functor of §8.11 and define a functor

$$\mathcal{J} : {}_{k[G], \underline{\phi}} \mathbf{mon} \longrightarrow \mathit{funct}_k^o({}_{k[G], \underline{\phi}} \mathbf{mon}, {}_k \mathbf{mod})$$

by $\mathcal{J}(M) = \mathit{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mon}}(-, M)$.

Then the category $\mathit{funct}_k^o({}_{k[G], \underline{\phi}} \mathbf{mon}, {}_k \mathbf{mod})$ is abelian. Furthermore both \mathcal{I} and \mathcal{J} are full embeddings (i.e. bijective on morphisms and hence injective on isomorphism classes of objects).

Proposition 8.14. For $M \in {}_{k[G], \underline{\phi}} \mathbf{mon}$ the functor $\mathcal{J}(M)$ in $\mathit{funct}_k^o({}_{k[G], \underline{\phi}} \mathbf{mon}, {}_k \mathbf{mod})$ is projective.

Definition 8.15. Let $M \in {}_{k[G], \underline{\phi}} \mathbf{mon}, V \in {}_{k[G], \underline{\phi}} \mathbf{mod}$. Define a k -linear isomorphism $\mathcal{K}_{M,V}$ of the form

$$\mathit{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mod}}(\mathcal{V}(M), V) \xrightarrow{\mathcal{K}_{M,V}} \mathit{Hom}_{\mathit{funct}_k^o({}_{k[G], \underline{\phi}} \mathbf{mon}, {}_k \mathbf{mod})}(\mathcal{J}(M), \mathcal{I}(V))$$

by sending $f : \mathcal{V}(M) \longrightarrow V$ to the natural transformation

$$\mathcal{K}_{M,V}(N) : \mathcal{J}(M)(N) \longrightarrow \mathcal{I}(V)(N)$$

given by $h \mapsto f \cdot \mathcal{V}(h)$ for all $N \in {}_{k[G], \underline{\phi}} \mathbf{mon}$

$$\mathit{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mon}}(N, M) \longrightarrow \mathit{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mod}}(\mathcal{V}(N), V).$$

The inverse isomorphism is given by $\mathcal{K}_{M,V}^{-1}(\phi) = \phi(M)(1_M)$ where 1_M denotes the identity morphism on M .

In fact \mathcal{K} is a functorial equivalence of the form

$$\mathcal{K} : \mathit{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mod}}(\mathcal{V}(-), -) \xrightarrow{\cong} \mathit{Hom}_{\mathit{funct}_k^o({}_{k[G], \underline{\phi}} \mathbf{mon}, {}_k \mathbf{mod})}(\mathcal{J}(-), \mathcal{I}(-))$$

Theorem 8.16.

Let

$$\dots \xrightarrow{\partial_i} M_i \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0 \xrightarrow{\epsilon} V \longrightarrow 0$$

be a chain complex with $M_i \in {}_{k[G], \underline{\phi}} \mathbf{mon}$ for $i \geq 0$, $V \in {}_{k[G], \underline{\phi}} \mathbf{mod}$, $\partial_i \in \mathit{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mon}}(M_{i+1}, M_i)$ and $\epsilon \in \mathit{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mod}}(\mathcal{V}(M_0), V)$. Then the following are equivalent:

- (i) $M_* \longrightarrow V$ is a ${}_{k[G], \underline{\phi}} \mathbf{mon}$ -resolution of V .
- (ii) The sequence

$$\dots \xrightarrow{\mathcal{J}(\partial_i)} \mathcal{J}(M_i) \xrightarrow{\mathcal{J}(\partial_{i-1})} \dots \xrightarrow{\mathcal{J}(\partial_1)} \mathcal{J}(M_1) \xrightarrow{\mathcal{J}(\partial_0)} \mathcal{J}(M_0) \xrightarrow{\mathcal{K}_{M_0,V}(\epsilon)} \mathcal{I}(V) \longrightarrow 0$$

is exact in $\mathit{funct}_k^o({}_{k[G], \underline{\phi}} \mathbf{mon}, {}_k \mathbf{mod})$.

Remark 8.17. Theorem 8.16 together with Proposition 8.13 and Proposition 8.14 imply that the map

$$(M_* \xrightarrow{\epsilon} V) \mapsto (\mathcal{J}(M_*) \xrightarrow{\mathcal{K}_{M_0, V}(\epsilon)} \mathcal{I}(V))$$

is a bijection between the $k[G], \underline{\phi}$ -**mon**-resolutions of V and the projective resolutions of $\mathcal{I}(V)$ consisting of objects from the subcategory $\mathcal{J}(k[G], \underline{\phi}$ -**mon**).

We are now going to pass from functors to a category of modules.

8.18. *The functor Φ_M*

Let $M \in k[G], \underline{\phi}$ -**mon** and let $\mathcal{A}_M = \text{Hom}_{k[G], \underline{\phi}\text{-mon}}(M, M)$, the ring of endomorphisms on M under composition. In the present context \mathcal{A}_M is a finitely generated k -algebra.

I shall show that there is an equivalence of categories between $\text{funct}_k^o(k[G], \underline{\phi}\text{-mon}, k\text{-mod})$ and the category of right modules $\text{mod}_{\mathcal{A}_M}$ for a suitable choice of M .

We have a functor

$$\Phi_M : \text{funct}_k^o(k[G], \underline{\phi}\text{-mon}, k\text{-mod}) \longrightarrow \text{mod}_{\mathcal{A}_M}$$

given by $\Phi(\mathcal{F}) = \mathcal{F}(M)$. Right multiplication by $z \in \mathcal{A}_M$ on $v \in \mathcal{F}(M)$ is given by

$$v \# z = \mathcal{F}(z)(v)$$

where $\mathcal{F}(z) : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$ is the left k -module morphism obtained by applying \mathcal{F} to the endomorphism z . This is a right- \mathcal{A}_M action since

$$v \# (zz_1) = \mathcal{F}(zz_1)(v) = (\mathcal{F}(z_1) \cdot \mathcal{F}(z))(v) = \mathcal{F}(z_1)(\mathcal{F}(z)(v)) = (v \# z) \# z_1.$$

In the other direction define a functor

$$\Psi_M : \text{mod}_{\mathcal{A}_M} \longrightarrow \text{funct}_k^o(k[G], \underline{\phi}\text{-mon}, k\text{-mod}),$$

for $P \in \text{mod}_{\mathcal{A}_M}$, by

$$\Psi_M(P) = \text{Hom}_{\mathcal{A}_M}(\text{Hom}_{k[G], \underline{\phi}\text{-mon}}(M, -), P).$$

Here, for $N \in k[G], \underline{\phi}$ -**mon**, $\text{Hom}_{k[G], \underline{\phi}\text{-mon}}(M, -)$ is a right \mathcal{A}_M -module via pre-composition by endomorphisms of M . For a homomorphism of \mathcal{A}_M -modules $f : P \longrightarrow Q$ the map $\Psi_M(f)$ is given by composition with f .

Next we consider the composite functor

$$\Phi_M \cdot \Psi_M : \text{mod}_{\mathcal{A}_M} \longrightarrow \text{mod}_{\mathcal{A}_M}.$$

This is given by $P \mapsto \text{Hom}_{\mathcal{A}_M}(\text{Hom}_{k[G], \underline{\phi}\text{-mon}}(M, M), P) = \text{Hom}_{\mathcal{A}_M}(\mathcal{A}_M, P)$ so that there is an obvious natural transformation $\eta : 1 \xrightarrow{\cong} \Phi_M \cdot \Psi_M$ such that $\eta(P)$ is an isomorphism for each module P .

Now consider the composite functor

$$\Psi_M \cdot \Phi_M : \text{funct}_k^o(k[G], \underline{\phi}\text{-mon}, k\text{-mod}) \longrightarrow \text{funct}_k^o(k[G], \underline{\phi}\text{-mon}, k\text{-mod}).$$

For a functor \mathcal{F} we shall define a natural transformation

$$\epsilon_{\mathcal{F}} : \mathcal{F} \longrightarrow \text{Hom}_{\mathcal{A}_M}(\text{Hom}_{k[G], \underline{\phi}}^{\mathbf{mon}}(M, -), \mathcal{F}(M)) = \Psi_M \cdot \Phi_M(\mathcal{F}).$$

For $N \in k[G], \underline{\phi} \mathbf{mon}$ we define

$$\epsilon_{\mathcal{F}}(N) : \mathcal{F}(N) \longrightarrow \text{Hom}_{\mathcal{A}_M}(\text{Hom}_{k[G], \underline{\phi}}^{\mathbf{mon}}(M, N), \mathcal{F}(M))$$

by the formula $v \mapsto (f \mapsto \mathcal{F}(f)(v))$.

Theorem 8.19.

Let $S \in k[G], \underline{\phi} \mathbf{mon}$ be the finite $(G, \underline{\phi})$ -lineable k -module given by

$$S = \bigoplus_{(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)} \underline{\text{Ind}}_H^G(k_{\phi}).$$

Then, in the notation of §8.18,

$$\Phi_S : \text{funct}_k^o(k[G], \underline{\phi} \mathbf{mon}, k \mathbf{mod}) \longrightarrow \mathbf{mod}_{\mathcal{A}_S}$$

and

$$\Psi_S : \mathbf{mod}_{\mathcal{A}_S} \longrightarrow \text{funct}_k^o(k[G], \underline{\phi} \mathbf{mon}, k \mathbf{mod})$$

are inverse equivalences of categories. In fact, the natural transformations η and ϵ are isomorphisms of functors when $M = S$.

Remark 8.20. Theorem 8.19 is true when S is replaced by any M which is the direct sum of $\underline{\text{Ind}}_H^G(k_{\phi})$'s containing at least one pair (H, ϕ) from each G -orbit of $\mathcal{M}_{\underline{\phi}}(G)$. That is, for any $(G, \underline{\phi})$ -lineable k -module containing

$$\bigoplus_{(H, \phi) \in G \backslash \mathcal{M}_{\underline{\phi}}(G)} \underline{\text{Ind}}_H^G(k_{\phi})$$

as a summand. This remark is established by Morita theory.

8.21. The bar resolution

We begin this section by recalling the two-sided bar-resolution for A -modules. Let k be a commutative Noetherian ring and let A be a (not necessarily commutative) k -algebra. For each integer $p \geq 0$ set

$$B_p(M, A, N) = M \otimes_k A \otimes_k A \otimes_k \dots \otimes_k A \otimes_k N$$

in which there are p copies of A and $M \in \mathbf{mod}_A, N \in_A \mathbf{mod}$. Define

$$d : B_1(M, A, N) \longrightarrow B_0(M, A, N)$$

by $d(m \otimes a \otimes n) = m \cdot a \otimes n - m \otimes a \cdot n$. For $p \geq 2$ define

$$d : B_p(M, A, N) \longrightarrow B_{p-1}(M, A, N)$$

by

$$\begin{aligned} & d(m \otimes a_1 \otimes \dots \otimes a_p \otimes n) \\ &= m \cdot a_1 \otimes \dots \otimes a_p \otimes n \\ &+ \sum_{i=1}^{p-1} (-1)^i m \otimes a_1 \otimes \dots \otimes a_i \cdot a_{i+1} \otimes \dots \otimes a_p \otimes n \\ &+ (-1)^p m \otimes a_1 \otimes \dots \otimes a_p \cdot n. \end{aligned}$$

Setting $N = A$ we define

$$\epsilon : B_0(M, A, A) = M \otimes_k A \longrightarrow M$$

by $\epsilon(m \otimes a) = m \cdot a$.

The following result is well-known.

Proposition 8.22.

In the situation of §8.21 the chain complex

$$\dots \longrightarrow B_p(M, A, A) \xrightarrow{d} B_{p-1}(M, A, A) \xrightarrow{d} \dots$$

$$\dots \longrightarrow B_1(M, A, A) \xrightarrow{d} B_0(M, A, A) \xrightarrow{\epsilon} M \longrightarrow 0$$

is a free right- A -module resolution of M .

8.23. As in §8.15 and §8.18, let $M \in {}_{k[G], \underline{\phi}} \mathbf{mon}$, $V \in {}_{k[G], \underline{\phi}} \mathbf{mod}$ and let $\mathcal{A}_M = \text{Hom}_{k[G], \underline{\phi}}(M, M)$, the ring of endomorphisms on M under composition. For $i \geq 0$ define $\tilde{M}_{M,i} \in {}_k \mathbf{mod}$ by (i copies of \mathcal{A}_M)

$$\tilde{M}_{M,i} = \text{Hom}_{k[G], \underline{\phi}}(\mathcal{V}(M), V) \otimes_k \mathcal{A}_M \otimes_k \dots \otimes_k \mathcal{A}_M$$

and set

$$\underline{M}_{M,i} = \tilde{M}_{M,i} \otimes_k \text{Hom}_{k[G], \underline{\phi}}(-, M).$$

Hence $\underline{M}_{M,i} \in \text{funct}_k^o({}_{k[G], \underline{\phi}} \mathbf{mon}, {}_k \mathbf{mod})$ and in fact the values of this functor are not merely objects in ${}_k \mathbf{mod}$ because they have a natural right \mathcal{A}_M -module structure, defined as in §8.18.

If $i \geq 1$ we defined natural transformations $d_{M,0}, d_{M,1}, \dots, d_{M,i}$ in the following way. Define

$$d_{M,0} : \underline{M}_{M,i} \longrightarrow \underline{M}_{M,i-1}$$

by

$$d_{M,0}(f \otimes \alpha_1 \otimes \dots \otimes \alpha_i \otimes u) = f(- \cdot \alpha_1) \otimes \alpha_2 \dots \otimes \alpha_i \otimes u.$$

The map $f(- \cdot \alpha_1) : \mathcal{V}(M) \longrightarrow V$ is a ${}_{k[G], \underline{\phi}} \mathbf{mod}$ -homomorphism since α_i acts on the right of M .

For $1 \leq j \leq i-1$ we define

$$d_{M,j} : \underline{M}_{M,i} \longrightarrow \underline{M}_{M,i-1}$$

by

$$d_{M,j}(f \otimes \alpha_1 \otimes \dots \otimes \alpha_i \otimes u) = f \otimes \alpha_1 \dots \otimes \alpha_j \alpha_{j+1} \otimes \dots \otimes \alpha_i \otimes u.$$

Finally

$$d_{M,i} : \underline{M}_{M,i} \longrightarrow \underline{M}_{M,i-1}$$

is given by

$$d_i(M)(f \otimes \alpha_1 \otimes \dots \otimes \alpha_i \otimes u) = f \otimes \alpha_1 \otimes \dots \otimes \alpha_{i-1} \otimes \alpha_i \cdot u.$$

Since u is a ${}_{k[G], \underline{\phi}} \mathbf{mon}$ -morphism so is $\alpha_i \cdot u$ because

$$(\alpha_i \cdot u)(\alpha m) = \alpha_i(u(\alpha m)) = \alpha_i(\alpha u(m)) = \alpha \alpha_i(u(m)) = \alpha(\alpha_i \cdot u)(m)$$

since α_i is a ${}_{k[G], \underline{\phi}} \mathbf{mon}$ endomorphism of M .

Next we define a natural transformation

$$\epsilon_M : \underline{M}_{M,0} \longrightarrow \mathcal{I}(V) = \mathrm{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mod}}(\mathcal{V}(-), V)$$

by sending $f \otimes u \in \underline{M}_{M,0}$ to $f \cdot \mathcal{V}(u) \in \mathcal{I}(V)$.

Finally we define

$$d_M = \sum_{j=0}^i (-1)^j d_{M,j} : \underline{M}_{M,i} \longrightarrow \underline{M}_{M,i-1}.$$

Theorem 8.24.

The sequence

$$\dots \xrightarrow{d_M} \underline{M}_{M,i}(M) \xrightarrow{d_M} \underline{M}_{M,i-1}(M) \dots \xrightarrow{d_M} \underline{M}_{M,0}(M) \xrightarrow{\epsilon_M} \mathcal{I}(V)(M) \longrightarrow 0$$

is the right \mathcal{A}_M -module bar resolution of $\mathcal{I}(V)(M)$.

8.25. The functorial monomial resolution of V

Let V be a finite rank k -lattice with a left G -action. Let $M \in {}_{k[G], \underline{\phi}} \mathbf{mon}$ and $W \in {}_k \mathbf{lat}$. Define another object $W \otimes_k M \in {}_{k[G], \underline{\phi}} \mathbf{mon}$ by letting G act only on the M -factor, $g(w \otimes m) = w \otimes gm$, and defining the Lines of $W \otimes_k M$ to consist of the one-dimensional subspaces $\langle w \otimes L \rangle$ where $w \in W$, runs through a k -basis of W , and L is a Line of M . Therefore, if $M' \in {}_{k[G], \underline{\phi}} \mathbf{mon}$ we have an isomorphism

$$\mathrm{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mon}}(W \otimes_k M, M') \xrightarrow{\cong} W \otimes_k \mathrm{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mon}}(M, M')$$

providing that W is finite-dimensional. Similarly we have an isomorphism

$$\mathrm{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mon}}(M, W \otimes_k M') \xrightarrow{\cong} W \otimes_k \mathrm{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mon}}(M, M')$$

when W is finite dimensional.

As in Theorem 8.19, let $S \in {}_{k[G], \underline{\phi}} \mathbf{mon}$ be the finite $(G, \underline{\phi})$ -Line Bundle over k given by

$$S = \bigoplus_{(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)} \underline{\mathrm{Ind}}_H^G(k_\phi).$$

As in §8.23, for $i \geq 0$ we have $\tilde{M}_{S,i} \in {}_k \mathbf{mod}$ by (i copies of \mathcal{A}_S)

$$\tilde{M}_{S,i} = \mathrm{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mod}}(\mathcal{V}(S), V) \otimes_k \mathcal{A}_S \otimes_k \dots \otimes_k \mathcal{A}_S,$$

which is a finite dimensional k -lattice. As a k -basis for $\tilde{M}_{S,i}$ we take the tensor product of the direct sum of bases for each $V^{(H, \phi)}$ and a basis for each \mathcal{A}_S -factor given by the basis of the hyperHecke algebra. Therefore we may form $\tilde{M}_{S,i} \otimes_k S \in {}_{k[G], \underline{\phi}} \mathbf{mon}$.

Recall from §8.11 that

$$\mathrm{Hom}_{{}_{k[G], \underline{\phi}} \mathbf{mod}}(\mathcal{V}(S), V) = \mathcal{I}(V)(S) \cong \bigoplus_{(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)} V^{(H, \phi)}.$$

We have morphisms in ${}_{k[G], \underline{\phi}} \mathbf{mon}$ for $i \geq 1$

$$d_0, d_1, \dots, d_i : \tilde{M}_{S,i} \otimes_k S \longrightarrow \tilde{M}_{S,i-1} \otimes_k S$$

defined on

$$f \otimes \alpha_1 \otimes \dots \otimes \alpha_i \otimes s \in \text{Hom}_{k[G, \underline{\phi}]\text{-mod}}(\mathcal{V}(S), V) \otimes_k \mathcal{A}_S^{\otimes i} \otimes_k S$$

by

$$d_0(f \otimes \alpha_1 \otimes \dots \otimes \alpha_i \otimes s) = f \cdot \mathcal{V}(\alpha_1) \otimes \alpha_2 \otimes \dots \otimes \alpha_i \otimes s,$$

and for $1 \leq j \leq i - 1$

$$d_j(f \otimes \alpha_1 \otimes \dots \otimes \alpha_i \otimes s) = f \otimes \alpha_1 \otimes \dots \otimes \alpha_j \alpha_{j+1} \dots \otimes \alpha_i \otimes s$$

$$d_i(f \otimes \alpha_1 \otimes \dots \otimes \alpha_i \otimes s) = f \otimes \alpha_1 \otimes \dots \otimes \alpha_{i-1} \otimes \alpha_i(s).$$

Setting $d = \sum_{j=0}^i (-1)^j d_j$ gives a morphism in ${}_{k[G, \underline{\phi}]\mathbf{mon}}$

$$d : \tilde{M}_{S,i} \otimes_k S \longrightarrow \tilde{M}_{S,i-1} \otimes_k S$$

for all $i \geq 1$. In addition we define a homomorphism in ${}_{k[G, \underline{\phi}]\mathbf{mod}}$

$$\epsilon : \tilde{M}_{S,0} \otimes_k S = \text{Hom}_{k[G, \underline{\phi}]\mathbf{mod}}(\mathcal{V}(S), V) \otimes_k S \longrightarrow V$$

by $\epsilon(f \otimes s) = f(s)$.

The chain complex

$$\dots \xrightarrow{d} \tilde{M}_{S,i} \otimes_k S \xrightarrow{d} \dots \xrightarrow{d} \tilde{M}_{S,1} \otimes_k S \xrightarrow{d} \tilde{M}_{S,0} \otimes_k S \xrightarrow{\epsilon} V \longrightarrow 0$$

satisfies the conditions of Theorem 8.16. Therefore, abbreviating $\tilde{M}_{S,i} \otimes_k S$ to M_i pro tem, this chain complex is a ${}_{k[G, \underline{\phi}]\mathbf{mon}}$ -resolution of V if and only if the sequence

$$\dots \xrightarrow{\mathcal{J}(d)} \mathcal{J}(M_i) \xrightarrow{\mathcal{J}(d)} \dots \xrightarrow{\mathcal{J}(d)} \mathcal{J}(M_1) \xrightarrow{\mathcal{J}(d)} \mathcal{J}(M_0) \xrightarrow{\mathcal{K}_{M_0, V}(\epsilon)} \mathcal{I}(V) \longrightarrow 0$$

is exact in $\text{funct}_k^o({}_{k[G, \underline{\phi}]\mathbf{mon}, k}\mathbf{mod})$. By Theorem 8.19 this chain complex of functors is exact if and only if the result of applying Φ_S to it is exact in the category $\mathbf{mod}_{\mathcal{A}_S}$. However, by Theorem 8.24 with $M = S$ the resulting chain complex in $\mathbf{mod}_{\mathcal{A}_S}$ is the bar resolution, which is exact.

We have proved the following result.

Theorem 8.26. *Existence of the bar-monomial resolution*

Let k be a field. Then, in the notation of §8.25, The chain complex

$$\dots \xrightarrow{d} \tilde{M}_{S,i} \otimes_k S \xrightarrow{d} \dots \xrightarrow{d} \tilde{M}_{S,1} \otimes_k S \xrightarrow{d} \tilde{M}_{S,0} \otimes_k S \xrightarrow{\epsilon} V \longrightarrow 0$$

is a ${}_{k[G, \underline{\phi}]\mathbf{mon}}$ -resolution of V .

The bar-monomial resolution of Theorem 8.26 possesses a number of the usual naturality properties.

9. THE BAR-MONOMIAL RESOLUTION: II. THE COMPACT, OPEN MODULO THE CENTRE CASE

Let G be a locally profinite group and let k be an algebraically closed field. Let V be a k -representation of G with central character $\underline{\phi}$ and that V is a $\mathcal{M}_{cmc,\underline{\phi}}(G)$ -admissible representation as in Proposition 4.1.

Let $\mathcal{H}_{cmc}(G)$ be the hyperHecke algebra, introduced in §2. Let

$$M_c(\underline{n}, G) = \bigoplus_{\alpha \in \mathcal{A}, (H, \phi) \in \mathcal{M}_c(G)} \underline{n}_\alpha X_c(H, \phi)$$

be the left $k[G] \times \mathcal{H}_{cmc}(G)$ -module of Theorem 5.15 form some family of strictly positive integers, $\{\underline{n}_\alpha\}$.

Theorem 9.1. Replacing S of Theorem 8.19 by $M_c(\underline{n}, G)$ and replacing the ring \mathcal{A}_M (when $M = S$) by $\mathcal{H}_{cmc}(G)$ we may imitate the construction of §8.25 to construct a $k[G], \underline{\phi}$ -**mon**-resolution of V

$$\begin{aligned} \dots &\xrightarrow{d} \tilde{M}_{M_c(\underline{n}, G), i} \otimes_k M_c(\underline{n}, G) \xrightarrow{d} \dots \xrightarrow{d} \tilde{M}_{M_c(\underline{n}, G), 1} \otimes_k M_c(\underline{n}, G) \\ &\xrightarrow{d} \tilde{M}_{M_c(\underline{n}, G), 0} \otimes_k M_c(\underline{n}, G) \xrightarrow{\epsilon} V \longrightarrow 0 \end{aligned}$$

This result is proved using the analogues of Theorem 8.19 and Theorem 8.24.

Remark 9.2. In [15] this result was proved⁸ by reduction to the finite modulo the centre case. Also an explicit bare hands homological construction was given in the case of GL_2 of a local field. I think that the use of the hyperHecke algebra simplifies the construction both in the compact, open modulo the centre case of this section and the general case of the next.

10. THE MONOMIAL RESOLUTION IN THE GENERAL CASE

Once again let G be a locally profinite group and let k be an algebraically closed field. Let V be a k -representation of G with central character $\underline{\phi}$ and that V is a $\mathcal{M}_{cmc,\underline{\phi}}(G)$ -admissible representation as in Proposition 4.1.

First I shall recall the properties of Tammo tom Dieck's space $\underline{E}(G, \mathcal{C})$ ([15] Appendix IV) which is defined for a group G and a family of subgroups \mathcal{C} which is closed under conjugation and passage to subgroups. This space is a simplicial complex on which G acts simplicially in such a way that for any subgroup $H \in \mathcal{C}$ the fixed-point set $\underline{E}(G, \mathcal{C})^H$ is non-empty and contractible. In our case \mathcal{C} will be the family of compact, open modulo the centre subgroups.

$\underline{E}(G, \mathcal{C})$ is unique up to G -equivariant homotopy equivalence. In the case of GL_n of a local field, for example, the Bruhat-Tits building gives a finite-dimensional model for the tom Dieck space.

If the set of conjugacy classes maximal compact, open modulo the centre subgroups of G is finite, as in the case of $GL_n K$ for example, one can find

⁸I believe!

a local system which assigns to each compact, open modulo the centre J a $k[J]_{\underline{\phi}}$ **mon**-resolution of $\text{Res}_J^G V$

$$\begin{aligned} \dots &\xrightarrow{d} \tilde{M}_{M_c(\underline{n}, J), i} \otimes_k M_c(\underline{n}, J) \xrightarrow{d} \dots \xrightarrow{d} \tilde{M}_{M_c(\underline{n}, J), 1} \otimes_k M_c(\underline{n}, J) \\ &\xrightarrow{d} \tilde{M}_{M_c(\underline{n}, J), 0} \otimes_k M_c(\underline{n}, J) \xrightarrow{\epsilon} \text{Res}_J^G V \longrightarrow 0. \end{aligned}$$

Next one forms the double complex ([15] Chapter Four Theorem 3.2) given by the simplicial chain complex of the tom Dieck space in one grading and the compact, open modulo the centre $k[J]_{\underline{\phi}}$ **mon**-resolutions in the other grading. The contribution of the resolutions corresponding to the orbit of one J -fixed simplex gives the compactly supported induction of that resolution.

Theorem 10.1. ([15] Chapter Four Theorem 3.2)

Let V be a $\mathcal{M}_{cmc, \underline{\phi}}(G)$ -admissible representation as in Proposition 4.1. Then the total complex of the above double complex is $k[G]_{\underline{\phi}}$ **mon**-resolution of V .

11. THE BERNSTEIN CENTRE

Let \mathcal{A} be an abelian category then its centre $Z(\mathcal{A})$ is the ring of endomorphisms of the identity functor of \mathcal{A} . Explicitly, for each object A of \mathcal{A} there is given an endomorphism $z_A \in \text{Hom}_{\mathcal{A}}(A, A)$ such that for any $f \in \text{Hom}_{\mathcal{A}}(A, B)$ one has $z_B f = f z_A$.

If the category \mathcal{A} is the product of abelian categories $(\mathcal{A}_i)_{i \in \mathcal{I}}$ then one has $Z(\mathcal{A}) = \prod_{i \in \mathcal{I}} Z(\mathcal{A}_i)$.

Suppose the category \mathcal{A} admits direct sums indexed by \mathcal{I} such that any morphism $f : X \longrightarrow \bigoplus_{i \in \mathcal{I}} Y_i$ is zero if and only if all the projections

$$X \xrightarrow{f} \bigoplus_{i \in \mathcal{I}} Y_i \xrightarrow{pr_i} Y_i$$

are zero.

This property holds for the category of algebraic (i.e. smooth) representations of a reductive group over a non-Archimedean local field ([8] p.5).

Under the above condition \mathcal{A} is the product of full subcategories \mathcal{A}_i for $i \in \mathcal{I}$ such that

- (i) if $X \in \mathcal{A}_i$ and $Y \in \mathcal{A}_j$ then $\text{Hom}_{\mathcal{A}}(X, Y) = 0$ if $i \neq j$ and
- (ii) for all objects X we have $X = \bigoplus_{i \in \mathcal{I}} X_i$ with X_i in \mathcal{A}_i .

Definition 11.1. Once more let G be a locally profinite group, let k be an algebraically closed field and let \mathcal{H}_G be the associated Hecke algebra as in §6.4. Let \mathcal{A} be the category of smooth representations. Then the centre of \mathcal{A} is called the Bernstein centre.

For the general linear group over a local field the Bernstein centre was determined in [2] and [3]. The determination in the general case is described in [8]. Deligne's essay uses the description of smooth representations in term of modules over \mathcal{H}_G (see Appendix: Smooth representations and Hecke modules) and is stated in ([8] §2).

Remark 11.2. In this essay I cannot recall all the details of [8]. However in this section I intend to explain an approach to the Bernstein centre of the category of smooth representation which is based on the derived category of monomial representations - that is, the derived category of modules over the hyperHecke algebra. The approach uses the monocentric conditions which were introduced in §3.

11.3. Resolutions and the centre of \mathcal{A}

Suppose that \mathcal{A} is an abelian category and that \mathcal{B} is an additive category together with a forgetful functor $\nu : \mathcal{B} \rightarrow \mathcal{A}$ and suppose that for each object $V \in \text{Ob}(\mathcal{A})$ we have a \mathcal{B} -resolution of V . This means a chain complex in \mathcal{B}

$$\xrightarrow{d} M_i \xrightarrow{d} M_{i-1} \xrightarrow{d} \dots \xrightarrow{d} M_0 \rightarrow 0$$

such that

$$\rightarrow \nu(M_i) \rightarrow \nu(M_{i-1}) \rightarrow \dots \rightarrow \nu(M_0) \rightarrow V \rightarrow 0$$

is exact in \mathcal{A} . In addition suppose that the association $V \mapsto M_*$ is functorial into the derived category of \mathcal{B} . Thus any two choices of \mathcal{B} -resolution for V are chain homotopy equivalent in \mathcal{B} and any morphism $f : V \rightarrow V'$ in \mathcal{A} induces a \mathcal{B} -chain map, f_* unique up to chain homotopy, between the resolutions.

Now consider a family giving an element in the centre of \mathcal{A} which yields $z_V : V \rightarrow V$ and $z_{V'} : V' \rightarrow V'$ satisfying $fz_V = z_{V'}f$ for all f . Fix resolutions for V and V' . Then z_V induces a chain map $(z_V)_*$ on M_* and another $(z_{V'})_*$ on M'_* . The morphism f induces a chain map $f_* : M_* \rightarrow M'_*$ and because $f_*(z_V)_*$ is chain homotopic to $(z_{V'})_*f_*$ the pair of \mathcal{A} -morphisms $\nu(f_i)\nu(z_V)_i$ and $\nu(z_{V'})_i\nu(f_i)$ for $i = 0, 1$ induce $fz_V = z_{V'}f$ and so $\nu(z_V)_i$ and $\nu(z_{V'})_i$ for $i = 0, 1$ induce the elements $z_V, z_{V'}$ of the central family.

Conversely the degree 0 and 1, for any choice of resolution of V determine a central morphism z_V . When \mathcal{A} is the category of smooth representations of G the morphisms $(z_V)_i$ for $i = 0, 1$ are described in terms of elements of the hyperHecke algebra satisfying the monocentric conditions of Remark 3.6.

Question 11.4. (i) Very briefly, the method of [8] uses the connection between Hecke modules and smooth representation to reduce determination of the Bernstein centre to the calculation of the centre of the algebra $\lim_{\leftarrow} \mathcal{H}_G e$ given by inverse limit over idempotents, e , of \mathcal{H}_G .

In view of the occurrence of the clearly related Hecke and hyperHecke algebras into the topic of this section, I am very curious to further understand the dictionary between the two approaches.

(ii) The conditions of Remark 3.6 would be most explicit in the case of $G = GL_2K$ where K is a non-Archimedean local field which is given by “explicit” homological algebra in ([15] Chapter Two).

Is it possible to describe the dictionary of (i) in this case?

12. APPENDIX: COMPARISON OF INDUCTIONS

In the case of finite groups this Appendix compares the “tensor product of modules” model of an induced representation with the “function space” model⁹.

Suppose that $H \subseteq G$ are finite groups and that W is a vector space over an algebraically closed field k together with a left H -action given by a homomorphism

$$\phi : H \longrightarrow \text{Aut}_k(W).$$

In this case the functional model for the induced representation is given by the k -vector space of functions $X_{(H,\phi)}$ consisting of functions of the form $f : G \longrightarrow W$ such that $f(hg) = \phi(h)(f(g))$. The left G -action on these functions is given by $(g \cdot f)(x) = f(xg)$ since $((g_1g_2) \cdot f)(x) = f(xg_1g_2)$ but if $\xi(y) = f(yg_2)$ then $(g_1 \cdot \xi)(z) = \xi(zg_1) = f(zg_1g_2)$ so that

$$g_1 \cdot (g_2 \cdot f) = (g_1g_2) \cdot f$$

as required.

For $w \in W$ we have a function f_w , supported in H and satisfying $(h \cdot f_w) = f_{\phi(h)(w)}$ for $h \in H$ so that $f_w(1) = w$. We have a left $k[H]$ -module map

$$f : W \longrightarrow X_{(H,\phi)}$$

defined by $w \mapsto f_w$. This makes sense because

$$f_w(x) = \begin{cases} \phi(x)(w) & \text{if } x \in H, \\ 0 & \text{if } x \notin H \end{cases}$$

so that

$$f_{\phi(h)(w)}(x) = \begin{cases} \phi(x)(\phi(h)(w)) = \phi(xh)(w) & \text{if } x \in H, \\ 0 & \text{if } x \notin H \end{cases}.$$

The map f induces a left $k[G]$ -module map, which is an isomorphism,

$$\hat{f} : \text{Ind}_H^G(W) = k[G] \otimes_{k[H]} W \xrightarrow{\cong} X_{(H,\phi)}$$

given by $\hat{f}(g \otimes_{k[H]} w) = g \cdot f_w$. This is well-defined, as is well-known, because

$$\hat{f}(g \otimes \phi(h)(w)) = g \cdot f_{\phi(h)(w)} = g \cdot (h \cdot f_w) = \hat{f}(gh \otimes_{k[H]} w).$$

Henceforth, in this Appendix, I shall consider only the case when $\dim_k(W) = 1$. In this case $W = k_\phi$ will denote the H -representation given by the action $h \cdot v = \phi(h)v$ for $h \in H, v \in k$.

⁹In ([15] Chapter Two, Definition 1.1) my unreliable typography resulted in a superfluous suffix “-1” which gives the right action. This this essay I have been more careful to give the correct formula for the left action, since left actions are my usual preference.

As in Definition §2, write $[(K, \psi), g, (H, \phi)]$ for any triple consisting of $g \in G$, characters ϕ, ψ on subgroups $H, K \leq G$, respectively such that

$$(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

which means that $K \leq g^{-1}Hg$ and that $\psi(k) = \phi(h)$ where $k = g^{-1}hg$ for $h \in H, k \in K$.

We have a well-defined left $k[G]$ -module homomorphism

$$[(K, \psi), g, (H, \phi)] : k[G] \otimes_{k[K]} k_\psi \longrightarrow k[G] \otimes_{k[H]} k_\phi$$

given by the formula $[(K, \psi), g, (H, \phi)](g' \otimes_{k[K]} v) = g'g^{-1} \otimes_{k[H]} v$.

This is well-defined because, for $k = g^{-1}hg \in K$ we have

$$\begin{aligned} [(K, \psi), g, (H, \phi)](g'k \otimes_{k[K]} \psi(k)^{-1}(v)) &= g'kg^{-1} \otimes_{k[H]} \psi(k)^{-1}(v) \\ &= g'g^{-1}(gkg^{-1}) \otimes_{k[H]} \psi(k)^{-1}(v) \\ &= g'g^{-1} \otimes_{k[H]} \phi(h)(\psi(k)^{-1}(v)) \\ &= g'g^{-1} \otimes_{k[H]} v \\ &= [(K, \psi), g, (H, \phi)](g' \otimes_{k[K]} v). \end{aligned}$$

Being well-defined, it is clearly a left $k[G]$ -module homomorphism.

When two of these triples correspond to $k[G]$ -modules homomorphisms which are composable we have

$$\begin{aligned} &[(H, \phi), g_1, (J, \mu)][(K, \psi), g, (H, \phi)](g' \otimes_{k[K]} v) \\ &= [(H, \phi), g_1, (J, \mu)](g'g^{-1} \otimes_{k[H]} v) \\ &= g'g^{-1}g_1^{-1} \otimes_{k[J]} v \\ &= g'(g_1g)^{-1} \otimes_{k[J]} v \\ &= [(K, \phi), g_1g, (J, \mu)](g' \otimes_{k[K]} v), \end{aligned}$$

which makes sense because

$$(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi)) \leq (g^{-1}g_1^{-1}Jg_1g, (g_1g)^*(\phi)).$$

In order to define a left $k[G]$ -homomorphism

$$[(K, \psi), g, (H, \phi)] : X_{(K, \psi)} \longrightarrow X_{(H, \phi)}$$

satisfying the relation

$$\hat{f} \cdot [(K, \psi), g, (H, \phi)] = [(K, \psi), g, (H, \phi)] \cdot \hat{f} : k[G] \otimes_{k[K]} k_\psi \longrightarrow X_{(H, \phi)}$$

we set

$$[(K, \psi), g, (H, \phi)](g_1 \cdot f_v) = (g_1g^{-1}) \cdot f_v.$$

This is well-defined because $g_1 \cdot f_v = g_2 \cdot f_v \in X_{(K, \psi)}$ if and only if $(g_1 \cdot f_v)(x) = (g_2 \cdot f_v)(x)$ for all $x \in G$ and so, for $v \neq 0$,

$$f_v(xg_1) = \begin{cases} \psi(xg_1)(v) & \text{if } xg_1 \in K \\ 0 & \text{if } xg_1 \notin K \end{cases}$$

equals

$$f_v(xg_2) = \begin{cases} \psi(xg_2)(v) & \text{if } xg_2 \in K \\ 0 & \text{if } xg_2^{-1} \notin K. \end{cases}$$

Hence $k = (xg_1)^{-1}xg_2 = g_1^{-1}g_2 \in K$ and

$$\psi(xg_1)(v) = \psi(xg_2)(v) = \psi(xg_1)(\psi(g_1^{-1}g_2)(v))$$

so that $\psi(g_1^{-1}g_2)(v) = v$.

Conjugating by g we have $gkg^{-1} = gg_1^{-1}g_2g^{-1} \in H$ which implies that

$$(xg_1g^{-1})^{-1}xg_2g^{-1} = gg_1^{-1}x^{-1}xg_2g^{-1} = gg_1^{-1}g_2g^{-1} \in H$$

and so $xg_1g^{-1} \in H$ if and only if $xg_1g^{-1}gg_1^{-1}g_2^{-1}g^{-1} = xg_2^{-1}g^{-1} \in H$. Also

$$\begin{aligned} \phi(xg_2g^{-1})(v) &= \phi(xg_1g^{-1}gg_1^{-1}g_2g^{-1})(v) \\ &= \phi(xg_1g^{-1})(\phi(gg_1^{-1}g_2g^{-1})(v)) \\ &= \phi(xg_1g^{-1})(\psi(g_1^{-1}g_2)(v)) \\ &= \phi(xg_1g^{-1})(v). \end{aligned}$$

Therefore

$$[(K, \psi), g, (H, \phi)](g_1 \cdot f_v) = [(K, \psi), g, (H, \phi)](g_2 \cdot f_v),$$

as required in order for the left $k[G]$ -module homomorphism to be well-defined.

It is easy to see that transporting the map $[(K, \psi), g, (H, \phi)]$ from the tensor product model of the induced representation to the function space model gives the left $k[G]$ -homomorphism whose well-definedness we have just verified.

Among the left $k[G]$ -maps

$$k[G] \otimes_{k[K]} k_\psi \longrightarrow k[G] \otimes_{k[H]} k_\phi$$

we have the relations, $h \in H, k \in K$

$$[(K, \psi), gk, (H, \phi)] = [(K, \psi), g, (H, \phi)] \cdot (1 \otimes_{k[K]} \psi(k^{-1}))$$

and

$$[(K, \psi), hg, (H, \phi)] = (1 \otimes_{k[H]} \phi(h^{-1})) \cdot [(K, \psi), g, (H, \phi)].$$

This is because

$$\begin{aligned}
g_1(gk)^{-1} \otimes_{k[H]} v &= g_1 g^{-1} g k^{-1} g^{-1} \otimes_{k[H]} v \\
&= g_1 g^{-1} \otimes_{k[H]} \phi(gk^{-1} g^{-1})(v) \\
&= g_1 g^{-1} \otimes_{k[H]} \psi(k^{-1})(v)
\end{aligned}$$

and

$$g_1(hg)^{-1} \otimes_{k[H]} v = g_1 g^{-1} h^{-1} \otimes_{k[H]} v g_1 g^{-1} \otimes_{k[H]} \phi(h^{-1})v.$$

Transporting these relations by \hat{f} yields

$$[(K, \psi), gk, (H, \phi)] = [(K, \psi), g, (H, \phi)] \cdot (g \cdot f_v \mapsto g \cdot f_{\psi(k^{-1})(v)})$$

and

$$[(K, \psi), hg, (H, \phi)] = (g \cdot f_v \mapsto g \cdot f_{\psi(h^{-1})(v)}) \cdot [(K, \psi), g, (H, \phi)]$$

because, for example, $g \otimes_{k[K]} v \longrightarrow g \otimes_{k[K]} \psi(k^{-1})(v)$ is transported to

$$g \cdot f_v \mapsto g \cdot f_{\psi(k^{-1})(v)} = gk^{-1} \cdot f_v.$$

12.1. Now we shall transport the Double Coset Formula over to the functional models.

The Double Coset Formula ([14] Theorem 1.2.40) is an isomorphism describing the restriction of an induced representation. It is a consequence of the J -orbit structure of the left action of a subgroup $J \subseteq G$ on G/H . This is a left $k[J]$ -isomorphism of the form

$$\text{Res}_J^G \text{Ind}_H^G(k_\phi) \xrightarrow{\alpha} \bigoplus_{z \in J \backslash G/H} \text{Ind}_{J \cap zHz^{-1}}^J((z^{-1})^*(k_\phi))$$

given by $\alpha(g \otimes_H v) = j \otimes_{J \cap zHz^{-1}} \phi(h)(v)$ for $g = jzh, j \in J, h \in H$. The inverse of α is given by $\alpha^{-1}(j \otimes_{J \cap zHz^{-1}} v) = jz \otimes_H (v)$. This makes sense for α because if, for example, $jzh = j'zh'$ then $(j')^{-1}j = zh'h^{-1}z^{-1} \in J \cap zHz^{-1}$ so that

$$\begin{aligned}
&j' \otimes_{J \cap zHz^{-1}} \phi(h')(v) \\
&= j'(j')^{-1}j \otimes_{J \cap zHz^{-1}} \phi(h'h^{-1})^{-1}(\phi(h')(v)) \\
&= j \otimes_{J \cap zHz^{-1}} \phi(h)(v)
\end{aligned}$$

and for α^{-1} ,

$$\begin{aligned}
\alpha^{-1}(\alpha(g \otimes_H v)) &= \alpha^{-1}(j \otimes_{J \cap zHz^{-1}} \phi(h)(v)) \\
&= jz \otimes_{k[H]} \phi(h)(v) \\
&= g \otimes_H v.
\end{aligned}$$

In terms of functional models for induced representations we have a left $k[J]$ -module isomorphism, obtained by transporting the above isomorphism via the appropriate map \hat{f} introduced above, of the form

$$a : \text{Res}_J^G X_{(H,\phi)} \xrightarrow{\cong} \bigoplus_{z \in J \backslash G/H} X_{(J \cap zHz^{-1}, (z^{-1})^*(\phi))}$$

whose $X_{(J \cap zHz^{-1}, (z^{-1})^*(\phi))}$ -component equals

$$a(g \cdot f_v) = j \cdot f_{\phi(h)(v)} \in X_{(J \cap zHz^{-1}, (z^{-1})^*(\phi))} \text{ if } g = jzh.$$

The inverse, a^{-1} , maps $j \cdot f_v \in X_{(J \cap zHz^{-1}, (z^{-1})^*(\phi))}$ by the formula

$$a^{-1}(j \cdot f_v) = jz \cdot f_v \in X_{(H,\phi)}.$$

Thus a is clearly a left $k[J]$ -module map.

13. APPENDIX: SMOOTH REPRESENTATIONS AND HECKE MODULES

In this Appendix, for my convenience, representations are complex representations.

Now let Γ be a compact totally disconnected group. Denote by $\hat{\Gamma}$ the set of equivalence classes of finite-dimensional irreducible representations of Γ whose kernel is open - and hence of finite index in Γ .

Suppose now that Γ is finite and (π, V) is a representation of Γ on a possible infinite dimensional vector space V . If $\rho \in \hat{\Gamma}$ let $V(\rho)$ be the sum of all invariant subspaces of V that are isomorphic as Γ -modules to V_ρ . $V(\rho)$ is the ρ -isotypic subspace of V . We have

$$V \cong \bigoplus_{\rho \in \hat{\Gamma}} V_\rho.$$

Now we generalise this to smooth representations of a totally disconnected locally compact group. Choose a compact open subgroup K of G . The compact open normal subgroups of K form a basis of neighbourhoods of the identity in K . Let $\rho \in \hat{K}$ then the kernel of ρ is K_ρ a compact open normal subgroup of finite index.

Proposition 13.1. ([6] *Proposition 4.2.2*)

Let (π, V) be a smooth representation of G . Then

$$V \cong \bigoplus_{\rho \in \hat{K}} V_\rho.$$

The representation π is admissible if and only if each $V(\rho)$ is finite-dimensional.

Let (π, V) be a smooth representation of G . If $\hat{v} : V \rightarrow \mathbb{C}$ is a linear functional we write $\langle v, \hat{v} \rangle = \hat{v}(v)$ for $v \in V$. We say \hat{v} is smooth if there exists an open neighbourhood U of $1 \in G$ such that for all $g \in U$

$$\langle \pi(g)(v), \hat{v} \rangle = \hat{v}(v).$$

Let \hat{V} denote the space of smooth linear functionals on V .

Define the contragredient representation $(\hat{\pi}, \hat{V})$ is defined by

$$\langle v, \hat{\pi}(g)(\hat{v}) \rangle = \langle \pi(g^{-1})(v), \hat{v} \rangle.$$

The contragredient representation of a smooth representation is a smooth representation. Also

$$\hat{V} \cong \bigoplus_{\rho \in \hat{K}} V_\rho^*$$

where V_ρ^* is the dual space of V_ρ .

Since the dual of a finite-dimensional V_ρ is again finite-dimensional the contragredient of an admissible representation is also admissible. Also $\hat{\pi} = \pi$.

If X is a totally disconnected space a complex valued function f on X is smooth if it is locally constant. Let \mathcal{H}_G be, as before, the space of smooth compactly supported complex-valued functions on $X = G$. Assuming G is unimodular \mathcal{H}_G is an algebra without unit under the convolution product

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(gh^{-1})\phi_2(h)dh.$$

This is the Hecke algebra - an idempotent algebra (see §6).

If $\phi \in \mathcal{H}$ define $\pi(\phi) \in \text{End}(V)$ with V as above

$$\pi(\phi)(v) = \int_G \phi(g)\pi(g)(v)dg.$$

Then

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1) \cdot \pi(\phi_2)$$

so that V is an \mathcal{H} -representation.

The integral defining ϕ may be replaced by a finite sum as follows. Choose an open subgroup K_0 fixing v . Choosing K_0 small enough we may assume that the support of ϕ is contained in a finite union of left cosets $\{g_i K_0 \mid 1 \leq i \leq t\}$. Then

$$\pi(\phi)(v) = \frac{1}{\text{vol}(K_0)} \sum_{i=1}^t \phi(g_i)\pi(g_i)(v).$$

Finite group example:

Let (π, V) be a finite-dimensional representation of a finite group G . Write \mathcal{H} for the space of functions from G to \mathbb{C} . If $\phi_1, \phi_2 \in \mathcal{H}$ define $\phi_1 * \phi_2 \in \mathcal{H}$ by

$$(\phi_1 * \phi_2)(g) = \sum_{h \in G} \phi_1(gh^{-1})\phi_2(h).$$

For $\phi \in \mathcal{H}$ define $\pi(\phi) \in \text{End}_{\mathbb{C}}(V)$ by

$$\pi(\phi)(v) = \sum_{g \in G} \phi(g)\pi(g)(v).$$

Hence

$$\begin{aligned}
& \pi(\phi_1(\pi(\phi_2)(v))) \\
&= \pi(\phi_1)(\sum_{g \in G} \phi_2(g)\pi(g)(v)) \\
&= \sum_{g \in G} \phi_2(g)\pi(\phi_1(\pi(g)(v))) \\
&= \sum_{g \in G} \phi_2(g) \sum_{\tilde{g} \in G} \phi_1(\tilde{g})(\pi(\tilde{g}(\pi(g)(v)))) \\
&= \sum_{g, \tilde{g} \in G} \phi_2(g)\phi_1(\tilde{g})(\pi(\tilde{g}g)(v)).
\end{aligned}$$

Now

$$\begin{aligned}
& \pi(\phi_1 * \phi_2)(v) \\
&= \sum_{g_1 \in G} (\phi_1 * \phi_2)(g_1)\pi(g_1)(v) \\
&= \sum_{g_1, h \in G} \phi_1(h_1 h^{-1})\phi_2(h)\pi(g_1)(v).
\end{aligned}$$

Setting $g = h$, $\tilde{g}g = g_1$ shows that

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1 \cdot \pi(\phi_2)).$$

Also $\mathcal{H} \cong \mathbb{C}[G]$ because if $f_g(x) = 0$ if $g \neq x$ and $f_g(g) = 1$ then

$$f_g * f_{g'} = f_{gg'}.$$

Proposition 13.2. ([6] Proposition 4.2.3)

Let (π, V) be a smooth non-zero representation of G . Then equivalent are:

- (i) π is irreducible.
- (ii) V is a simple \mathcal{H} -module.
- (iii) V^{K_0} is either zero or simple as an \mathcal{H}_{K_0} -module for all open subgroups K_0 . Here $\mathcal{H}_{K_0} = e_{K_0} * \mathcal{H} * e_{K_0}$.

Schur's Lemma holds ([6] §4.2.4) for (π, V) an irreducible admissible representation of a totally disconnected group G .

Proposition 13.3. ([6] Proposition 4.2.5)

Let (π, V) be an admissible representation of the totally disconnected locally compact group G with contragredient $(\hat{\pi}, \hat{V})$. Let $K_0 \subseteq G$ be a compact open subgroup. Then the canonical pairing between V and \hat{V} induces a non-degenerate pairing between V^{K_0} and \hat{V}^{K_0} .

The trace

As with representations of finite groups the character of an admissible representation of a totally disconnected locally compact group G is an important invariant. It is a distribution. It is a theorem of Harish-Chandra that if G is a reductive p -adic group then the character is in fact a locally integrable function defined on a dense subset of G .

We shall define the character as a distribution on $\mathcal{H}_G = C_c^\infty(G)$. Suppose that U is a finite-dimensional vector space and let $f : U \rightarrow U$ be a linear map. Suppose $\text{Im}(f) \subseteq U_0 \subseteq U$. Then we have

$$\text{Trace}(f : U_0 \rightarrow U_0) = \text{Trace}(f : U \rightarrow U).$$

Therefore we may define the trace of any endomorphism f of V which has finite rank by choosing any finite-dimensional U_0 such that $\text{Im}(f) \subseteq U_0 \subseteq V$ and by defining

$$\text{Trace}(f) = \text{Trace}(f : U_0 \rightarrow U_0).$$

Now let (π, V) be an admissible representation of G . Let $\phi \in \mathcal{H}_G$. Since ϕ is compactly supported and locally constant there exists a compact open K_0 such that $\phi \in \mathcal{H}_{K_0}$. The endomorphism $\pi(\phi)$ has image in V^{K_0} which is finite-dimensional - by admissibility - so we define the trace distribution

$$\chi_V : \mathcal{H} \rightarrow \mathbb{C}$$

by

$$\chi_V(\phi) = \text{Trace}(\pi(\phi)).$$

Proposition 13.4. ([6] *Proposition 4.2.6*)

Let R be an algebra over a field k . Let E_1 and E_2 be simple R -modules that are finite-dimensional over k . For each $\phi \in R$ if

$$\text{Trace}((\phi \cdot -) : E_1 \rightarrow E_1) = \text{Trace}((\phi \cdot -) : E_2 \rightarrow E_2)$$

then the E_i are isomorphic R -modules.

Proposition 13.5. ([6] *Proposition 4.2.7*)

Let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of G (as above) such that, for each compact open K_1 , $V_1^{K_1} \cong V_2^{K_1}$ as \mathcal{H}_{K_1} -modules then $(\pi_1, V_1) \cong (\pi_2, V_2)$.

Theorem 13.6. ([6] *Theorem 4.2.1*)

Let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of G (as above) such that $\chi_{V_1} = \chi_{V_2}$ then $(\pi_1, V_1) \cong (\pi_2, V_2)$.

From this one sees that the contragredient of an admissible irreducible (π, V) of $GL_n K$ (K a p -adic local field) is given by $\pi_1(g) = \pi((g^{-1})^{tr})$ on the same vector space V .

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