Monomial Resolutions of Admissible Representations

Victor Snaith

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For details see post-retirement math on webpage: http://victor-snaith.staff.shef.ac.uk

§1: A motivating remark

The Langlands programme (LP) concerns topological groups G acting linearly on vector spaces V over k, algebraically closed field (not necessarily characteristic zero).

What sort of G? (i) Locally p-adic Lie groups like $GL_n\mathbb{Q}_p$, (ii) adèlic Lie groups $GL_2\mathbb{A}_{\mathbb{Q}}$ and (iii) fundamental groups of schemes.

What sort of representations V? Respectively: (i) admissible (i.e. generalising f.d. reps of finite groups), (ii) automorphic (i.e. restricted tensor products of admissibles twisted by algebras of differential operators "at ∞", (iii) of cohomological origin.

LP - what's all the fuss?

LP predicts bijections between representation theoretic data in important but previously unrelated areas:

E.g. K a local (or adèles of a number) field

$$\left\{ \begin{array}{c} admissible\ irreducible\ reps\ V\ of\ GL_nK \\ \\ L-functions\ and\ epsilon\ factors\ of\ V \end{array} \right\}$$

$$\pi \ \updownarrow$$

$$\left\{ \begin{array}{c} Weil-Deligne\ reps\ \pi(V)\ of\ K,\ \mathrm{dim}(\pi(V)) = n \\ \\ L-functions\ and\ epsilon\ factors\ of\ \pi(V) \end{array} \right\}$$

ADVERT BREAK:

Remark: When $n = 1$ this bijection is the class field theory of Takagi and Artin.

Remark: LP for number fields implies inter alia Artin's conjecture on the holomorphicity of L-functions (in analytic number theory this is often a substitute for the Riemann Hypothesis).

ADVERT BREAK OVER:

Fix G a locally p-adic Lie group e.g. $GL_n\mathbb{Q}_p$

$Z(G) \leq H \leq G$ a compact, open modulo the centre subgroup

$\phi : H \longrightarrow k^* = k - \{0\}$ continuous character i.e. $\phi \in \hat{H}$, group of cts characters

Fix $\underline{\phi} : Z(G) \longrightarrow k^*$, a central character

We have the category ${}_{k[G,\underline{\phi}]}\mathrm{mod}$ of continuous k-representations on which the centre acts via $\underline{\phi}$.

We also have ${}_{k[G,\underline{\phi}]}\mathrm{mon}$ an additive category

CONSTRUCTED FROM 1-DIMENSIONAL DATA.

Theorem 1

There is a functorial embedding - called the monomial resolution - of ${}_{k[G,\underline{\phi}]}\mathrm{mod}$ into the derived category of ${}_{k[G,\underline{\phi}]}\mathrm{mon}$.

Theorem 2

Theorem 1 has an analogue for

(i) automorphic representations and

(ii) Weil-Deligne representations.

§2: Monomial resolutions

Admissible: restriction to compact mod centre subgroups H is countable direct sum of finite dimensional irreducibles with finite multiplicity (slightly different when char(k) > 0).

e.g. $H = K^* \cdot GL_n\mathcal{O}_K \subset GL_nK$ where $\mathcal{O}_K$ denotes the valuation ring of local field K.

Let $\mathcal{M}_{G,\underline{\phi}}$ denote the set of pairs (J, ϕ) with $J \subseteq G$ a compact open modulo the centre subgroup containing $Z(G)$, $\phi \in \hat{J}$ with $\mathrm{Res}^J_{Z(G)}(\phi) = \underline{\phi}$.

The set $\mathcal{M}_{G,\underline{\phi}}$ is a G-poset via G-conjugation

Associated to (J, ϕ) is the $k[J]$-module k_ϕ given by k on which J acts via $j(z) = \phi(j) \cdot z$

An G-Line Bundle* is $M \in_{k[G,\underline{\phi}]}$ mod together with a decomposition into the direct sum of one-dimensional subspaces

$$M = \oplus_{\alpha \in \mathcal{A}} M_\alpha$$

M_α's are permuted by the G-action and the stabiliser M_α is H_α, compact modulo the centre

H_α acts on M_α via a character ϕ_α

The M_α's are called the Lines of M.

The pair (H_α, ϕ_α) is called the stabilising pair of M_α.

*Auto-referential apology: Apologies to topologists. Robert Boltje coined the term when introducing ${}_{k[G]}$mon for finite G.

For a pair $(H, \phi) \in \mathcal{M}_{G,\underline{\phi}}$ the G-Line Bundle denoted by $\underline{\text{Ind}}_H^G(k_\phi)$ is given by the compactly induced representation $c - \text{Ind}_H^G(k_\phi)$. As in the finite case has a k-basis indexed by G/H.

For each $(J, \phi) \in \mathcal{M}_{G,\underline{\phi}}$ set

$$M^{((J,\phi))} = \oplus_{\alpha \in \mathcal{A},\ (J,\phi) \leq (H_\alpha, \phi_\alpha)} M_\alpha,$$

the (J, ϕ)-fixed points of M.

A morphism

$$f : M = \oplus_{\alpha \in \mathcal{A}} M_\alpha \longrightarrow \oplus_{\beta \in \mathcal{B}} M'_\beta = M'$$

between two G-Line Bundles is a continuous $k[G]$-module homomorphism such that

$$f(M^{((J,\phi))}) \subseteq (M')^{((J,\phi))}$$

for all $(J, \phi) \in \mathcal{M}_{G,\underline{\phi}}$.

This defines an additive (but NOT abelian) category ${}_{k[G,\underline{\phi}]}\text{mon}$

A monomial complex is a chain complex of continuous $k[G]$-modules

$$C_* : \quad \ldots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \ldots$$

in which each C_n is an G-Line Bundle and each d is a morphism.

Let $V \in_{k[G,\underline{\phi}]}$ mod, not necessarily irreducible.

$$V^{(H,\phi)} = \{v \in V \mid h(v) = \phi(h) \cdot v \text{ for all } h \in H\}$$

Remark: If V is an automorphic representation of $GL_2\mathbb{A}_{\mathbb{Q}}$, $H = \Gamma_0(N)$ and ϕ is a Hecke character of level N then spaces of modular forms give examples of $V^{(H,\phi)}$'s.

A chain complex of continuous $k[G]$-modules

$$\ldots \stackrel{d}{\longrightarrow} C_n \stackrel{d}{\longrightarrow} C_{n-1} \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} C_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is called a monomial resolution of V if:

C_n a G-Line Bundle, d a morphism

$$\epsilon(C_0^{((J,\phi))}) \subseteq V^{(J,\phi)} \ (J,\phi) \in \mathcal{M}_{G,\underline{\phi}}$$

and for all $(J,\phi) \in \mathcal{M}_{G,\underline{\phi}}$

$$\ldots \stackrel{d}{\longrightarrow} C_1^{((J,\phi))} \stackrel{d}{\longrightarrow} C_0^{((J,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(J,\phi)} \longrightarrow 0$$

is an exact complex of k-modules.

Monomial resolutions are unique up to chain homotopy (need more for e.g. GL_nK, $n > 2$)

§3: The bar-monomial resolution

Let G be a group which is finite modulo the centre with central character $\underline{\phi}$.

$\mathcal{V}$: forgetful functor (Line Bundles to G-modules)

$\mathcal{A}_S = \mathrm{Hom}_{k[G,\underline{\phi}]-mon}(S, S)$

$$W_{S,i} = \mathrm{Hom}_{k[G,\underline{\phi}]-mod}(\mathcal{V}(S), V) \otimes \mathcal{A}_S^{\otimes^i},$$

$W_{S,i} \otimes S$ has left G-action only on S-factor (Lines = (k-basis) $\otimes$ Line of S)

$(W_{S,*} \otimes S)^{((H,\phi))}$ *well-defined!*

left $k[G,\underline{\phi}]$-monomial morphisms, defined by the obvious formulae,

$$d_0, d_1, \ldots, d_i : W_i \otimes S \longrightarrow W_{i-1} \otimes S$$

for $i \geq 1$

left $k[G, \underline{\phi}]$-module homomorphism

$$\epsilon : \mathrm{Hom}_{k[G,\underline{\phi}]-mod}(\mathcal{V}(S), V) \otimes S \longrightarrow V$$

given by $\epsilon(f \otimes s) = f(s)$. Let d be given by the alternating sum $d = \sum_{j=0}^{i} (-1)^j d_j$.

Choose S to be the direct sum of one copy of each $\underline{\mathrm{Ind}}_H^G(\phi)$ for $(H, \phi) \in G \backslash \mathcal{M}_{G,\underline{\phi}}$.

Theorem 3 *(The bar-monomial resolution)*

$$W_{S,*}(S) \otimes S \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is a canonical left $k[G, \underline{\phi}]$-monomial resolution of V, which is natural with respect to homomorphisms of groups G and V.

Proved using a full embedding into the category of $\mathcal{A}_S$-modules. $\square$

§4: Sketch proof of Theorem 1

Start with V and its restrictions to compact open modulo the centre subgroups.

Tammo tom Dieck gives us: a canonical simplicial complex $\underline{E}(G,\mathcal{C})$ on which G acts simplicially in such a way that, for every compact modulo the centre subgroup $H \subseteq G$, the H-fixed subcomplex $\underline{E}(G,\mathcal{C})^H$ is non-empty and contractible.

E.g. $G = GL_nK$, K local then $\underline{E}(G,\mathcal{C})$ is the Bruhat-Tits building.

Promote the bar-monomial resolution to a local system of monomial complexes on $\underline{E}(G,\mathcal{C})$. Together with the internal differentials from $\underline{E}(G,\mathcal{C})$ construct a double-complex in ${}_{k[G,\underline{\phi}]}\text{mon}$.

The total complex is the monomial resolution.

Further aspects of monomial resolutions:

(i) Hecke operators and monomial resolutions

(ii) L-functions, epsilon factors and Tate's thesis applied Line by Line

(iii) Galois base change and Shintani's correspondence

To keep it short let's look at (i): The monomial resolution of an automorphic representation gives exact sequences

$$\ldots \stackrel{d}{\longrightarrow} C_1^{((J,\phi))} \stackrel{d}{\longrightarrow} C_0^{((J,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(J,\phi)} \longrightarrow 0$$

The $V^{(J,\phi)}$ include spaces of classical modular forms on which the famous Hecke operators act.

$$[JgH] : V^{(H,\phi)} \longrightarrow V^{(J,\phi')}$$

The $[JgH]$'s originate in the Mackey double coset formula (DCF)

$$\text{Res}^G_J \text{Ind}^G_H(k_\phi) \cong \oplus_{z \in J\backslash G/H} \text{Ind}^J_{J \bigcap zHz^{-1}}((z^{-1})^*(k_\phi))$$

from the term corresponding to JgH.

DCF applies to $\text{Res}^G_J \underline{\text{Ind}}^G_H(k_\phi)$.

Therefore often these extend canonically (up to homotopy) to the entire exact sequence!