MONOMIAL RESOLUTIONS FOR ADMISSIBLE REPRESENTATIONS OF GL_2 OF A LOCAL FIELD

VICTOR P. SNAITH

Contents

1. Introduction

This is a temporary introduction, appropriate for this version of the paper, which only gives the construction of monomial resolutions for admissible representations of SL_2K, GL_2K^0, GL_2K^+ and quasi-monomial resolutions for admissible representations of GL_2K over the complex numbers in the case when K is a p-adic local field.

At one point I have used the topological action on classical projective space to obtain the monomial resolutions in the finite-modulo-the-centre case. In view of recent interest by Breuil et al in representations of GL_2K over $\overline{\mathbb{Q}}_p$ I should point out that the use of topology in §5 can be completely avoided, at the cost of rehearsing the algebraic treatment of [7] (see [72]). Therefore I contend that the constructions give below work for admissible (in the classical Langlandsian sense which is used here) over any complete algebraically closed field of characteristic zero. In fact, at the expense of the appearance of unbounded monomial resolutions in §5 the constructions will also work for representations over any algebraically closed field of positive characteristic.

I have stopped this manuscript temporarily at this point because the next phase branches in three directions (i) towards GL_nK with $n > 2$ which will

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require an account of Tits building together with a more rigid (in the sense of categorical homotopy theory) topological construction of the finite-modulo-the-centre monomial resolutions (ii) towards applications which will require several motivating sections about the $GL_n\mathbb{F}_q$ case, Kondo Gauss sums and L-functions via monomial resolutions and (iii) towards global admissible representations, modular forms and number theory (for which there is still much to be worked out, even for GL_2).

We retirees work slowly and travel little. But I hope to make this paper the basis for some lectures at Sheffield University in the Fall of 2011.

2. Relative monomial resolutions of admissible representations

Throughout this section, let G be an arbitrary locally compact Hausdorff group such that the compact open subgroups form a basis for the neighborhoods of the identity. In particular we are thinking of locally p-adic Lie groups such as GL_2K where K is a p-adic local field. The subgroups of such a group which we shall concentrate on are the subgroups $H \subseteq G$ which are compact or compact open modulo the centre $Z(G)$. This means that the image of H in $G/Z(G)$, with the induced topology, is compact or compact open, respectively. Since we shall be emphasising GL_2K whose centre is the group K^* consisting of the scalar matrices a good example of a compact open modulo the centre subgroup is $K^* \cdot GL_2\mathcal{O}_K \subset GL_2K$ where $\mathcal{O}_K$ denotes the valuation ring of K.

If $H \subseteq G$ is a compact modulo the centre subgroup then we write $\hat{H}$ for the multiplicative group of continuous group homomorphisms $\hat{H} = \mathrm{Hom}_{cts}(H, \mathbb{C}^*)$ where $\mathbb{C}^*$ has the discrete topology. The example $\hat{K^*}$ is easy to describe since there is a topological isomorphism

$$K^* \cong \mathcal{O}_K^* \times \mathbb{Z}\langle \pi_K \rangle$$

where π_K is a uniformiser of K. A continuous homomorphism in this case means a homomorphism which is of finite order when restricted to the group of units $\mathcal{O}_K^*$ and on the infinite cyclic group generated by π_K it is given by $\pi_K^n \mapsto x^n$ for some $x \in \mathbb{C}^*$.

Definition 2.1. *Monomial modules and complexes*

Let $\mathcal{M}_G$ denote the set of pairs (J, ϕ) with $J \subseteq G$ a compact modulo the centre subgroup and $\phi \in \hat{J}$ endowed with the usual partial order in which $(J, \phi) \leq (J', \phi')$ if and only if $J \subseteq J'$ and $Res_J^{J'}(\phi') = \phi$. Also G acts on $\mathcal{M}_G$ (on the left) by the formula $g(J, \phi) = (gJg^{-1}, (g^{-1})^*(\phi))$ where $(g^{-1})^*(\phi)(gjg^{-1}) = \phi(j)$. The G-stabiliser of (J, ϕ) under this action is denoted by $N_G(J, \phi)$ and the G-orbit of (J, ϕ) is denoted by $(J, \phi)^G$. Associated to (J, ϕ) is the $\mathbb{C}[J]$-module $\mathbb{C}_\phi$ given by $\mathbb{C}$ on which J acts via $j(z) = \phi(j) \cdot z$ for all $z \in \mathbb{C}$.

An G-Line Bundle[1] is an continuous $\mathbb{C}[G]$-module M together with a decomposition into the direct sum of one-dimensional complex subspaces

$$M = \oplus_{\alpha \in A} \ M_\alpha$$

where the M_α's are permuted by the G-action and the stabiliser of each M_α is a pair $(H_\alpha, \phi_\alpha) \in \mathcal{M}_G$. In particular each H_α is compact modulo the centre. The subspaces M_α's are called the Lines of M. The pair (H_α, ϕ_α) is called the stabilising pair of M_α.

For a pair $(H, \phi) \in \mathcal{M}_G$ the G-Line Bundle denoted by $\underline{\mathrm{Ind}}_H^G(\phi)$ is given the direct sum of lines L_g for $g \in G/H$ where L_g is $\mathbb{C}$ upon which $ghg^{-1} \in gHg^{-1}$ acts by the formula $ghg^{-1} \cdot z = \phi(h)z$ and $g \in G$ sends $L_{g'}$ to $L_{gg'}$ by the identity map on $\mathbb{C}$. As in the case of linear representations this construction with be called the G-Line Bundle induced from (H, ϕ). The stabilising pair of L_g is $g(H, \phi) = (gHg^{-1}, (g^{-1})^*(\phi))$.

For each $(J, \phi) \in \mathcal{M}_G$ set

$$M^{((J,\phi))} = \oplus_{\alpha \in A, \ (J,\phi) \leq (H_\alpha, \phi_\alpha)} \ M_\alpha,$$

which is a subspace of M called the (J, ϕ)-fixed points of M. A morphism

$$f : M = \oplus_{\alpha \in A} \ M_\alpha \longrightarrow \oplus_{\beta \in B} \ M'_\beta = M'$$

between two G-Line Bundles is a continuous $\mathbb{C}[G]$-module homomorphism such that

$$f(M^{((J,\phi))}) \subseteq (M')^{((J,\phi))}$$

for all $(J, \phi) \in \mathcal{M}_G$. This defines a category ${}_{\mathbb{C}[G]}\mathrm{mon}$. This is an additive category in which $Hom_{{}_{\mathbb{C}[G]}\mathrm{mon}}(M, M')$ is an complex vector space.

A monomial complex is a chain complex of continuous $\mathbb{C}[G]$-modules

$$C_* : \qquad \ldots \stackrel{d}{\longrightarrow} C_n \stackrel{d}{\longrightarrow} C_{n-1} \stackrel{d}{\longrightarrow} C_{n-2} \stackrel{d}{\longrightarrow} \ldots$$

in which each C_n is an G-Line Bundle and each d is a morphism.

Let V be an admissible representation of G. For $(J, \phi) \in \mathcal{M}_G$ define $V^{(J,\phi)}$ to be the subspace

$$V^{(J,\phi)} = \{v \in V \mid j(v) = \phi(j) \cdot v \text{ for all } j \in J\}.$$

An exact chain complex of continuous $\mathbb{C}[G]$-modules of the form

$$\ldots \stackrel{d}{\longrightarrow} C_n \stackrel{d}{\longrightarrow} C_{n-1} \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} C_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is called a relative monomial resolution of V if each C_n is an G-Line Bundle, each d is a morphism and, in addition, $\epsilon(C_0^{((J,\phi))}) \subseteq V^{(J,\phi)}$ for each $(J, \phi) \in \mathcal{M}_G$.

This relative monomial resolution of V is called a monomial resolution if, in addition, the complex of vector spaces

$$\ldots \stackrel{d}{\longrightarrow} C_n^{((J,\phi))} \stackrel{d}{\longrightarrow} C_{n-1}^{((J,\phi))} \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} C_0^{((J,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(J,\phi)} \longrightarrow 0$$

[1]The capital letters are chosen there to distinguish the line from the familiar vector bundle terminology.

is exact for each $(J, \phi) \in \mathcal{M}_G$.

Note that if $C^{((J))}$ and $V^{(J)}$ mean, respectively, the sum over ϕ's of lines with stabiliser greater than or equal to J and the sum of the $V^{(J,\phi)}$'s then monomial exactness is equivalent to exactness of

$$\ldots \stackrel{d}{\longrightarrow} C_n^{((J))} \stackrel{d}{\longrightarrow} C_{n-1}^{((J))} \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} C_0^{((J))} \stackrel{\epsilon}{\longrightarrow} V^{(J)} \longrightarrow 0$$

for each $J \subseteq G$ with J compact modulo the centre.

2.2. *A construction with a finite modulo the centre group*

Let G be a locally compact Hausdorff group as introduced above which sits in an extension of topological groups of the form

$$Z(G) \longrightarrow G \longrightarrow G/Z(G)$$

in which $G/Z(G)$ is finite, $Z(G)$ being the centre of G. For convenience, assume further that there is a topological isomorphism $Z(G) \cong K^*$ for some p-adic local field K. This ensures that for all $H \subseteq G$ the characters in $\hat{H}$ are also continuous with respect to the classical topology on the complex numbers.

Let V be a finite-dimensional complex vector space with the classical topology and let G act continuously on V to give an irreducible continuous representation. Choose a basis $u_1, \ldots u_n$ for V with respect to which the representation V corresponds to a continuous homomorphism

$$\pi_V : G \longrightarrow GL_n\mathbb{C}.$$

Let $T^n \subseteq GL_n\mathbb{C}$ denote the subgroup of diagonal matrices whose normaliser $N_{GL_n\mathbb{C}}T^n$ is given by $T^n \cdot \Sigma_n$ where Σ_n denotes the subgroup of permutation matrices (i.e. those having one 1 and $n-1$ zeroes on each row).

Hence G acts via π by left multiplication on $GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n$. Let

$$(C_*(GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n), d)$$

denote the integral singular chain complex of $GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n$ with the discrete topology. Thus $C_m(X)$ is the free abelian group on continuous maps of the form $f : \Delta^m \longrightarrow X$. Set

$$C_*(V) = C_*(GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n) \otimes_{\mathbb{Z}} V, d \otimes 1).$$

This is a chain complex of continuous $\mathbb{C}[G]$-modules where G acts diagonally on the tensor products.

Choose an basis $\{u_i \mid 1 \leq i \leq n\}$ as above then, depending on the choice of basis, there is a set of lines in this complex defined in the following manner. Let

$$f : \Delta^m \longrightarrow GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n$$

be an m-simplex. Choose any coset

$$\tau_f N_{GL_n\mathbb{C}}T^n \in GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n$$

such that $f(z) = \tau_f N_{GL_n\mathbb{C}}T^n$ for a choice of an interior point $z \in \Delta^m$. Set L_f equal to the set of lines

$$\{\langle f \otimes \tau_f(u_i)\rangle \subset C_m(GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n; \mathbb{C}) \otimes_{\mathbb{C}} V,\ 1 \leq i \leq n\}.$$

The set of lines in dimension m consists of the union of the L_f's as f varies through singular m-simplices.

Proposition 2.3.
(i) In §2.2 L_f depends only on f and the choice of basis for V.
(ii) Also for each $m \geq 0$, as complex vector spaces,

$$\oplus_{L \in L_f} L = C_m(V)$$

where f varies through continuous maps of the form

$$f : \Delta^m \longrightarrow GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n.$$

Proof

For (i), to see that the set of lines is independent of choices once the basis has been chosen we observe that the elements of the diagonal matrices map each line generated by a member of the basis into itself. The normaliser $N_{GL_n\mathbb{C}}T^n$ is generated by these plus the matrices which permute these lines among themselves. Hence L_f depends only of the coset $\tau_f N_\infty(V)$ once one has chosen the interior point $z \in \Delta^m$.

The set of lines does not vary with the choice of $z \in \Delta^n$ because any two such points are connected by a continuous path which yields a continuous path between the two resulting cosets so that the "permutation part" of these cosets are constant throughout the path.

Part (ii) amounts to showing that the $f \otimes \tau_f(u_i)$'s give a basis for $C_m(V)$ over the complex numbers. However by construction the $f \otimes u_i$'s do give a basis for $C_m(V)$ and $\tau_f \in GL_n\mathbb{C}$, which completes the proof. $\square$

Proposition 2.4.
The action of $h \in H$ on $C_n(V)$ permutes the set of lines. In addition, the centre $Z(G)$ maps each line to itself so that the stabilising pair of each line has the form $(J, \psi) \in \hat{J}$ where $Z(G) \subseteq J$.

Proof

The action of h sends f to $h \cdot f$ and hence gives a bijection between L_f and $L_{h \cdot f}$. Furthermore, by Schur's Lemma, $\pi(Z(G))$ consists of scalar matrices so that $Z(G)$ acts trivially on $GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n$ and sends each $\tau_f(u_i)$ to a multiple of itself. $\square$

Theorem 2.5.
In the notation of §2.2

$$H_i(C_*(V), d \otimes 1) = \begin{cases} 0 & \text{if } i > 0, \\ V & \text{if } i = 0. \end{cases}$$

Proof

By the polar decomposition of complex general linear groups [29] the space $GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n$ is homotopy equivalent to the quotient $U_n/N_{U_n}T^n$ of the unitary group of $n \times n$ matrices by the normaliser of its maximal torus of diagonal matrices. However, for each $n \geq 1$, this space has the rational homology of a point [70]. □

Theorem 2.6.

The complex of continuous $\mathbb{C}[H]$-modules $C_*(V)$ defined in §2.2, augmented by the canonical map, gives a relative monomial resolution of V

$$\ldots \stackrel{d}{\longrightarrow} C_3(V) \stackrel{d}{\longrightarrow} C_2(V) \stackrel{d}{\longrightarrow} C_1(V) \stackrel{d}{\longrightarrow} C_0(V) \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0.$$

Proof

The exactness part of this result was proved in §2.2 and Theorem 2.5.

For the remainder we must show that if we choose a finite modulo the centre subgroup $J \subseteq G$ and a character $\phi \in \hat{J}$ then, in the notation of §2.1, for each $n \geq 0$ the boundary

$$d \otimes 1 : C_n(V) \longrightarrow C_{n-1}(V)$$

maps $C_n(V)^{((J,\phi))}$ to $C_{n-1}(V)^{((J,\phi))}$ and $\epsilon(C_0^{((J,\phi))}) \subseteq V^{(J,\phi)}$.

Choose any

$$f : \Delta^m \longrightarrow GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n$$

such that $f \otimes \tau_f(u_i)$, which is a basis element for one of the set of lines in L_f, belongs to a line with stabilising pair greater than or equal to (J, ϕ). This is the same as saying that for all $j \in J$

$$j(f) \otimes j(\tau_f(u_i)) = f \otimes \phi(j)\tau_f(u_i).$$

This implies that $j(f) = f$ and $j(\tau_f(u_i)) = \phi(j)\tau_f(u_i)$.

Therefore, if $z_m \in \Delta^m$ is the chosen interior point,

$$\tau_f N_{GL_n\mathbb{C}}T^n = f(z_m) = j(f(z_m)) = j(\tau_f N_{GL_n\mathbb{C}}T^n)$$

so that the homomorphism

$$\lambda : J \longrightarrow GL_n\mathbb{C}$$

defined by $\lambda(j)(v) = \tau_f^{-1}(j(\tau_f(v)))$ is a continuous homomorphism whose image lies in $N_{GL_n\mathbb{C}}T^n$. By Schur's Lemma λ maps $Z(G) \bigcap J$ to the subgroup of scalar matrices and the image of $\lambda(J)$ in $PGL_n\mathbb{C}$ is finite.

Let $\{\Delta_k^{m-1} \mid 0 \leq k \leq m\}$ be the faces of Δ^m and let $z_{k,m} \in \Delta_k^{m-1}$ be the chosen interior point. The boundary of f is $df = \sum_{k=0}^{m} (-1)^k (f|\Delta_k^{m-1})$. Therefore it will suffice for each k to show that $(f|\Delta_k^{m-1}) \otimes \tau_f(u_i)$ generates one of the lines in the set

$$\{\langle (f|\Delta_k^{m-1}) \otimes \tau_{(f|\Delta_k^{m-1})}(u_s)\rangle \mid \text{for some s}\}$$

whose stabiliser is greater than or equal to (J,ϕ). The condition on the stabiliser pair is automatic because J fixes f and therefore fixes its restriction to each boundary face.

Choose a path $z(t)$ from z_m to $z_{k,m}$ in Δ^m, in the interior except for the endpoint, so that $f(z(t))$ is a path from the coset $\tau_f N_{GL_n\mathbb{C}}T^n$ to the coset $\tau_{(f|\Delta_k^{m-1})}N_{GL_n\mathbb{C}}T^n$. We may lift this path to $\tau(t) \in GL_n\mathbb{C}$ such that $\tau(0) = \tau_f$ and

$$\tau(1)N_{GL_n\mathbb{C}}T^n = \tau_{(f|\Delta_k^{m-1})}N_{GL_n\mathbb{C}}T^n.$$

Since $\tau(1)$ and $\tau_{(f|\Delta_k^{m-1})}$ define the same coset they also define the same set of lines

$$\{\langle (f|\Delta_k^{m-1}) \otimes \tau(1)(u_s)\rangle\}$$

and

$$\{\langle (f|\Delta_k^{m-1}) \otimes \tau_{(f|\Delta_k^{m-1}))}(u_s)\rangle\}.$$

We shall complete the proof by showing that the line generated by the element $(f|\Delta_k^{m-1}) \otimes \tau(1)(u_i)$ is stabilised by (J,ϕ).

Consider the homomorphism

$$\lambda_t : J \longrightarrow N_{GL_n\mathbb{C}}T^n$$

given by replacing τ_f by $\tau(t)$ in the formula for λ. The image of λ_t lies in the subgroup $N_{GL_n\mathbb{C}}T^n$ because J fixes $f(z(t))$ for all t, since $z(t) \in \Delta^m$. In terms of matrices, the condition that $\lambda_t(j)(u_i) = \phi(j)u_i$ is equivalent to the only non-zero entry on the i-th row of $\lambda_t(j)$ being given by $j \mapsto \phi(j)$ in the (i,i)-th entry.

For each t the homomorphism λ_t factorises through the finite group $J/J\bigcap Z(G)$. Therefore λ_t, by continuity, is constant as t varies and so the (i,i)-th entry in both λ_0 and λ_1 is given by $j \mapsto \phi(j)$, as required. $\square$

Remark 2.7. Since G acts on $GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n$ through a finite quotient we may triangulate this space in such a way that G acts simplicially. Replacing the simplicial m-chains by the free abelian group in the m-simplices we obtain a finitely generated, chain homotopy equivalent $\mathbb{Z}[G]$-subcomplex of $C_*(GL_n\mathbb{C}/N_{GL_n\mathbb{C}}T^n)$. Using this chain complex in the construction of Theorem 2.6 yields a relative monomial resolution of finite length.

3. Monomial resolutions in the finite modulo centre case

3.1. Suppose that G is a locally p-adic group which is finite modulo the centre and suppose that V is a finite dimensional, irreducible complex representation of G with central character $\underline{\phi}$ on $Z(G)$. We shall construct a monomial resolution of V by induction.

Hypothesis H_t**:**
Let

$$M_* \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

be a chain complex of continuous $\mathbb{C}[G]$-modules such that:

(a) For $i \geq 0$ each M_i is a finite dimensional complex vector space with the structure of a $\mathbb{C}[G]$-Line Bundle in which each stabiliser pair satisfies $(Z(G), \underline{\phi}) \leq (H, \phi)$. Suppose further that M_* is a $\mathbb{C}[G]$-monomial complex and that $\epsilon(M_0^{((H,\phi))}) \subseteq V^{(H,\phi)}$ for each $(H, \phi) \in \mathcal{M}_G$ satisfying $(Z(G), \underline{\phi}) \leq (H, \phi)$.

(b) The complex of vector spaces

$$M_*^{((Z(G),\underline{\phi}))} \stackrel{\epsilon}{\longrightarrow} V^{(Z(G),\underline{\phi})} = V \longrightarrow 0$$

is exact.

(c) For $t \geq -1$

$$M_t \longrightarrow M_{t-1} \longrightarrow \ldots \longrightarrow M_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is a $\mathbb{C}[G]$-monomially exact chain complex. When $t = -1$ this condition is empty.

By induction on t we shall construct a $\mathbb{C}[G]$-monomial resolution of V. We begin by constructing a complex which satisfies Hypothesis H_0. Throughout what follows all stabiliser pairs (H, ϕ) which occur will be greater than or equal to $(Z(G), \underline{\phi})$, which ensures that the index of H in G is finite.

Lemma 3.2.

Let $\phi \in \hat{H}$ satisfy $(Z(G), \underline{\phi}) \leq (H, \phi)$ and let ϕ also denote the associated one-dimensional $\mathbb{C}[H]$-module. Let $\mathrm{Ind}_H^G(\phi)$ be the induced $\mathbb{C}[G]$-module. Then there exists a finite-dimensional $\mathbb{C}[G]$-Line Bundle M together with a $\mathbb{C}[G]$-module epimorphism $\alpha : M \longrightarrow \mathrm{Ind}_H^G(\phi)$ such that

$$\alpha(M^{((J,\psi))}) = \mathrm{Ind}_H^G(\phi)^{(J,\psi)}$$

for all $(J, \psi) \in \mathcal{M}_G$ satisfying $(Z(G), \underline{\phi}) \leq (J, \psi)$.

Proof

Another proof of this result is given as Step (b) of the proof of Theorem 5.2.

By the double coset formula ([72] pp. 19-20 (proof))

$$\mathrm{Res}_J^G(\mathrm{Ind}_H^G(\phi)) \cong \oplus_{z \in J \backslash G/H} \ \mathrm{Ind}_{J \cap zHz^{-1}}^J((z^{-1})^*(\phi))$$

so that, by Frobenius reciprocity,

$$\mathrm{Ind}_H^G(\phi)^{(J,\psi)} \cong \ \oplus_{z \in J \backslash G/H, \ \mathrm{Res}^J_{J \cap zHz^{-1}}(\psi) = (z^{-1})^*(\phi)} \langle e(z, \psi) \rangle$$

where $\langle e(z, \psi) \rangle$ is the one-dimensional subspace generated by

$$e(z, \psi) = \sum_{u \in J/J \cap zHz^{-1}} uz \otimes_{\mathbb{C}[H]} \psi(u)^{-1}.$$

Observe that, if $w = zhz^{-1} \in J \bigcap zHz^{-1}$ then

$$\begin{aligned} uwz \otimes_{\mathbb{C}[H]} \psi(uw)^{-1} &= uzh \otimes_{\mathbb{C}[H]} \psi(w)^{-1}\psi(u)^{-1} \\ &= uz \otimes_{\mathbb{C}[H]} \phi(h)\psi(w)^{-1}\psi(u)^{-1} \\ &= uz \otimes_{\mathbb{C}[H]} \psi(u)^{-1} \end{aligned}$$

since $\psi(w) = (z^{-1})^*(\phi)(w) = \phi(h)$. Therefore $e(z, \psi) \in \mathrm{Ind}_H^G(\phi)$ is well-defined. Clearly, if $j \in J$, then

$$\begin{aligned} je(z,\psi) &= \textstyle\sum_{u \in J/J \cap zHz^{-1}} \ juz \otimes_{\mathbb{C}[H]} \psi(u)^{-1} \\ &= \textstyle\sum_{ju \in J/J \cap zHz^{-1}} \ (ju)z \otimes_{\mathbb{C}[H]} \psi(j)\psi(ju)^{-1} \\ &= \psi(j)e(z,\psi) \end{aligned}$$

so that $e(z, \psi) \in \mathrm{Ind}_H^G(\phi)^{(J,\psi)}$.

As in Definition 2.1, define the $\mathbb{C}[G]$-Line Bundle $\underline{\mathrm{Ind}}_J^G(\psi)$ to be the usual induced $\mathbb{C}[G]$-module $\mathrm{Ind}_J^G(\psi)$ with Lines given by $\langle g \otimes_{K[J]} 1 \rangle$ for $g \in G/J$, whose stabilising pair is $(gJg^{-1}, (g^{-1})^*(\psi))$.

Let $\mathcal{S}$ denote a set of representatives for the G-conjugacy classes of pairs in $\mathcal{M}_G$. Define a $\mathbb{C}[G]$-Line Bundle M by the direct sum

$$M = \oplus_{(J,\psi) \in \mathbb{S}} \ \oplus_{z \in J \backslash G/H, \ \mathrm{Res}^J_{J \cap zHz^{-1}}(\psi) = (z^{-1})^*(\phi)} \ \underline{\mathrm{Ind}}_J^G(\psi)$$

and define a $\mathbb{C}[G]$-module epimorphism

$$\alpha : M \longrightarrow \underline{\mathrm{Ind}}_H^G(\phi)$$

by $\alpha(1 \otimes_{\mathbb{C}[J]} 1) = e(z, \psi)$ on the summand indexed by (J, ψ) and z. By construction α satisfies

$$\alpha(M^{((J,\psi))}) = \mathrm{Ind}_H^G(\phi)^{(J,\psi)}$$

for all $(J, \psi) \in \mathcal{M}_G$ - a condition which is fulfilled vacuously if $(Z(G), \underline{\phi}) \not\leq (J, \psi)$. $\square$

Corollary 3.3.

Let V be a finite dimensional, irreducible $\mathbb{C}[G]$-module as in §3.1. Then there exists a $\mathbb{C}[G]$-Line Bundle M together with a $\mathbb{C}[G]$-module epimorphism $\beta : M \longrightarrow V$ such that

$$\beta(M^{((J,\psi))}) = V^{(J,\psi)}$$

for all $(J, \psi) \in \mathcal{M}_G$ - a condition which is fulfilled vacuously if $(Z(G), \underline{\phi}) \not\leq (J, \psi)$.

Proof

Since V is an irreducible $\mathbb{C}[G]$-module there is a short exact sequence of $\mathbb{C}[G]$-modules of the form

$$0 \longrightarrow W \longrightarrow \mathrm{Ind}_H^G(\phi) \stackrel{\pi}{\longrightarrow} V \longrightarrow 0.$$

Since $\mathbb{C}$ is a field of characteristic zero we have a short exact sequence

$$0 \longrightarrow W^{(J,\psi)} \longrightarrow \mathrm{Ind}_H^G(\phi)^{(J,\psi)} \longrightarrow V^{(J,\psi)} \longrightarrow 0$$

for each $(J, \psi) \in \mathcal{M}_G$. Therefore, we may take M as in Lemma 3.2 and $\beta = \pi \cdot \alpha$. $\square$

3.4. *The inductive first step - establishing* H_0

Let $C_* \xrightarrow{\epsilon} V \longrightarrow 0$ be a relative $\mathbb{C}[G]$-monomial resolution in which each $\mathbb{C}[G]$-Line Bundle is a finite dimensional vector space and in which each stabilising pair is greater than or equal to $(Z(G), \underline{\phi})$. If β is as in Corollary 3.3 then

$$\ldots \longrightarrow C_t \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{d\oplus 0} C_0 \oplus M \xrightarrow{\epsilon+\beta} V \longrightarrow 0$$

satisfies H_0(a) and (c) of §3.1. Inductively we may modify this complex by adding $\mathbb{C}[G]$-Line Bundles of the form $\mathrm{Ind}_{Z(G)}^G(\underline{\phi})$ to $C_1, C_2, \ldots$ with differentials extended on the additional summands in such a way as to make the complex exact. Since every stabilising pair is greater than or equal to $(Z(G), \underline{\phi})$ this process yields a complex which satisfies H_0(a), (b) and (c).

3.5. Let $M = \oplus L_i$ be a $\mathbb{C}[G]$-Line Bundle. If $v \in L$, where L is a Line of M and $\mathrm{stab}_G(L) = (J, \psi)$, we may construct from v a vector $w \in M^{(H,\phi)}$ by the formula

$$w = \sum_{h \in H} \phi(h)^{-1} h \cdot v.$$

Let $x_1, \ldots, x_t$ be a set of coset representatives for $H/H \bigcap J$. Therefore

$$\begin{aligned} w &= \textstyle\sum_{h\in H} \ \phi(h)^{-1} h \cdot v \\ &= \textstyle\sum_{i=1}^t \ \sum_{g\in H\cap J} \ \phi(x_i)^{-1}\phi(g)^{-1} x_i g \cdot v \\ &= \textstyle\sum_{i=1}^t \ \sum_{g\in H\cap J} \ \phi(x_i)^{-1}\phi(g)^{-1}\psi(g) x_i \cdot v \\ &= \textstyle(\sum_{g\in H\cap J} \ \phi(g)^{-1}\psi(g)) \sum_{i=1}^t \ \phi(x_i)^{-1} x_i \cdot v \\ &= \begin{cases} 0 & \text{if } \mathrm{Res}^H_{H\cap J}(\phi) \neq \mathrm{Res}^H_{H\cap J}(\psi), \\ |H \cap J| \phi(g)^{-1}\psi(g)) \sum_{i=1}^t \ \phi(x_i)^{-1} x_i \cdot v & \text{otherwise.} \end{cases} \end{aligned}$$

Proposition 3.6.

In the conditions of §3.5 suppose that $z = \sum z_i \in M^{(H,\phi)}$ with $z_i \in L_i$. Then $z_i = 0$ unless $\mathrm{stab}_G(L_i) = (H_i, \phi_i)$ and $\mathrm{Res}^H_{H\cap H_i}(\phi) = \mathrm{Res}^H_{H\cap H_i}(\phi_i)$.

Proof

By the calculation of §3.5

$$z = \sum_{i\in H-\text{orbit}, \ \mathrm{Res}^H_{H\cap H_i}(\phi)=\mathrm{Res}^H_{H\cap H_i}(\phi_i)} \frac{|H \cap H_i|}{|H|} \sum_{x_i \in H/H\cap H_i} \phi(x_i)^{-1} x_i \cdot z_i.$$

□

3.7. *The inductive first step - establishing H_t*

If (H,ϕ) and (J,ψ) lie in $\mathcal{M}_G$ with $(H,\phi) \leq (J,\psi)$ define the length $\mathcal{L}((H,\phi),(J,\psi))$ to be the largest i such that there is a chain of strict inclusions

$$(H,\phi) = (J_0,\psi_0) < (J_1,\psi_1) < \ldots < (J_i,\psi_i) = (J,\psi).$$

Let

$$M_* \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

be a chain complex of continuous $\mathbb{C}[G]$-modules satisfying H_{t-1} for $t \geq 1$. Therefore

$$M_{t-1} \longrightarrow M_{t-2} \longrightarrow \ldots \longrightarrow M_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is a $\mathbb{C}[G]$-monomially exact chain complex. Suppose further that, for some integer $n \geq 0$

$$M_t^{((H,\phi))} \longrightarrow M_{t-1}^{((H,\phi))} \longrightarrow \ldots \longrightarrow M_0^{((H,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(H,\phi)} \longrightarrow 0$$

is exact for all (H,ϕ) greater than or equal to $(Z(G),\underline{\phi})$ such that $\mathcal{L}((Z(G),\underline{\phi}),(H,\phi)) \leq n$. Condition (b) of H_{t-1} ensures that a maximal such an n exists.

Consider (H,ϕ) greater than or equal to $(Z(G),\underline{\phi})$ with $\mathcal{L}((Z(G),\underline{\phi}),(H,\phi)) = n+1$ for which the finite dimensional vector space $\mathrm{Ker}(M_{t-1}^{((H,\phi))} \longrightarrow M_{t-2}^{((H,\phi))})$ is not equal to $d(M_t^{((H,\phi))})$. Choose $z \in M_{t-1}^{((H,\phi))}$ such that $d(z) = 0$ (or $\epsilon(z) = 0$ if $t = 1$). Since

$$M_t^{(H,\phi)} \longrightarrow M_{t-1}^{(H,\phi)} \longrightarrow \ldots \longrightarrow M_0^{(H,\phi)} \stackrel{\epsilon}{\longrightarrow} V^{(H,\phi)} \longrightarrow 0$$

is exact there exists $w \in M_t^{(H,\phi)}$ such that $d(w) = z$.

By Proposition 3.6 we may write $w = \sum w_i \in M^{(H,\phi)}$ with $w_i \in L_i$ where L_i is a Line in M_t with $\mathrm{stab}_G(L_i) = (H_i,\phi_i)$ and $w_i = 0$ unless $\mathrm{Res}^H_{H\cap H_i}(\phi) = \mathrm{Res}^H_{H\cap H_i}(\phi_i)$. More precisely we have the formula

$$w = \sum_{i\in H-\text{orbit},\ \mathrm{Res}^H_{H\cap H_i}(\phi)=\mathrm{Res}^H_{H\cap H_i}(\phi_i)} \frac{|H\cap H_i|}{|H|} \sum_{x_i\in H/H\cap H_i} \phi(x_i)^{-1}x_i \cdot w_i.$$

When $H \bigcap H_i = H$ we see that $(H,\phi) \leq (H_i,\phi_i)$ and so $w_i \in M_t^{((H,\phi))}$ with

$$w_i = \frac{|H\cap H_i|}{|H|} \sum_{x_i\in H/H\cap H_i} \phi(x_i)^{-1}x_i \cdot w_i.$$

Let w' denote the sum of all the terms in the expression for w with $H \bigcap H_i = H$ so that $w - w' = w'' \in M_t^{(H,\phi)}$ satisfies $d(w-w') = z - d(w') \in M_{t-1}^{((H,\phi))}$. Replace M_t by $M_t' = M_t \oplus \underline{\mathrm{Ind}}_H^G(\phi)$ and extend the differential on the extra summand by $d(1 \otimes_H 1) = z - d(w')$.

By induction on the dimension of the quotient of $\mathrm{Ker}(M_{t-1}^{((H,\phi))} \longrightarrow M_{t-2}^{((H,\phi))})$ by the image of $M_t'^{((H,\phi))}$ and on $\mathcal{L}((Z(G),\underline{\phi}),(H,\phi))$ we obtain a chain complex satisfying H_t(a) and (c). This complex may be modified to satisfy H_t(a), (b) and (c) by use of $\underline{\mathrm{Ind}}_{Z(G)}^{G}(\underline{\phi})$'s as in §3.4.

This completes the construction of a (possibly infinite length) $\mathbb{C}[G]$-monomial resolution for V.

Theorem 3.8.

Suppose that G is a locally p-adic group which is finite modulo the centre and suppose that V is a finite dimensional, irreducible complex representation of G with central character $\underline{\phi}$ on $Z(G)$. Then there exists a $\mathbb{C}[G]$-monomial resolution for V which satisfies Hypothesis H_t of §3.1 for all $t \geq -1$.

4. Explicit Brauer Induction in the finite modulo the centre case

4.1. In this section I shall begin by recalling various the natural Explicit Brauer Induction formulae associated to finite dimensional complex representations of a finite group G. The objective is to extend the formula of ([7]; see also [6], [9] and [72]) to the case when G is a locally p-adic group which is finite modulo the centre and V is a finite dimensional, continuous, irreducible complex representation of G with central character $\underline{\phi}$ on $Z(G)$.

Historically the first of these Explicit Brauer Induction formulae appeared in [70]. It may succinctly be described as the Euler characteristic $\sum_i (-1)^i [C_i(V)]$, in the Grothendieck group $R_+(G)$ (introduced below) of a finite length, finitely generated relative monomial resolution of V as constructed in Theorem 2.6 (see Remark 2.7 for the finiteness conditions). Motivated by the search for an algebraic approach a formula with superior properties was discovered by Robert Boltje [7]. The two formulae are related by a simple equation which is derived in [11]. The formula of [7] is constructed in [76] by means of the topology of group actions on projective space, which is similar to the topological method used in [70] (and in Theorem 2.6). Both these actions are trivial on the centre $Z(G)$ with the result that the formula of [7] may be extended to the finite modulo the centre case by the construction of [76].

Such an extension of the formula will be necessary (at one critical concluding point) in the next section in order to construct a finitely generated, bounded $\mathbb{C}[G]$-monomial resolution of V in the finite modulo the centre case by a generalisation of the method of ([10] §6).

4.2. *Brauer Induction formula*

Let G be a finite group. Let $R(G) = K_0(\mathbb{C}[G])$ denote the Grothendieck group of the category of finite-dimensional left $\mathbb{C}[G]$-modules. Every such module yields a matrix representation $\rho : G \longrightarrow GL_n\mathbb{C}$ which is conjugate to a (unique) unitary representation $\rho : G \longrightarrow U(n)$. Since such a representation is determined by its character, we may identify $R(G)$ with the character ring of G, the free abelian group on the set of irreducible characters on G. The set of linear characters (i.e one-dimensional representations) $G \longrightarrow U(1) = S^1$ will be denoted by $\hat{G}$. Brauer proved [21] that every unitary representation $\rho : G \longrightarrow U(n)$ is an integral sum of representations which are induced from linear characters on subgroups of G, so

$$\rho = \sum_i n_i \mathrm{Ind}_{H_i}^G \phi_i \in R(G)$$

with $H_i \subseteq G$, $\phi_i \in \hat{H}_i$.

Let $R_+(G)$ denote the free abelian group on the G-conjugacy classes $(H, \phi)^G$ of pairs $(H, \phi) \in \mathcal{M}_G$ so $H \subseteq G$ and $\phi \in \hat{H}$. If $J \subseteq G$ we have a restriction homomorphism

$$\mathrm{Res}_J^G : R_+(G) \longrightarrow R_+(J)$$

defined by

$$\mathrm{Res}_J^G((H, \phi)^G) = \sum_{z \in J \backslash G/H} (J \bigcap zHz^{-1}, (z^{-1})^* \phi)^J$$

where $(z^{-1})^* \phi(zhz^{-1}) = \phi(h)$. If $\pi : G \longrightarrow K$ is a surjection there is an inflation homomorphism

$$\mathrm{Inf}_K^G : R_+(K) \longrightarrow R_+(G)$$

given by $\mathrm{Inf}_K^G(H, \phi)^K = (\pi^{-1}(H), \phi \cdot \pi)^G$. By means of these maps $R_+(-)$ gives a functor from finite groups to abelian groups. In fact, we even obtain a Mackey functor from finite groups to the category of rings ([6], [7], [11], [70], [71], [72]).

I am going to use the Explicit Brauer Induction formula of [7] (see [72]; also [59]) which is tautologous on one-dimensional representations and natural with respect to restriction and inflation, namely

$$a_G(\rho) = \sum_{(H_0,\phi_0)<(H_1,\phi_1)<\ldots<(H_r,\phi_r)} (-1)^r \frac{|H_0|}{|G|} \langle \mathrm{Res}_{H_r}^G(\rho), \phi_r \rangle (H_0, \phi_0)^G \in R_+(G).$$

Here $\langle \mathrm{Res}_{H_r}^G(\rho), \phi_r \rangle = \dim_{\mathbb{C}}(\rho^{(H_r,\phi_r)})$ and each pair (H_i, ϕ_i) lies in $\mathcal{M}_G$. The formula for $a_G(V)$ may succinctly be described as the Euler characteristic $\sum_i (-1)^i [M_i(V)]$, in $R_+(G)$ of a finite length, finitely generated relative monomial resolution of V as constructed in the next section. For finite groups it is characterised by the following result.

Theorem 4.3.

Let G be a finite group.

(i) The map $\rho \mapsto a_G(\rho)$ defines an additive homomorphism of the form

$$a_G : R(G) \longrightarrow R_+(G).$$

(ii) This homomorphism is characterised by the properties that

$$a_G(\phi) = (G, \phi)^G \text{ if } \phi \in \hat{G}$$

and a_G is natural with respect to restriction and inflation homomorphisms.

(iii) In addition, if $b_G : R_+(G) \longrightarrow R(G)$ is the additive homomorphism defined by $b_G(H,\phi)^G) = \mathrm{Ind}_H^G(\phi)$ then $b_G(a_G(\rho)) = \rho$ for all $\rho \in R(G)$.

Example 4.4. *([59] §§2.5-2.6)*

Let Q_{4n} denote the generalised quaternion group

$$Q_{4n} = \left\langle x, y \mid x^n = y^2,\ y^4 = 1, yxy^{-1} = x^{-1} \right\rangle$$

and let Ψ denote the symplectic representation

$$\Psi : Q_{4n} \longrightarrow Sp(1) \subset \mathbf{H}^*$$

given by $\Psi(x) = \xi_{2n}$ and $\Psi(y) = j$. Here $\xi_n = e^{2\pi\sqrt{-1}/n}$ and j is the usual quaternion. Let $\rho = c(\Psi) : Q_{4n} \longrightarrow U(2)$ denote the complexification of Ψ. Then

$$a_{Q_{4n}}(\rho) = \begin{cases} (\langle x\rangle, \phi_x)^{Q_{4n}} + (\langle y\rangle, \rho_y)^{Q_{4n}} + (\langle y\rangle, \overline{\rho}_y)^{Q_{4n}} - (\langle y^2\rangle, \chi)^{Q_{4n}} & n \textit{ odd}, \\ (\langle x\rangle, \phi_x)^{Q_{4n}} + (\langle y\rangle, \rho_y)^{Q_{4n}} + (\langle xy\rangle, \rho_{xy})^{Q_{4n}} - (\langle y^2\rangle, \chi)^{Q_{4n}} & n \textit{ even}, \end{cases}$$

where $\rho_x(x) = \xi_{2n}$.

Remark 4.5. As we shall see when we come to Symonds' topological formula [76] for $a_G(\rho)$, the formula for $a_{Q_{4n}}(c(\Psi))$ is determined by the projective representation associated to $c(\Psi)$ which is the same projective representation as the one associated to the dihedral representation

$$\nu : D_{2n} \longrightarrow U(2)$$

given by

$$\nu(x) = \begin{pmatrix} \xi_n & 0 \\ 0 & \overline{\xi}_n \end{pmatrix}, \ \nu(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where

$$D_{2n} = < x, y \mid x^n = 1 = y^2, yxy = x^{-1} > .$$

This implies that $a_{D_{2n}}(\nu)$ is also given by the formulae of Example 4.4.

4.6. *Extending a_G to the finite modulo centre case*

Suppose that G is a locally p-adic group which is finite modulo the centre and suppose that V is a finite dimensional (not necessarily irreducible) complex representation of G. We shall assume that the centre $Z(G)$ acts on V be multiplication by the central character $\underline{\phi} \in \hat{\underline{Z}}(G)$. Let $\mathbb{P}(V)$ denote the projective space of V with the associated left action by G. Note that the action by G factors through the finite group $G/Z(G)$. Therefore we may give $\mathbb{P}(V)$ the structure of an equivariant, finite CW complex with an admissible G-action (i.e. the stabiliser of a cell as a set also stabilises it pointwise). Consider the formal sum

$$L_G(V) = \sum_{\sigma \in G \backslash \mathbb{P}(V)} (-1)^{\dim(\sigma)} \ (Stab_G(\tilde{\sigma}), \phi_{\tilde{\sigma}})$$

which lies in $\mathbb{Z}\langle \mathcal{M}_G \rangle$, the free abelian from on $\mathcal{M}_G$. Here the sum is over the cells σ of $\mathbb{P}(V)$, $\tilde{\sigma}$ is a cell of $\mathbb{P}(V)$ which lies above σ and $\phi_{\tilde{\sigma}}$ is the character by means of which the stabiliser $Stab_G(\tilde{\sigma})$ acts on any line corresponding to a point of $\tilde{\sigma}$.

Now consider the same sum as representing an element of $R_+(G)$

$$L_G(V) = \sum_{\sigma \in G \backslash \mathbb{P}(V)} (-1)^{\dim(\sigma)} \ (Stab_G(\tilde{\sigma}), \phi_{\tilde{\sigma}})^G \in R_+(G).$$

Since $(J, \phi)^G$ is the G-conjugacy class of the pair (J, ϕ) we can group together all the terms with conjugate pairs $(Stab_G(\tilde{\sigma}), \phi_{\tilde{\sigma}})$ to obtain

$$L_G(V) = \sum_{(J,\phi)^G \in G \backslash \mathcal{M}_G} \alpha_{(J,\phi)^G}(V) \ (J, \phi)^G \in R_+(G).$$

As explained in [70] (see also ([76] §2)) the integer $\alpha_{(J,\phi)^G}(V)$ is equal to the Euler characteristic in compactly supported cohomology of the stratum of $G \backslash \mathbb{P}(V)$ of type $(J, \phi)^G$, which is independent of the G-CW structure chosen on projective space, depending only on the projective representation associated to V.

When G is a finite group (in which case we discard the hypothesis concerning the existence of a central character) the main result of [76] states that

$$L_G(V) = a_G(V) \in R_+(G).$$

En route the topological construction shows immediately the integrality of the formula for a_G, which apparently has rational coefficients (see also [59]).

In the finite modulo the centre case we have the following result.

Theorem 4.7.
Let G be a locally p-adic group which is finite modulo the centre and let V, W be a finite dimensional complex representation of G with the same central character $\underline{\phi}$. Then
(i) $L_G(V \oplus W) = L_G(V) + L_G(W) \in R_+(G)$,
(ii) if $\alpha_{(J,\phi)^G}(V)$ is non-zero then $(Z(G), \underline{\phi}) \leq (J, \phi)$ and
(iii) $b_G(L_G(V)) = V \in R(G)$,
(iv) $L_G(V) \in R_+(G)$ is natural with respect to restriction and inflation.

Proof
Parts (ii) and (iv) are obvious from the topological construction on $L_G(V)$. Parts (i) and (iii) following from the corresponding results for finite groups in [76]. Let H denote the maximal compact subgroup of G. Then the restriction of V to H factors through a finite quotient of H. Therefore (i) and (ii) are true for $\mathrm{Res}^G_H(V)$.

For part (iii) write the equation $b_H(L_H(\mathrm{Res}^G_H(V))) = \mathrm{Res}^G_H(V) \in R(H)$ in the equivalent form as an H-isomorphism $\gamma : V_1 \stackrel{\cong}{\longrightarrow} V_2$ between H-representations (rather than virtual representations) where the V_i are restrictions of G representations with central character $\underline{\phi}$. Since G is generated by H and $Z(G)$, γ is also a G-isomorphism, which is equivalent to part (iii) for G.

Similarly part (iv) follows for G from the case for H by virtue of part (ii). $\square$

5. Bounded monomial resolutions in the finite modulo centre case

5.1. Suppose that G is a locally p-adic group which is finite modulo the centre and suppose that V is a finite dimensional, irreducible complex representation of G with central character $\underline{\phi}$ on $Z(G)$. We shall construct a finite length, finite type monomial resolution of V by modifying the proof for finite groups which is given in ([10] §6).

This is an involved induction of the "homological algebra" type which produces a monomial resolution unique up to chain homotopy equivalence.

We say that $(H, \phi) \in \mathcal{M}_G$ is V-admissible if $V^{(H,\phi)} \neq 0$ and $(Z(G), \underline{\phi}) \leq (H, \phi)$. Let $\mathcal{S}(V)$ denote the set of non-zero subspaces of V and let $A(V) \subseteq \mathcal{M}_G$ denote the set of V-admissible pairs. Define maps

$$F_V : A(V) \longrightarrow \mathcal{S}(V) \text{ and } P_V : \mathcal{S}(V) \longrightarrow A(V)$$

by the formulae

$$F_V(H, \phi) = V^{(H,\phi)} \text{ and } P_V(W) = \sup\{(H, \phi) \mid W \subseteq V^{(H,\phi)}\}.$$

Usually suprema do not exist in $\mathcal{M}_G$ but $P_V(W)$ exists in this context. Firstly $W \subseteq V^{(Z(G),\underline{\phi})}$. On the other hand, if $W \subseteq V^{(H,\phi)}$ and $W \subseteq V^{(H',\phi')}$ then $W \subseteq V^{(H'',\phi'')}$ where H'' is the subgroup generated by H and H' and ϕ'' is a character which extends both ϕ' and ϕ. This extension exists since $H/Z(G)$ and $H'/Z(G)$ are both finite and $\mathbb{C}^*$ is an injective abelian group.

Both $\mathcal{S}(V)$ and $A(V)$ are posets with G-action with $W \subseteq F_V(P_V(W))$ and $(H,\phi) \leq P_V(F_V(H,\phi))$. Define the V-closure $\mathrm{cl}_V(H,\phi)$ by

$$\mathrm{cl}_V(H,\phi) = P_V(F_V(H,\phi))$$

and say that (H,ϕ) is V-closed if $(H,\phi) = \mathrm{cl}_V(H,\phi)$. Hence $\mathrm{cl}_V(H,\phi)$ is the largest pair (H',ϕ') such that $V^{(H,\phi)} = V^{(H',\phi')}$. Closure commutes with the G-action, is idempotent and order-increasing. Let Cl(V) $\subseteq$ A(V) denote the subset of closed pairs.

For $(H,\phi) \in$ Cl(V) define the V-depth $d_V(H,\phi)$ to be the largest integer $n \geq 0$ such that there exists a strictly increasing chain in Cl(V) of length n of the form

$$(H,\phi) = (H_0,\phi_0) < (H_1,\phi_1) < \cdots < (H_{n-1},\phi_{n-1}) < (H_n,\phi_n).$$

Therefore $d_V(H,\phi) = 0$ if and only if (H,ϕ) is maximal in Cl(V).

Theorem 5.2.

In the situation and notation of §5.1 there exists a $\mathbb{C}[G]$-monomial resolution

$$M_* \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

such that:

(i) For $i \geq 0$, M_i has no Line with stabiliser pair $(H,\phi) \notin \mathrm{Cl}(V)$.

(ii) For $i \geq 0$, M_i has no Line with stabiliser pair $(H,\phi) \in \mathrm{Cl}(V)$ and $d_V(H,\phi) < i$.

In particular $M_i = 0$ for all $i > \max\{d_V(H,\phi) \mid (H,\phi) \in \mathrm{Cl(V)}\}$.

Proof

By induction on n we shall show that there exists a chain complex

$$M_n \stackrel{\partial_{n-1}}{\longrightarrow} M_{n-1} \stackrel{\partial_{n-2}}{\longrightarrow} \ldots \stackrel{\partial_0}{\longrightarrow} M_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

in which each M_i is a $\mathbb{C}[G]$-Line Bundle, each ∂_i is a morphism and ϵ is a homomorphism of $\mathbb{C}[G]$modules such that the following conditions are satisfied:

(A_n) For $0 \leq i \leq n$, M_i has no Line with stabiliser pair $(H,\phi) \notin \mathrm{Cl}(V)$.

(B_n) For $0 \leq i \leq n$, M_i has no Line with stabiliser pair $(H,\phi) \in \mathrm{Cl}(V)$ and $d_V(H,\phi) < i$.

(C_n) The sequence of vector spaces

$$M_n^{((H,\phi))} \stackrel{\partial_{n-1}}{\longrightarrow} M_{n-1}^{((H,\phi))} \stackrel{\partial_{n-2}}{\longrightarrow} \ldots \stackrel{\partial_0}{\longrightarrow} M_0^{((H,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(H,\phi)} \longrightarrow 0$$

is exact for all $(H,\phi) \in \mathcal{M}_G$.

(D_n) The sequence of vector spaces

$$0 \longrightarrow M_n^{((H,\phi))} \stackrel{\partial_{n-1}}{\longrightarrow} M_{n-1}^{((H,\phi))} \stackrel{\partial_{n-2}}{\longrightarrow} \ldots \stackrel{\partial_0}{\longrightarrow} M_0^{((H,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(H,\phi)} \longrightarrow 0$$

is exact for all $(H, \phi) \in \mathrm{Cl}(V)$ with $d_V(H, \phi) \leq n$.

Note that for $n > \max(d_V(H, \phi) \mid (H, \phi) \in \mathrm{Cl}(V))$ the properties (A_n) and (B_n) imply that $M_n = 0$ and property (C_n) implies that

$$M_* \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is a $\mathbb{C}[G]$-monomial resolution satisfying conditions (i) and (ii) of Theorem 5.2. By $\mathbb{C}[G]$-equivariance it suffices to prove (A_n)-(D_n) for one pair (H, ϕ) in each G-orbit.

Step (a): We show that if (A_n) , (B_n) and (D_n) hold then it suffices to prove (C_n) only for $(H, \phi) \in \mathrm{Cl}(V)$. Let $(H, \phi) \in \mathcal{M}_G$ such that $(Z(G), \underline{\phi}) \leq (H, \phi)$. If $(H, \phi) \notin A(V)$ then no larger (H', ϕ') is V-admissible and since $Cl(V) \subseteq A(V)$ we have $M_i^{((H,\phi))} = 0$ for $0 \leq i \leq n$ by (A_n). Also $V^{(H,\phi)} = 0$. Now suppose that $(H, \phi) \in A(V)$ then we must prove that the sequence in (C_n) is exact, assuming that it is exact for all $(H, \phi) \in \mathrm{Cl}(V)$. We have $V^{H,\phi)} = V^{\mathrm{cl}_V(H,\phi)}$ by the definition of closure. In addition, $M_i^{((H,\phi))} = M_i^{(\mathrm{cl}_V(H,\phi))}$ for all $0 \leq i \leq n$. This is seen as follows: $(H, \phi) \leq \mathrm{cl}_V(H, \phi)$ implies that $M_i^{(\mathrm{cl}_V(H,\phi))} \subseteq M_i^{((H,\phi))}$ and for all $(H', \phi') \geq (H, \phi)$ we have no Lines in M_i with stabiliser pair (H', ϕ') unless $(H', \phi') \in \mathrm{Cl}(V)$, by (A_n), but in this case we have $(H', \phi') = \mathrm{cl_V(H'}, \phi') \geq \mathrm{cl_V(H}, \phi) = (\mathrm{H}, \phi)$ so that $M_i^{((H',\phi'))} \subseteq M_i^{(\mathrm{cl}_V(H,\phi))}$. Therefore the sequences in (C_n) for (H, ϕ) and for its closure coincide but the latter is exact by assumption.

Step (b): We start the induction on n by defining $\epsilon : M_0 \longrightarrow V$. For each (H, ϕ) we need to define the set of Lines in M_0 whose stabiliser pair is G-conjugate to (H, ϕ). If $(H, \phi) \notin \mathrm{Cl}(V)$ we shall define the set of such Lines to be empty. If $(H, \phi) \in \mathrm{Cl}(V)$ define this set of lines to be given by the Line Bundle $\underline{\mathrm{Ind}}_H^G(V^{(H,\phi)})$ where $V^{(H,\phi)}$ is viewed as a $\mathbb{C}[H]$-Line Bundle with any choice of decomposition into Lines with the H-action $hv = \phi(h) \cdot v$, which was given on $V^{(H,\phi)}$ already. Define ϵ on this sub-Line Bundle of M_0

$$\underline{\mathrm{Ind}}_H^G(V^{(H,\phi)}) \longrightarrow V$$

by $\epsilon(g \otimes_{\mathbb{C}[H]} v) = g \cdot v$. This satisfies both (A_0) and (B_0).

Next we show that (D_0) holds. Suppose that (H, ϕ) is a maximal element in $\mathrm{Cl}(V)$, which implies that (H, ϕ) is maximal in $A(V)$ because if there were a larger pair in $A(V)$ its closure would be in $\mathrm{Cl}(V)$ and larger than (H, ϕ). Therefore the normaliser of (H, ϕ) must equal H. For the normaliser is $\mathrm{stab}_G(H, \phi)$ and so if it is greater than H the character ϕ may be extended ϕ' on to $H' > H$ with an abelian quotient group H'/H and $V^{(H,\phi)} \neq 0$. By Clifford theory $V^{(H',\phi')} \neq 0$ and therefore $(H', \phi') > (H, \phi)$ in $A(V)$ - a contradiction. By maximality in $\mathrm{Cl}(V)$, by (A_0) and by the condition $H = \mathrm{stab}_G(H, \phi)$ we find that $M_0^{((H,\phi))} = 1 \otimes_{\mathbb{C}[H]} V^{(H,\phi)}$ so that $\epsilon : M^{((H,\phi))} \longrightarrow V^{(H,\phi)}$ is an isomorphism. Finally, by step (a), we must show that $\epsilon : M^{((H,\phi))} \longrightarrow V^{(H,\phi)}$ is surjective for any $(H,\phi) \in \mathrm{Cl}(V)$, which is clear by construction.

Step (c): Next we assume that we have already constructed a chain complex

$$M_n \stackrel{\partial_{n-1}}{\longrightarrow} M_{n-1} \stackrel{\partial_{n-2}}{\longrightarrow} \ldots \stackrel{\partial_0}{\longrightarrow} M_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

such that (A_n), (B_n), (C_n) and (D_n) hold. When $n = 0$ we interpret (M_n, ∂_{n-1}) as (V, ϵ). We shall define $\partial_n : M_{n+1} \longrightarrow M_n$ by defining the Lines of M_{n+1} whose stabiliser pair is G-conjugate to (H,ϕ) and specifying ∂_n on the direct sum of those Lines. If $(H,\phi) \notin \mathrm{Cl}(V)$ we set the sum of these Lines to be zero, which assures that (A_{n+1}) holds. If $(H,\phi) \in \mathrm{Cl}(V)$ and $d_V(H,\phi) \leq n$ we also set the sum of these Lines to be zero, so that (B_{n+1}) follows from (B_n). Moreover (D_n) and (A_{n+1}) implies (C_{n+1}) for all (H,ϕ) with $d_V(H,\phi) \leq n$. Also (D_n) implies (D_{n+1}) for all $(H,\phi) \in \mathrm{Cl}(V)$ with $d_V(H,\phi) \leq n$.

If $(H,\phi) \in \mathrm{Cl}(V)$ with $d_V(H,\phi) > n+1$ we define the direct sum of Lines in M_{n+1} with stabiliser pair conjugate to (H,ϕ) to be $\underline{\mathrm{Ind}}_H^G(\Omega_n^{(H,\phi)})$ with $\Omega_n^{(H,\phi)} = \mathrm{Ker}(\partial_{n-1} : M_n^{((H,\phi))} \longrightarrow M_{n-1}^{((H,\phi))})$ considered as an H-Line Bundle with any chosen decomposition and we set $\partial_n(g \otimes_{\mathbb{C}[H]} v) = gv$. Clearly $\partial_{n-1} \cdot \partial_n = 0$ as defined so far and ∂_n is a morphism by Frobenius reciprocity in the category of Line Bundles and morphisms [10]. Also

$$\mathrm{Ker}(\partial_{n-1} : M_n^{((H,\phi))} \longrightarrow M_{n-1}^{((H,\phi))}) = \Omega_n^{(H,\phi)} = \partial_n(1 \otimes_{\mathbb{C}[H]} \Omega_n^{(H,\phi)} \subseteq \mathrm{Im}(\partial_n)$$

shows that (C_{n+1}) holds for this pair (H,ϕ) (and its G-conjugates) and (D_{n+1}) is vacuously true for it.

It remains to deal with the case when $d_V(H,\phi) = n+1$ and only (D_{n+1}) requires to be proved since (C_{n+1}) is vacuously satisfied in this case. We shall show in Lemma 5.3 that

$$\Omega_n^{(H,\phi)} = \mathrm{Ker}(\partial_{n-1} : M_n^{((H,\phi))} \longrightarrow M_{n-1}^{((H,\phi))}) \cong \mathrm{Ind}_H^{\mathrm{stab}_G(H,\phi)}(L_{(H,\phi)})$$

as $\mathbb{C}[\mathrm{stab}_G(H,\phi)]$-modules for some $\mathbb{C}[H]$-submodule $L_{(H,\phi)} \subseteq \Omega_n^{(H,\phi)}$. Since H acts on $L_{(H,\phi)}$ via multiplication by ϕ we may choose a decomposition for $L_{(H,\phi)}$ as a direct sum of (H,ϕ)-Lines. Then we define the direct sum of Lines in M_{n+1} whose stabilisers are G-conjugate to (H,ϕ) to be given by $\underline{\mathrm{Ind}}_H^G(L_{(H,\phi)})$ and define ∂_n on $\underline{\mathrm{Ind}}_H^G(L_{(H,\phi)})$ by $\partial_n(g \otimes_{\mathbb{C}[H]} v) = gv$. Then ∂_n is a morphism, by Frobenius reciprocity in the category of Line Bundles and morphisms, and (D_{n+1}) holds because, by (A_{n+1}) and (B_{n+1}),

$$M_{n+1}^{((H,\phi))} = (\underline{\mathrm{Ind}}_H^G(L_{(H,\phi)}))^{((H,\phi))} = \oplus_{g \in \mathrm{stab}_G(H,\phi)/H}\ s \otimes_{\mathbb{C}[H]} L_{(H,\phi)},$$

which shows that ∂_n induces an isomorphism $M_{n+1}^{((H,\phi))} \longrightarrow \Omega_n^{(H,\phi)}$.

The proof will be completed by Lemma 5.3. □

Lemma 5.3.

Suppose we have a chain complex as at the start of the proof of Theorem 5.2

$$M_n \stackrel{\partial_{n-1}}{\longrightarrow} M_{n-1} \stackrel{\partial_{n-2}}{\longrightarrow} \ldots \stackrel{\partial_0}{\longrightarrow} M_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

such that conditions (A_n), (B_n), (C_n) and (D_n) hold. Let $(H,\phi) \in \mathrm{Cl}(V)$ with $d_V(H,\phi) = n+1$. Then the class $\theta \in K_0(\mathbb{C}[N])$ of the $\mathbb{C}[\mathrm{stab}_G(H,\phi)]$-module

$$\mathrm{Ker}(\partial_{n-1} : M_n^{((H,\phi))} \longrightarrow M_{n-1}^{((H,\phi))}) = \Omega_n^{(H,\phi)}$$

is a (possibly zero) multiple of the character of $\mathrm{Ind}_H^{\mathrm{stab}_G(H,\phi)}(\phi)$.

Proof

Set $N = \mathrm{stab}_G(H,\phi)$. Then, by (C_n) for (H,ϕ) we have an exact sequence of $\mathbb{C}[N]$-modules

$$0 \longrightarrow \Omega_n^{(H,\phi)} \longrightarrow M_n^{((H,\phi))} \stackrel{\partial_{n-1}}{\longrightarrow} M_{n-1}^{((H,\phi))} \stackrel{\partial_{n-2}}{\longrightarrow} \ldots \stackrel{\partial_0}{\longrightarrow} M_0^{((H,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(H,\phi)} \longrightarrow 0.$$

Denote by $\chi_i \in K_0(\mathbb{C}[N])$ the character of $M_i^{((H,\phi))}$ and ν the class of $V^{(H,\phi)}$. Therefore we obtain

$$\nu = (-1)^{n+1}\theta + \sum_{i=0}^{n} (-1)^i \chi_i \in K_0(\mathbb{C}[N]).$$

At this point the proof for G finite ([10] §6) uses the existence of the Explicit Brauer Induction maps a_N and b_N of Theorem 4.3, which is replaced in the finite modulo the centre case by L_N and b_N of Theorem 4.7.

Now consider the class of the $\mathbb{C}[N]$-Line Bundle $M_i^{((H,\phi))}$ which satisfies $\chi_i = b_N(M_i^{((H,\phi))})$, by definition. We also have $b_N(L_N(\nu)) = \nu$ so that we obtain

$$(-1)^{n+1}\theta = b_N(L_N(\nu) - \sum_{i=0}^{n} (-1)^i \chi_i) \in K_0(\mathbb{C}[N]).$$

Observe next that all the stabiliser pairs of the Lines of $M_i^{((H,\phi))}$ for $0 \leq i \leq n$ have the form $(H' \bigcap N, \mathrm{Res}^{H'}_{H'\cap N}(\phi'))$ for some $(H',\phi') \in \mathcal{M}_G$ such that $(H,\phi) \leq (H',\phi')$. Hence in the free abelian group $R_+(N)$ on the N-conjugacy classes of $\mathcal{M}_N$ the class of $M_i^{((H,\phi))} \in R_+(N)$ may have non-zero coefficients only at basis elements $(K,\psi) \in N\backslash\mathcal{M}_N$ with $(H,\phi) \leq (K,\psi)$. By a basic property of L_N (analogous to Theorem 4.7(ii)), the same is true of $L_N(\nu)$ since ν restricts to a multiple of ϕ on H. Therefore we may write

$$L_N(\nu) - \sum_{i=0}^{n} (-1)^i M_i^{((H,\phi))} = \sum_{\substack{(K,\psi)_N \in \mathcal{M}_N/N \\ (H,\phi)\leq(K,\psi)}} \alpha_{(K,\psi)^N} \cdot (K,\psi)^N \in R_+(N)$$

where each $\alpha_{(K,\psi)^N}$ is an integer.

We shall show that $\alpha_{(K,\psi)^N} = 0$ for all $(H,\phi) < (K,\psi)$, which concludes the proof.

Assume that $\alpha_{(K_0,\psi_0)^N} \neq 0$ for some $(H,\phi) < (K_0,\psi_0)$ and assume also that (K_0,ψ_0) is maximal amongst pairs satisfying this condition.

Recall ([7]; see also [72]) that there is a (non-symmetric) bilinear form on $R_+(N)$ and maximality of (K_0,ψ_0) yields

$$((K_0,\psi_0)^N, \sum_{\substack{(K,\psi)^N \in N\backslash \mathcal{M}_N \\ (H,\phi)\leq (K,\psi)}} \alpha_{(K,\psi)^N} \cdot (K,\psi)^N)_N$$

$$= \sum_{\substack{(K,\psi)^N \in N\backslash \mathcal{M}_N \\ (H,\phi)\leq (K,\psi)}} \alpha_{(K,\psi)^N}((K_0,\psi_0)^N, (K,\psi)^N)_N$$

$$= \alpha_{(K_0,\psi_0)^N}((K_0,\psi_0)^N, (K_0,\psi_0)^N)_N$$

$$= \alpha_{(K_0,\psi_0)^N}[\mathrm{stab}_N(K_0,\psi_0) : K_0] \neq 0.$$

On the other hand, adjointness properties of L_G (analogous to adjointness for a_G in the finite case) and the bilinear form yield

$$((K_0,\psi_0)^N, L_N(\nu) - \sum_{i=0}^n (-1)^i M_i^{((H,\phi))})_N$$

$$= ((K_0,\psi_0)^N, L_N(\nu))_N - \sum_{i=0}^n (-1)^i((K_0,\psi_0)^N, M_i^{((H,\phi))})_N$$

$$= (\mathrm{Ind}_{K_0}^N(\psi_0), \nu)_N - \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}[N]-mon}(\underline{\mathrm{Ind}}_{K_0}^N(\psi_0), M_i^{((H,\phi))}))$$

$$= (\psi_0, \mathrm{Res}_{K_0}^N(\nu))_{K_0} - \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}}(M_i^{((K_0,\psi_0))})$$

$$= \dim_{\mathbb{C}}(V^{(K_0,\psi_0)}) - \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}}(M_i^{((K_0,\psi_0))}).$$

The lemma will be proved by showing that this last expression is zero. If (K_0,ψ_0) is not $V^{(H,\phi)}$-admissible then (A_n) implies that every term in this sum vanishes. If (K_0,ψ_0) is $V^{(H,\phi)}$-admissible then we have $cl_V(H,\phi) = (H,\phi) < (K_0,\psi_0) \leq cl_V(K_0,\psi_0)$. This implies that $d_V(cl_V(K_0,\psi_0)) < d_V(H,\phi) = n+1$. However (D_n) implies that the chain complex

$$0 \longrightarrow M_n^{(cl_V(K_0,\psi_0))} \stackrel{\partial_{n-1}}{\longrightarrow} M_{n-1}^{(cl_V(K_0,\psi_0))} \stackrel{\partial_{n-2}}{\longrightarrow} \ldots$$

$$\stackrel{\partial_0}{\longrightarrow} M_0^{(cl_V(K_0,\psi_0))} \stackrel{\epsilon}{\longrightarrow} V^{cl_V(K_0,\psi_0)} = V^{(K_0,\psi_0)} \longrightarrow 0$$

is exact. In addition, by an argument used in the proof of Theorem 5.2(a) we have $M_i^{(cl_V(K_0,\psi_0))} = M_i^{((K_0,\psi_0))}$ for $0 \leq i \leq n$, which completes the proof. $\square$

Now we are going to establish the finite-modulo-centre versions of universal properties of $\mathbb{C}[G]$-monomial resolutions and then proceed to the construction for GL_2K with K local.

Lemma 5.4.

Assume we are in the finite modulo the centre case with the usual assumptions about the stabilising pairs which appear each satisfying $(Z(G), \underline{\phi}) \leq (H, \phi)$. Let

$$M \xrightarrow{g} N \xleftarrow{f} P$$

be a diagram with $M, P \in_{\mathbb{C}[G]}$-mon, $N \in_{\mathbb{C}[G]}$-mod with f, g both $\mathbb{C}[G]$-module homomorphisms. In particular, we allow f, g to be morphisms in $_{\mathbb{C}[G]}$-mon. Assume that

$$f(P^{((H,\phi))}) \subseteq g(M^{((H,\phi))})$$

for all $(H, \phi) \in \mathcal{M}_G$ with $(Z(G), \underline{\phi}) \leq (H, \phi)$. Then there exists a $_{\mathbb{C}[G]}$-mon morphism $h : P \longrightarrow M$ such that $g \cdot h = f$.

Proof

We may suppose that P is indecomposable and set $P = \underline{\mathrm{Ind}}_H^G(\phi)$. Consider the following diagram

$$M^{((H,\phi))} \xrightarrow{g} g(M^{((H,\phi))}) \xleftarrow{f} 1 \otimes_{\mathbb{C}[H]} \mathbb{C}_\phi \subseteq P^{((H,\phi))}.$$

Choose a $\mathbb{C}$-linear map $h' : 1 \otimes_{\mathbb{C}[H]} \mathbb{C}_\phi \longrightarrow M^{((H,\phi))}$ such that $g \cdot h' = f$ then h' is automatically a $\mathbb{C}[H]$-module map which therefore extends uniquely, by Frobenius reciprocity, to a morphism $h : P \longrightarrow M$ which also satisfies $g \cdot h = f$. $\square$

Proposition 5.5.

Assume we are in the finite modulo the centre case with the usual assumptions about the stabilising pairs which appear all satisfying $(Z(G), \underline{\phi}) \leq (H, \phi)$. Let

$$\ldots \longrightarrow M_n \xrightarrow{\partial_{n-1}} M_{n-1} \xrightarrow{\partial_{n-2}} \ldots \xrightarrow{\partial_0} M_0 \xrightarrow{\epsilon} V \longrightarrow 0$$

be a $\mathbb{C}[G]$-monomial resolution of V and let

$$\ldots \longrightarrow C_n \xrightarrow{\partial'_{n-1}} C_{n-1} \xrightarrow{\partial'_{n-2}} \ldots \xrightarrow{\partial'_0} C_0 \xrightarrow{\epsilon'} V \longrightarrow 0$$

a chain complex where each ∂'_i is a morphism of $\mathbb{C}[G]$-Line Bundles and ϵ' is a $\mathbb{C}[G]$-module homomorphism such that $\epsilon'(C_0^{((H,\phi))}) \subseteq V^{(H,\phi)}$ for each $(H, \phi) \in \mathcal{M}_G$ as in §5.1. Then there exists a chain map of morphisms of $\mathbb{C}[G]$-Line Bundles $\{f_i : C_i \longrightarrow M_i, i \geq 0\}$ such that

$$\epsilon \cdot f_0 = \epsilon',\ f_{i-1} \cdot \partial'_i = \partial_i \cdot f_i \text{ for all } i \geq 1.$$

In addition, if $\{f'_i : C_i \longrightarrow M_i, i \geq 0\}$ is another chain map of morphisms of $\mathbb{C}[G]$-Line Bundles such that $\epsilon \cdot f_0 = \epsilon \cdot f'_0$ then then exists a chain homotopy of morphisms of $\mathbb{C}[G]$-Line Bundles $\{s_i : C_i \longrightarrow M_{i+1},$ for all $i \geq 0\}$ such that $\partial_i \cdot s_i + s_{i-1} \cdot \partial'_i = f_i - f'_i$ for all $i \geq 1$ and $f_0 - f'_0 = \partial_0 \cdot s_0$.

Proof

This is the usual homological algebra argument using Lemma 5.4. $\square$

6. Monomial resolutions for GL_2K

Let K be a local field. In this section I shall use the well-known action of GL_2K on its tree ([65] p.69) to extend the finite modulo the centre monomial resolutions of Theorem 5.2 to a relative monomial resolution of an admissible irreducible representation of GL_2K which is unique up to chain homotopy in the monomial category and which satisfy a modified form of monomial exactness, which I shall christen a quasi-monomial resolution.

6.1. *The GL_2K-action on its tree*

A lattice in $K \oplus K$ is any finitely generated $\mathcal{O}_K$-submodule which generates $K \oplus K$ as a K-vector space. If $x \in K^*$ and L is a lattice then so also is xL. The homothety class of L is the orbit of L in the set of lattices under this K^*-action. The set of classes of lattices gives rise to a tree ([65] Chapter II) with a right GL_2K-action.

Let $H_1 = GL_2O_K$ and

$$H_2 = \begin{pmatrix} \pi_K & 0 \\ 0 & 1 \end{pmatrix} H_1 \begin{pmatrix} \pi_K & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

which are two of the maximal compact subgroups. All other maximal compact subgroups of GL_2K are conjugate to H_1. Explicitly we have

$$H_2 = \{ \begin{pmatrix} a & b\pi_K \\ c\pi_K^{-1} & d \end{pmatrix} \mid a,b,c,d \in \mathcal{O}_K,\ ad - bc \in \mathcal{O}_K^* \}.$$

By ([65] p.69 et seq) if $L = O_K \oplus O_K$ and $L' = O_K \oplus O_K\pi_K$ then $\mathrm{Stab}_{GL_2K}(L) = H_1 \cdot K^*$ and $\mathrm{Stab}_{GL_2K}(L') = H_2 \cdot K^*$ where GL_2K acts by right multiplication on the vector space $V = K \oplus K$. This fact will enable us to calculate some normalisers.

If $XH_1X^{-1} = H_1$ then $((L)X)H_1 = (L)H_1X = (L)X$ but from the tree structure each homothety class of a lattice is stabilised by a different maximal compact subgroup so that $H_1 \cdot K^*$ stabilises L and $(L)X$ and so $(L)X = L$ and $X \in H_1 \cdot K^*$. This shows that $N_{GL_2K}H_1 = H_1 \cdot K^*$. Similarly for H_2.

If $YH_1 \bigcap H_2Y^{-1} = H_1 \bigcap H_2$ then $(L)Y = (L)YH_1 \bigcap H_2$ and $(L')Y = (L')YH_1 \bigcap H_2$. Also $(H_1 \bigcap H_2) \cdot K^* \subseteq \mathrm{Stab}_{GL_2K}(L) \bigcap \mathrm{Stab}_{GL_2K}(L')$. Now the distance from L to L' is one [65], so they are adjacent in the graph, and the (pointwise) stabiliser of the edge they define is precisely $(H_1 \bigcap H_2) \cdot K^*$. Furthermore this is the only edge that this subgroup stabilises. But $(L)Y$ and $(L')Y$ are also adjacent and $H_1 \bigcap H_2 \cdot K^*$ also stabilises this edge so the edges coincide. This coincidence can happen in two ways. If the ordered pair $((L)Y, (L')Y)$ is equal to (L, L') then $Y \in H_1 \bigcap H_2 \cdot K^*$. On the other hand it is possible that $((L)Y, (L')Y)$ is equal to (L', L). In fact, the matrix calculation

$$\begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix} \begin{pmatrix} \alpha & \pi_K\beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & \pi_K \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \delta & \pi_K\gamma \\ \beta & \alpha \end{pmatrix}$$

shows that $H_1 \bigcap H_2$ is normalised by the matrix

$$\begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix},$$

which does not belong to $H_1 \bigcap H_2$. Therefore

$$N_{GL_2K}(H_1 \bigcap H_2) = \langle (H_1 \bigcap H_2) \cdot K^*, \begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix} \rangle.$$

This group stabilises the edge $\{L, L'\}$ but only the subgroup of index two $(H_1 \bigcap H_2) \cdot K^*$ maps the the ordered pair (L, L') to itself by the identity, other matrices interchange the order.

Therefore the Weyl groups of H_1, H_2 given by $N_{GL_2K}H_i/H_i$ are both isomorphic to $K^*/O_K^* \cong \mathbb{Z}$ generated by the scalar matrix π_K. The Weyl group $N_{GL_2K}(H_1 \bigcap H_2)/H_1 \bigcap H_2$ is isomorphic to the infinite cyclic group generated by

$$u = \begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix}$$

which contains a subgroup of index two given by $\langle u^2 \rangle = K^*/O_K^* \cong \mathbb{Z}$.

Now supose that $Z(H_1 \bigcap H_2)Z^{-1} \subset H_1$ then $(L)Z = (L)ZH_1$ and the preceding argument shows that $Z \in H_1 \cdot K^*$. Hence the coset space

$$\frac{H_1 \cdot K^*}{H_1 \cap H_2} \cong \frac{H_1}{H_1 \cap H_2} \times \mathbb{Z}$$

and $\frac{H_1}{H_1 \cap H_2}$ is in one-one correspondence with $\mathbb{P}^1(\mathcal{O}_K/(\pi_K))$ because this coset is isomorphic to the orbit of the edge LL' under the action of $\mathrm{Stab}_{GL_2K}(L)$ ([65] p.72).

The correspondence between $\mathbb{P}^1(\mathcal{O}_K/(\pi_K))$ and the set of lattices of distance one from L is described as follows in ([65] p.72). Let $L'' \subset L$ be such that $L/L'' \cong O_K/(\pi_K) \cong k$. Then we have a short exact sequence

$$0 \longrightarrow k \cong L''/\pi_K L \longrightarrow L/\pi_K L \cong k \oplus k \longrightarrow k \longrightarrow 0$$

which associates to L'' a linear subspace in $k \oplus k$ and hence a point in $\mathbb{P}^1(k)$.

Also, via the left action on lattices, since $H_1 \bigcap H_2$ stabilises the edge through L and L' we get a bijection

$$H_1 \cap H_2 \backslash GL_2\mathcal{O}_K \leftrightarrow \mathbb{P}^1(k).$$

The transpose of this bijection is given explicitly as follows. Suppose, for $a, b, c, d, \alpha, \beta, \gamma, \delta \in O_K$, that

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, Y = \begin{pmatrix} \alpha & \beta\pi_K \\ \gamma & \delta \end{pmatrix}$$

with $X \in GL_2O_K$, $Y \in H_1 \bigcap H_2$. Then

$$XY = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta\pi_K \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta\pi_K + b\delta \\ ca + d\gamma & c\beta\pi_K + d\delta \end{pmatrix}$$

and because $\alpha, \delta \in O_K^*$ we have a well-defined element

$$\begin{pmatrix} \frac{\overline{b}}{\overline{d}} \end{pmatrix} = \begin{pmatrix} \frac{\overline{b\delta}}{\overline{d\delta}} \end{pmatrix} \in \mathbb{P}^1(k)$$

depending only on the coset $XH_1 \bigcap H_2$. Hence representatives of this coset are given by

$$\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $b \in O_K$ runs through a set of representatives for k.

6.2. *The simplicial complex of the tree*

Now we are ready to calculate the simplicial chain complex of the tree together with its GL_2K-action. I am going to transpose to a left action on the tree by GL_2K.

The 1-cells are clearly given by the induced representation

$$C_1 = \mathrm{Ind}_{N_{GL_2K}(H_1 \cap H_2)}^{GL_2K}(\mathbb{C}_\tau) = \mathbb{C}[GL_2K] \otimes_{\mathbb{C}[N_{GL_2K}(H_1 \cap H_2)]} \mathbb{C}_\tau$$

where $\mathbb{C}_\tau$ is a copy of the complex numbers on which $(H_1 \bigcap H_2) \cdot K^*$ acts trivially and $\begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix}$ acts like -1. Here we are using the "crude" algebraic induction in terms of tensor product over group rings. In other words, individual simplices are far apart, as they are in the distance function on lattice classes in ([65] pp.69-70).

This crude induction can be topologised to coincide with compact induction aince we are inducing from compact open modulo the centre subgroups.

The 0-cells are given by the induced representation

$$C_0 = \mathrm{Ind}_{N_{GL_2K}H_1}^{GL_2K}(\mathbb{C})$$

where $\mathbb{C}$ has the trivial action. The simplicial differential

$$d : C_1 \longrightarrow C_0$$

is a GL_2K-map and so is determined by a $N_{GL_2K}(H_1 \cap H_2)$-map from $\mathbb{C}_\tau$ to C_0. This map is easily seen to be given by

$$d(1) = 1 \otimes_{N_{GL_2K}H_1} 1 - \begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix} \otimes_{N_{GL_2K}H_1} 1.$$

If $X \in N_{GL_2K}(H_1 \cap H_2) \subset N_{GL_2K}H_1$ then

$$Xd(1) = X \otimes_{N_{GL_2K}H_1} 1 - X \begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix} \otimes_{N_{GL_2K}H_1} 1 = d(1) = d(X \cdot 1)$$

while

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix} d(1) &= \begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix} \otimes_{N_{GL_2K}H_1} 1 - 1 \otimes_{N_{GL_2K}H_1} 1 \\ &= -d(1) \\ &= d(\begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix} \cdot 1). \end{aligned}$$

Since a tree is contractible we have an exact sequence of $\mathbb{C}[GL_2K]$-modules

$$0 \longrightarrow C_1 \stackrel{d}{\longrightarrow} C_0 \stackrel{\epsilon}{\longrightarrow} \mathbb{C} \longrightarrow 0$$

where, for $Z \in GL_2K$,

$$d(Z \otimes_{N_{GL_2K}(H_1 \cap H_2)} 1) = Z \otimes_{N_{GL_2K}H_1} 1 - Z \begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix} \otimes_{N_{GL_2K}H_1} 1$$

and $\epsilon(Y \otimes_{N_{GL_2K}H_1} v) = v$.

The above action is not simplicial because the subgroup preserving a given 1-simplex does not act on it by the identity. For example, $\{L, L'\}$ is inverted by

$$\begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix}.$$

However, it is easy barycentrically to subdivide the simplicial tree by adding L'', the midpoint of $\{L, L'\}$, and all its GL_2K-translates. The stabiliser of L'' is $N_{GL_2K}(H_1 \bigcap H_2)$. The result is a one-dimensional simplicial complex with 1-simplices given by $\{L, L''\}$ and its GL_2K-translates. The stabiliser of $\{L, L''\}$ is $(H_1 \bigcap H_2)K^*$ and the resulting GL_2K-action is simplicial. The 0-cells are given by

$$\tilde{C}_0 = \mathrm{Ind}_{N_{GL_2K}(H_1)}^{GL_2K}(\mathbb{C}) \oplus \mathrm{Ind}_{N_{GL_2K}(H_1 \cap H_2)}^{GL_2K}(\mathbb{C})$$

while the 1-cells are

$$\tilde{C}_1 = \mathrm{Ind}_{(H_1 \cap H_2)K^*}^{GL_2K}(\mathbb{C}).$$

Therefore we have a short exact sequence of $\mathbb{C}[GL_2K]$-modules of the form

$$0 \longrightarrow \tilde{C}_1 \stackrel{d}{\longrightarrow} \tilde{C}_0 \stackrel{\epsilon}{\longrightarrow} \mathbb{C} \longrightarrow 0$$

in which

$$d(g \otimes_{(H_1 \cap H_2)K^*} v) = (g \otimes_{N_{GL_2K}(H_1)} v, -g \otimes_{N_{GL_2K}(H_1 \cap H_2)} v)$$

and

$$\epsilon(g_1 \otimes_{N_{GL_2K}(H_1)} v_1, g_2 \otimes_{N_{GL_2K}(H_1 \cap H_2)} v_2) = v_1 + v_2.$$

6.3. If V is a representation of GL_2K we have, using crude induction, an isomorphism of GL_2K-representations

$$\phi : \mathrm{Ind}_H^{GL_2K}(W) \otimes V \stackrel{\cong}{\longrightarrow} \mathrm{Ind}_H^{GL_2K}(W \otimes \mathrm{Res}_H^{GL_2K}(V))$$

given by $\phi((g \otimes_H w) \otimes v) = g \otimes_H (w \otimes g^{-1}v)$. This is well-defined because

$$\phi((gh \otimes_H h^{-1}w) \otimes v) = gh \otimes_H (h^{-1}w) \otimes h^{-1}g^{-1}v) = g \otimes_H (w \otimes g^{-1}v)$$

and is a GL_2K-map because

$$g'\phi((g \otimes_H w) \otimes v) = g'g \otimes_H (w \otimes g^{-1}v) = \phi(g'(g \otimes_H w) \otimes g'v).$$

We have $N_{GL_2K}H_1 \bigcap H_2 = \langle H_1 \bigcap H_2, u\rangle$ where

$$u = \begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix}, \quad u^2 = \begin{pmatrix} \pi_K^{-1} & 0 \\ 0 & \pi_K^{-1} \end{pmatrix} \in Z(GL_2K) = K^*$$

and $N_{GL_2K}H_1 = \langle H_1, u^2\rangle = H_1 \cdot K^*$.

The homomorphism

$$\begin{array}{c} \mathrm{Ind}_{(H_1 \cap H_2)K^*}^{GL_2K}(\mathbb{C}) \otimes V \\ d \otimes 1 \ \downarrow \\ \mathrm{Ind}_{N_{GL_2K}(H_1)}^{GL_2K}(\mathbb{C}) \otimes V \oplus \mathrm{Ind}_{N_{GL_2K}(H_1 \cap H_2)}^{GL_2K}(\mathbb{C}) \otimes V \end{array}$$

transforms under ϕ to

$$\begin{array}{c} \mathrm{Ind}_{(H_1 \cap H_2)K^*}^{GL_2K}(V) \\ \psi \ \downarrow \\ \mathrm{Ind}_{N_{GL_2K}(H_1)}^{GL_2K}(V) \oplus \mathrm{Ind}_{N_{GL_2K}(H_1 \cap H_2)}^{GL_2K}(V) \end{array}$$

given by

$$\psi(g \otimes_{(H_1 \cap H_2)K^*} v) = (g \otimes_{N_{GL_2K}(H_1)} v, -g \otimes_{N_{GL_2K}(H_1 \cap H_2)} v)$$

because

$$\psi(\phi((g \otimes_{(H_1\cap H_2)K^*} 1) \otimes gv))$$

$$= \psi(g \otimes_{(H_1\cap H_2)K^*} v)$$

$$= (g \otimes_{N_{GL_2K}(H_1)} v, -g \otimes_{N_{GL_2K}(H_1\cap H_2)} v)$$

while

$$\phi(d \otimes 1((g \otimes_{(H_1\cap H_2)K^*} 1) \otimes gv))$$

$$= \phi((g \otimes_{N_{GL_2K}(H_1)} 1) \otimes gv, -g \otimes_{N_{GL_2K}(H_1\cap H_2)} 1) \otimes gv)$$

$$= (g \otimes_{N_{GL_2K}(H_1)} v, -g \otimes_{N_{GL_2K}(H_1\cap H_2)} v).$$

6.4. *Covering ψ by a monomial-morphism*

As explained below, I also have "modulo the centre" monomial resolutions

$$M_{1,*} \stackrel{\epsilon_1}{\longrightarrow} V$$

for $\mathbb{C}[(H_1 \cap H_2)K^*]$,

$$M_{0,*} \stackrel{\epsilon_0}{\longrightarrow} V$$

for $\mathbb{C}[N_{GL_2K}(H_1)]$ and

$$M'_{0,*} \stackrel{\epsilon'_0}{\longrightarrow} V$$

for $\mathbb{C}[N_{GL_2K}(H_1 \cap H_2)]$.

Note that $u^2 \in Z(GL_2K)$ so that all characters we shall meet are given by $\underline{\phi}(u^2)$ on u^2.

The restriction of V to each of the compact modulo the centre subgroups $(H_1 \bigcap H_2)K^*$, $N_{GL_2K}H_1$ and $N_{GL_2K}(H_1 \bigcap H_2)$ is a countable direct sum of finite-dimensional complex representations where the group acts through a finite modulo the centre quotient with a common central character given by $\underline{\phi}$ on K^*. Therefore the monomial resolutions $M_{1,*}$, $M_{0,*}$ and $M'_{0,*}$ exist by forming the direct sum of the monomial resolutions of the finite modulo centre summands obtained by applying Theorem 5.2 to each summand.

The first objective is to produce a $\mathbb{C}[GL_2K]$-morphism

$$\psi_0 : \underline{\mathrm{Ind}}_{(H_1\cap H_2)K^*}^{GL_2K}(M_{1,0}) \longrightarrow \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(M_{0,0}) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1\cap H_2)}^{GL_2K}(M'_{0,0})$$

to commute with the augmentations. That is,

$$(\underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(\epsilon_0) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1\cap H_2)}^{GL_2K}(\epsilon'_0))\psi_0 = \psi\underline{\mathrm{Ind}}_{(H_1\cap H_2)K^*}^{GL_2K}(\epsilon_1)$$

as

$$\underline{\mathrm{Ind}}_{(H_1\cap H_2)K^*}^{GL_2K}(M_{1,0}) \longrightarrow \mathrm{Ind}_{N_{GL_2K}(H_1)}^{GL_2K}(V) \oplus \mathrm{Ind}_{N_{GL_2K}(H_1\cap H_2)}^{GL_2K}(V).$$

We begin by constructing a $\mathbb{C}[GL_2K]$-module homomorphism and then we sort out the behaviour of Lines under the map.

Start with a Line from $\underline{\mathrm{Ind}}_{(H_1\cap H_2)K^*}^{GL_2K}(M_{1,0})$ with stabiliser pair (J, ϕ) where ϕ is a character of J. Recall that $(Z(GL_2K), \underline{\phi}) \leq (J, \phi)$. Since we need only a representative from the GL_2K-orbit of the Line we may as well assume that

$J \subseteq (H_1 \cap H_2)K^*$ and the Line is generated by $1 \otimes_{(H_1\cap H_2)K^*} (1 \otimes_J 1)$ so that $j \in J$ acts on this line via $j(1 \otimes_{(H_1\cap H_2)K^*} (1 \otimes_J 1)) = \phi(j) \otimes_{(H_1\cap H_2)K^*} (1 \otimes_J 1)$ and $v = \epsilon_1(1 \otimes_J 1) \in V^{(J,\phi)}$.

Now consider the two terms in

$$\begin{aligned}\psi(\epsilon_1(1 \otimes_{(H_1\cap H_2)K^*} (1 \otimes_J 1)) &= \psi(1 \otimes_{(H_1\cap H_2)K^*} v) \\ &= 1 \otimes_{N_{GL_2K}(H_1)} v - 1 \otimes_{N_{GL_2K}(H_1\cap H_2)} v.\end{aligned}$$

The action of $j \in J \bigcap H_1 \bigcap H_2$ on each of these terms is by multiplication by $\phi(j)$. Therefore there exists $w \in M_{(0,0}^{((J,\phi))}$ and $w' \in {M'_{(0,0}}^{((J,\phi))}$ such that $\epsilon_0(w) = v, \epsilon'_0(w') = v$ so that

$$1 \otimes_{N_{GL_2K}(H_1)} \epsilon_0(w) - 1 \otimes_{N_{GL_2K}(H_1\cap H_2)} \epsilon'_0(w') = \psi(\epsilon_1(1 \otimes_{(H_1\cap H_2)K^*} (1 \otimes_J 1))).$$

Set

$$\psi_0(1 \otimes_{(H_1\cap H_2)K^*} (1 \otimes_J 1))) = 1 \otimes_{N_{GL_2K}(H_1)} w - 1 \otimes_{N_{GL_2K}(H_1\cap H_2)} w'.$$

This defines a $\mathbb{C}[GL_2K]$-morphism ψ_0 which commutes with augmentations and satisfies

$$\psi_0(\underline{\mathrm{Ind}}_{(H_1\cap H_2)K^*}^{GL_2K}(M_{1,0}^{((J,\phi))})) \subseteq \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(M_{0,0}^{((J,\phi))}) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1\cap H_2)}^{GL_2K}({M'_{0,0}}^{((J,\phi))})$$

Now, by induction, we construct similar chain maps for all $i \geq 0$

$$\psi_i : \underline{\mathrm{Ind}}_{(H_1\cap H_2)K^*}^{GL_2K}(M_{1,i}) \longrightarrow \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(M_{0,i}) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1\cap H_2)}^{GL_2K}(M'_{0,i})$$

which commute with the differentials and satisfy

$$\psi_i(\underline{\mathrm{Ind}}_{(H_1\cap H_2)K^*}^{GL_2K}(M_{1,i}^{((J,\phi))}) \subseteq \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(M_{0,i}^{((J,\phi))}) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1\cap H_2)}^{GL_2K}({M'_{0,i}}^{((J,\phi))}).$$

We start with a Line in $\underline{\mathrm{Ind}}_{(H_1\cap H_2)K^*}^{GL_2K}(M_{1,i})$ with $i \geq 1$ (up to GL_2K-conjugacy) generated by $1 \otimes_{(H_1\cap H_2)K^*} \otimes (1 \otimes_J 1)$. The stabilising pair of $\langle 1 \otimes_J 1\rangle$ is (J, ϕ) and the differential in $M_{1,*}$ induces a differential given by

$$d(1 \otimes_{(H_1\cap H_2)K^*} \otimes (1 \otimes_J 1)) = 1 \otimes_{(H_1\cap H_2)K^*} \otimes d(1 \otimes_J 1)$$

where $d(1 \otimes_J 1) \in M_{1,i-1}^{((J,\phi))}$. Also $d(1 \otimes_J 1)$ lies in the kernel of the differential if $i \geq 2$ and of the augmentation if $i = 1$. Therefore, by induction,

$$\psi_{i-1}(d(1 \otimes_{(H_1\cap H_2)K^*} \otimes (1 \otimes_J 1))) \in \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(M_{0,i-1}^{((J,\phi))}) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1\cap H_2)}^{GL_2K}({M'_{0,i-1}}^{((J,\phi))})$$

also lies in the kernel of the differential if $i \geq 2$ and of the augmentation if $i = 1$. By monomial exactness of $M_{0,*}$ there exists

$$(w, w') \in \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(M_{0,i}^{((J,\phi))}) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1\cap H_2)}^{GL_2K}({M'_{0,i}}^{((J,\phi))})$$

such that

$$d(w, w') = \psi_{i-1}(d(1 \otimes_{(H_1\cap H_2)K^*} \otimes (1 \otimes_J 1)))$$

and so we set $\psi_i(1 \otimes_{(H_1\cap H_2)K^*} \otimes (1 \otimes_J 1)) = (w, w')$.

Similarly any two constructions of ψ_* will be chain homotopic as monomial-morphisms because the monomial resolutions $M_{0,*}$ and $M_{1,*}$ are unique up to chain homotopy by Proposition 5.5.

6.5. *Some elementary homological algebra*

We now consider the chain complex $\underline{M}_* \longrightarrow V$ in which for $i \geq 0$

$$\underline{M}_i = \underline{\mathrm{Ind}}_{(H_1 \cap H_2)K^*}^{GL_2K}(M_{1,i-1}) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(M_{0,i}) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1 \cap H_2)}^{GL_2K}(M'_{0,i})$$

with differential

$$d(w_{1,i-1}, w_{0,i}, w'_{0,i}) = (d(w_{1,i-1}), d(w_{0,i}, w'_{0,i}) + (-1)^i \psi_{i-1}(w_{1,i-1})).$$

This is a chain complex because

$$dd(w_{1,i-1}, w_{0,i})$$

$$= (dd(w_{1,i-1}), dd(w_{0,i}, w'_{0,i}) + (-1)^i d\psi_i(w_{1,i-1}) + (-1)^{i-1}\psi_{i-2}(dw_{1,i-1}))$$

$$= (0, (-1)^i d\psi_{i-1}(w_{1,i-1}) + (-1)^{i-1}\psi_{i-2}d(w_{1,i-1}))$$

which is zero because $d\psi_{i-1} = \psi_{i-2}d$, by construction.

If we have two chain complexes

$$\ldots \longrightarrow A_i \longrightarrow A_{i-1} \longrightarrow \ldots \longrightarrow A_{-1} \longrightarrow 0$$

and

$$\ldots \longrightarrow B_i \longrightarrow B_{i-1} \longrightarrow \ldots \longrightarrow B_{-1} \longrightarrow 0$$

with a chain map f_* between them such that

$$0 \longrightarrow A_{-1} \stackrel{f_{-1}}{\longrightarrow} B_{-1} \longrightarrow V \longrightarrow 0$$

is a short exact sequence, consider the mapping cone chain complex $N_i = A_{i-1} \oplus B_i$ with differential $d(a_{i-1}, b_i) = (d(a_{i-1}), d(b_i) + (-1)^i f_{i-1}(a_{i-1}))$. We have a short exact sequence of chain complexes

$$0 \longrightarrow B_* \longrightarrow N_* \longrightarrow N_*/B_* \longrightarrow 0$$

for $* \geq 0$. Since $N_i/B_i \cong A_{i-1}$ for $i \geq 1$ we have a long exact homology sequence of the form

$$\ldots \longrightarrow H_i(B) \longrightarrow H_i(N) \longrightarrow H_{i-1}(A) \stackrel{\partial}{\longrightarrow} H_{i-1}(B) \longrightarrow \ldots$$

where $\partial = (-1)^i f_{i-1}$ on $H_{i-1}(A)$. If A_*, B_* are exact (not just in positive dimensions) then we have $H_i(N_*) = 0$ for $i > 0$ while

$$0 \longrightarrow A_{-1} \stackrel{\partial}{\longrightarrow} B_{i-1} \longrightarrow H_0(N_*) \longrightarrow 0$$

yields an isomorphism $H_0(N_*) \cong V$ induced by

$$N_0 \longrightarrow B_0 \longrightarrow B/d(B_1) \cong B_{-1} \longrightarrow V.$$

6.6. *Monomial exactness*

Consider the chain complex

$$\ldots \longrightarrow \underline{M}_i \stackrel{d}{\longrightarrow} \underline{M}_{i-1} \longrightarrow \ldots \longrightarrow \underline{M}_0 \longrightarrow V \longrightarrow 0.$$

Each of the $\underline{M}_i$'s is a Line-bundle with Lines generated by $g \otimes_{(H_1 \bigcap H_2)K^*} L_1$, $g \otimes_{N_{GL_2K}(H_1)} L_0$ or $g \otimes_{N_{GL_2K}(H_1 \cap H_2)} L_0'$ with L_1, L_0, L_0' being Lines in $M_{1,*}$, $M_{0,*}$ or $M_{0,*}'$, respectively.

A Line of the form $g \otimes_{(H_1 \bigcap H_2)K^*} L_1$, $g \otimes_{N_{GL_2K}(H_1)} L_0$ or $g \otimes_{N_{GL_2K}(H_1 \cap H_2)} L_0'$ has stabiliser of the form $g(J', \phi')g^{-1}$ where $J' \subseteq (H_1 \bigcap H_2)K^*$, $N_{GL_2K}(H_1)$ or $N_{GL_2K}(H_1 \cap H_2)$, respectively.

Let $(J, \phi) \in \mathcal{M}_{GL_2K}$ with $(K^*, \underline{\phi}) \leq (J, \phi)$ and J being compact open modulo the centre K^*. We wish to examine exactness in the middle of

$$\underline{M}_{i+1}^{((J,\phi))} \longrightarrow \underline{M}_i^{((J,\phi))} \longrightarrow \underline{M}_{i-1}^{((J,\phi))}$$

for $i \leq 1$.

Consider the inclusions:

$$H_1K^* = N_{GL_2K}(H_1) \geq (H_1 \cap H_2)K^* \leq N_{GL_2K}(H_1 \cap H_2) = \langle H_1 \bigcap H_2, u \rangle.$$

Since J is compact open modulo the centre K^* and H_1 is a maximal compact subgroup to which all other such subgroups are conjugate we must have J GL_2K-conjugate to a subgroup of H_1K^*.

Therefore, up to GL_2K-conjugation, we must have one of the following cases:

Case A: $J \subseteq (H_1 \bigcap H_2)K^*$.

Case B: $J \subseteq H_1K^*$, J is conjugate to a subgroup of $\langle H_1 \bigcap H_2, u \rangle$ but is not conjugate to a subgroup of $(H_1 \bigcap H_2)K^*$.

Case C: $J \subseteq H_1K^*$, but J is not conjugate to a subgroup of either $\langle H_1 \bigcap H_2, u \rangle$ or $(H_1 \bigcap H_2)K^*$.

The following result shows that Cases A-C exhaust the possibilities.

Proposition 6.7.

In Case D there exists $g \in GL_2K$ such that $gJg^{-1} \subseteq (H_1 \bigcap H_2)K^*$.

Proof

Observe that $H_1K^* \bigcap \langle H_1 \bigcap H_2, u \rangle$ stabilises the two ends of the 1-simplex whose stabiliser is $(H_1 \bigcap H_2)K^*$. Pro tem, call this 1-simplex the canonical fundamental domain. Hence

$$H_1K^* \bigcap \langle H_1 \bigcap H_2, u \rangle = (H_1 \bigcap H_2)K^*.$$

Now we may assume that $J \subseteq \langle H_1 \bigcap H_2, u \rangle$ and that there exists $g \in GL_2K$ such that $gJg^{-1} \subseteq H_1K^*$. Hence J stabilises the end-point, β, of the canonical fundamental domain which was introduced during the barycentric subdivision and also stablises $g\alpha$ where α is the other end of the canonical fundamental domain. Since the tree contains no closed loops J stabilises all the 1-simplices between β and $g\alpha$. In particular J stabilises the canonical fundamental domain or its neighbour. In the first case

$$J \subseteq H_1K^* \bigcap \langle H_1 \bigcap H_2, u\rangle = (H_1 \bigcap H_2)K^*$$

and in the second case

$$u^{-1}Ju \subseteq H_1K^* \bigcap \langle H_1 \bigcap H_2, u\rangle = (H_1 \bigcap H_2)K^*.$$

□

6.8. *Monomial exactness continued*

Now let us examine $\underline{M}_i^{((J,\phi))}$ in Case A.

We have a short exact sequence of chain complexes

$$0 \longrightarrow \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(M_{0,*}) \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1 \cap H_2)}^{GL_2K}(M'_{0,*})$$

$$\longrightarrow \underline{M}_* \longrightarrow \underline{\mathrm{Ind}}_{(H_1 \cap H_2)K^*}^{GL_2K}(M_{1,*-1}) \longrightarrow 0$$

and taking the $((J,\phi))$-part yields a short exact sequence of the form

$$0 \longrightarrow \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1)}^{GL_2K}(M_{0,*})^{((J,\phi))} \oplus \underline{\mathrm{Ind}}_{N_{GL_2K}(H_1 \cap H_2)}^{GL_2K}(M'_{0,*})^{((J,\phi))}$$

$$\longrightarrow \underline{M}_*^{((J,\phi))} \longrightarrow \underline{\mathrm{Ind}}_{(H_1 \cap H_2)K^*}^{GL_2K}(M_{1,*-1})^{((J,\phi))} \longrightarrow 0$$

Let L be a Line in M so that $g \otimes_H L$ generates a Line in $\underline{Ind}_H^{GL_2K}(M)$. This Line lies in $\underline{Ind}_H^{GL_2K}(M)^{((J,\phi))}$ if and only if $g^{-1}Jg \subseteq H$ and $g^{-1}Jg$ acts on L via $g^*(\phi)$. That is, $g^{-1}jg(v) = \phi(j)v$ for $v \in L$.

Therefore the left-hand group in the short exact sequence is equal to

$$\bigoplus_{g^{-1}Jg \subseteq N_{GL_2K}(H_1)} g \otimes_{N_{GL_2K}(H_1)} M_{0,*}^{((g^{-1}Jg, g^*(\phi)))}$$

$$\bigoplus$$

$$\bigoplus_{g^{-1}Jg \subseteq N_{GL_2K}(H_1 \cap H_2)} g \otimes_{N_{GL_2K}(H_1 \cap H_2)} {M'_{0,*}}^{((g^{-1}Jg, g^*(\phi)))}$$

while the right-hand group is equal to

$$\bigoplus_{g^{-1}Jg \subseteq (H_1 \cap H_2)K^*} g \otimes_{(H_1 \cap H_2)K^*} M_{1,*-1}^{((g^{-1}Jg, g^*(\phi)))}.$$

These direct sums have to be interpreted with care. For example, that for the right-hand group means that we choose coset representatives $\{g_\alpha, \alpha \in \mathcal{A}\}$ then we form the direct sum over the g_α's such that $g_\alpha^{-1}Jg_\alpha \subseteq (H_1 \cap H_2)K^*$ of $g_\alpha \otimes_{(H_1 \cap H_2)K^*} L$ where L runs through the Lines of $M_{1,*-1}^{((g_\alpha^{-1}Jg_\alpha, g_\alpha^*(\phi)))}$. The differential on such a line maps it by $1 \otimes d$ to $g_\alpha \otimes_{(H_1 \cap H_2)K^*} d(L)$. Hence the complex is the direct sum of subcomplexes, one for each g_α.

Therefore, by monomial exactness of $M_{0,*}, M'_{0,*}, M_{1,*}$, we have $H_i(\underline{M}_*^{((J,\phi))}) = 0$ for $i \geq 2$ and there is an exact homology sequence of the form

$$0 \longrightarrow H_1(\underline{M}_*^{((J,\phi))}) \longrightarrow \oplus_{g^{-1}Jg \subseteq (H_1 \cap H_2)K^*} \; g \otimes_{(H_1 \cap H_2)K^*} V^{(g^{-1}Jg, g^*(\phi))}$$

$$\longrightarrow \oplus_{g^{-1}Jg \subseteq N_{GL_2K}(H_1)} \; g \otimes_{N_{GL_2K}(H_1)} V^{(g^{-1}Jg, g^*(\phi))}$$

$$\oplus \;\; \oplus_{g^{-1}Jg \subseteq N_{GL_2K}(H_1 \cap H_2)} \;\; g \otimes_{N_{GL_2K}(H_1 \cap H_2)} V^{(g^{-1}Jg, g^*(\phi))}$$

$$\longrightarrow H_0(\underline{M}_*^{((J,\phi))}) \longrightarrow 0.$$

Suppose that $g^{-1}Jg \subseteq (H_1 \cap H_2)K^*$ but that $g \notin (H_1 \cap H_2)K^*$. Then $gL \neq L$ where L denotes the canonical fundamental domain. On the other hand J fixes both L and gL. Since the tree has no closed loops this happens only if $J = \{1\}$. A similar argument applies to the other direct sums in the exact sequence, replacing the 1-simplex L by a vertex. Therefore if $J \neq \{1\}$ then the exact sequence takes the form

$$0 \longrightarrow H_1(\underline{M}_*^{((J,\phi))}) \longrightarrow V^{(J,\phi)} \longrightarrow V^{(J,\phi)} \oplus V^{J,\phi)} \longrightarrow H_0(\underline{M}_*^{((J,\phi))}) \longrightarrow 0.$$

In addition the map in the centre is given by $(v \mapsto (v, -v)$ so that

$$H_i(\underline{M}_*^{((J,\phi))}) = \begin{cases} V^{J,\phi)} & \text{if } i = 0, \\ \\ 0 & \text{otherwise.} \end{cases}$$

When $J = \{1\}$ the exact sequence becomes

$$0 \longrightarrow H_1(\underline{M}_*^{((\{1\},1))}) \longrightarrow \mathrm{Ind}_{(H_1 \cap H_2)K^*}^{GL_2K} V$$

$$\xrightarrow{\psi} \mathrm{Ind}_{N_{GL_2K}(H_1)}^{GL_2K} V \oplus \mathrm{Ind}_{N_{GL_2K}(H_1 \cap H_2)}^{GL_2K} V$$

$$\longrightarrow H_0(\underline{M}_*^{((J,\phi))}) \longrightarrow 0.$$

Therefore when J is trivial we also have

$$H_i(\underline{M}_*^{((\{1\},1))}) = \begin{cases} V & \text{if } i = 0, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Next we show that Case B cannot happen. Suppose otherwise and assume, by conjugation, that $J \subseteq \langle H_1 \bigcap H_2, u \rangle$ and $g^{-1}Jg \subseteq H_1K^*$. Therefore J fixes the vertex α of the canonical fundamental domain which was added upon barycentric subdivision. Also J fixes g β where β is the other vertex of the canonical fundamental domain. Since the tree has no closed loops J fixes all simplices between α and $g\beta$. Therefore, if L is the canonical fundamental domain then J fixes L or uL and so either $J \subseteq (H_1 \bigcap H_2)K^*$ or $uJu^{-1} \subseteq (H_1 \bigcap H_2)K^*$.

In Case C, by a similar argument we find that $H_i(\underline{M}_*^{((J,\phi))}) = 0$ for $i \neq 0$ and

$$H_0(\underline{M}_*^{((J,\phi))}) \cong \oplus_{g^{-1}Jg \subseteq N_{GL_2K}(H_1)} \; g \otimes_{N_{GL_2K}(H_1)} V^{(g^{-1}Jg, g^*(\phi))}.$$

However, if there exists $g \notin N_{GL_2K}(H_1)$ such that $gJg^{-1} \subseteq N_{GL_2K}(H_1)$ then J fixes β and $g\beta$. Therefore J fixes all simplices between β and $g\beta$ which include a translate of the canonical fundamental domain L so that J is subconjugate to $(H_1 \bigcap H_2)K^*$. Therefore there is only one coset in the above direct sum and $H_0(\underline{M}_*^{((J,\phi))}) \cong V^{(J,\phi)}$. *Therefore in all cases we have*

$$H_i(\underline{M}_*^{((J,\phi))}) = \begin{cases} V^{J,\phi)} & \text{if } i = 0, \\ & \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6.9. *Cuspidality, Jacquet modules etc*

If $H \subset GL_2K$, following ([26] p.16), define the subspace spanned by elements of the form $v - h \cdot v$

$$V(H) = \langle v - h \cdot v \text{ for } v \in V, h \in H \rangle.$$

If H is a compact open subgroup then $V = V^H \oplus V(H)$ ([26] Corollary 2, p.16).

In the case when K is a finite dimensional complex representation of $GL_2\mathbb{F}_q$ is cuspidal if and only if it is irreducible and $V^N = 0$, which is equivalent to $V/V(N) = 0$, the vanishing of the coinvariants ([80] p.111).

In the local case GL_2K the Jacquet module V_N of a smooth representation (π, V) is the coinvariant quotient $V_N = V/V(N)$. It is a smooth representation of the torus $\mathbb{T}$ of diagonal matrices. If (π, V) is irreducible and smooth then the non-vanishing of V_N is equivalent to V being a subrepresentation of $\mathrm{Ind}_B^{GL_2K}(\mathrm{Inf}_{\mathbb{T}}^B(\chi))$ for some character χ of $\mathbb{T}$ ([26] p.62).

An irreducible smooth representation (π, V) of GL_2K is cuspidal if $V_N = 0$. In the following result we observe that cuspidality implies if $V^N = 0$ in the case of a local field.

Remark 6.10. It may be that, in the locally p-adic case the correct condition for a monomial resolution is in terms of coinvariants $V(H)$ rather than invariants V^H.

Proposition 6.11.

Let K be a p-adic local field. Let V be a smooth irreducible representation of GL_2K. Then $V^N \bigcap V(N) = 0$. In particular, if V is cuspidal then $V^N = 0$.

Proof

We shall only need to consider V as a representation of N. Define a compact open subgroup of N by

$$H_r = \{ \begin{pmatrix} 1 & w \\ & \\ 0 & 1 \end{pmatrix} \mid w \in \mathcal{O}_K \pi_K^r \}.$$

Hence $H_r \subset H_{r-1}$ and $N = \bigcup H_r$. Therefore $V(N) = \bigcup V(H_r)$. Now given $v \in V^N \bigcap V(N)$ we have $v \in V(H_r)$ for some r. However $V^N \subseteq V^{H_r}$ so that $v \in V^{H_r} \bigcap V(H_r) = 0$ by ([26] Corollary 2, p.16). Here we have used that H_r is compact, open in N (but not in GL_2K) which is sufficient for our proof since we are applying ([26] Corollary 2, p.16) to V as an N-representation.

If V is cuspidal then $V(N) = V$ so $0 = V^N \bigcap V(N) = V^N$. □

7. Synopsis of results so far for GL_2K, SL_2K, GL_2K^0 and GL_2K^+

7.1. *Some subgroups of GL_2K*

Let K be a p-adic local field with valuation $v_K : K^* \longrightarrow \mathbb{Z}$. We have homomorphisms

$$\det : GL_2K \longrightarrow K^* \text{ and } v_K \cdot \det : GL_2K \longrightarrow \mathbb{Z}.$$

Following ([65] p.75) we may define subgroups of GL_2K denoted by SL_2K, GL_2K^0 and GL_2K^+ by

$$SL_2K = \mathrm{Ker}(\det),\ GL_2K^0 = \mathrm{Ker}(v_K \cdot \det),\ GL_2K^+ = \mathrm{Ker}(v_K \cdot \det \text{ modulo } 2)$$

so that

$$SL_2K \subset GL_2K^0 \subset GL_2K^+ \subset GL_2K.$$

As explained in ([65] pp.78/79) and in terms of Tits buildings (i.e. BN-pairs etc) in ([65] p.91) each of the first three groups acts transitively on the vertices of the tree and act on a 1-simplex between adjacent vertices simplicially (i.e. any element sending the 1-simplex to itself does so point-wise).

The following result summarises the properties of the quasi-monomial resolutions for representations of GL_2K.

Theorem 7.2.

Let V be an admissible irreducible representation of GL_2K. As in §6.1 let $H_1 = GL_2O_K$ and

$$H_2 = \{ \begin{pmatrix} a & b\pi_K \\ c\pi_K^{-1} & d \end{pmatrix} \mid a, b, c, d \in \mathcal{O}_K,\ ad - bc \in \mathcal{O}_K^* \}.$$

As in §6.3 set

$$u = \begin{pmatrix} 0 & 1 \\ \pi_K^{-1} & 0 \end{pmatrix} \in GL_2K.$$

(i) There exist monomial resolutions

$$M_{1,*} \stackrel{\epsilon_1}{\longrightarrow} V$$

for $\mathbb{C}[(H_1 \cap H_2)K^*]$,

$$M_{0,*} \stackrel{\epsilon_0}{\longrightarrow} V$$

for $\mathbb{C}[N_{GL_2K}(H_1)]$ and

$$M'_{0,*} \stackrel{\epsilon'_0}{\longrightarrow} V$$

for $\mathbb{C}[N_{GL_2K}(H_1 \cap H_2)]$ from which a monomial resolution

$$\underline{M}_* \longrightarrow V \longrightarrow 0$$

may be constructed by the process of §6.5.

(ii) This monomial resolution is unique up to chain homotopy of such resolutions.

(iii) The Line Bundles of $\underline{M}_*$ are admissible in the sense that the stabilising pair of any Line is compact modulo the centre $Z(GL_2K)$.

(iv) The analogous construction gives admissible monomial resolutions for any of the groups SL_2K, GL_2K^0 and GL_2K^+.

Theorem 7.3. *The functorial monomial bar resolution*

For each compact, open modulo the centre subgroup H of GL_nK with K local and admissible representation V (with a central character $\underline{\phi}$) of GL_nK there is a kH-monomial resolution

$$W_{*,H} \longrightarrow V \longrightarrow 0$$

which is covariantly functorial in H. That is, if $H \longrightarrow H'$ is a homomorphism then there is a kH-monomial chain map $kH$of monomial resolutions

$$i_{H,H'} : W_{*,H} \longrightarrow W_{*,H'}$$

commuting with the augmentation maps to V.

7.4. *An alternative construction of the GL_2K-monomial resolution*

Let G be a locally p-adic Lie group such as $G = GL_nK$ for K a local field. Let Y be a simplicial complex upon which G acts simplicially and in which the stabiliser $H_\sigma = \mathrm{stab}_G(\sigma)$ is compact, open modulo the centre of G. An example of this is GL_nK acting on a suitable subdivision of its building. Let V be an irreducible, admissible representation of G over k. For each simplex σ we have a kH_σ-bar-monomial resolution

$$W_{*,H_\sigma} \longrightarrow V \longrightarrow 0.$$

Form the graded k-vectorspace which in degree m is equal to

$$\underline{M}_m = \oplus_{\alpha+n=m} W_{\alpha,H_{\sigma^n}}.$$

If σ^{n-1} is a face of σ^n there is an inclusion $H_{\sigma^n} \subseteq H_{\sigma^{n-1}}$. Therefore there is a monomial chain map

$$i_{H_{\sigma^n},H_{\sigma^{n-1}}} : W_{*,H_{\sigma^n}} \longrightarrow W_{*,H_{\sigma^{n-1}}}$$

such that

$$i_{H_{\sigma^{n-1}},H_{\sigma^{n-2}}} i_{H_{\sigma^n},H_{\sigma^{n-1}}} = i_{H_{\sigma^n},H_{\sigma^{n-2}}}.$$

If σ^{n-1} is a face of σ^n let $d(\sigma^{n-1}, \sigma^n)$ denote the incidence degree of σ^{n-1} in σ^n; this is ± 1. In the simplicial chain complex of Y

$$d(\sigma^n) = \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n)\sigma^{n-1}.$$

For $x \in W_{\alpha, H_{\sigma^n}}$ write

$$d_Y(x) = \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) \ i_{H_{\sigma^n}, H_{\sigma^{n-1}}}(x).$$

Let $d_{\sigma^n} : W_{\alpha, H_{\sigma^n}} \longrightarrow W_{\alpha-1, H_{\sigma^n}}$ denote the differential in the monomial resolution.

Define $\underline{d} : \underline{M}_m \longrightarrow \underline{M}_{m-1}$ when $m = \alpha + n$ by

$$\underline{d}(x) = d_Y(x) + (-1)^n d_{\sigma^n}(x).$$

Theorem 7.5.

In §7.4 $(\underline{M}_*, \underline{d})$ is a kG-monomial complex when $G = GL_n K$ and Y is the building of G.

When $G = GL_2 K$ $(\underline{M}_*, \underline{d})$ is chain homotopy equivalent to the monomial resolution of V previously denoted by $\underline{M}_*$.

Conjecture 7.6. For $n \geq 3$, K local and $G = GL_n K$

$$\longrightarrow \underline{M}_i \stackrel{d}{\longrightarrow} \underline{M}_{i-1} \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} \underline{M}_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is a kG-monomial resolution. That is, for each $(H, \phi) \in \mathcal{M}_G$

$$\longrightarrow \underline{M}_i^{((H,\phi))} \stackrel{d}{\longrightarrow} \underline{M}_{i-1}^{((H,\phi))} \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} \underline{M}_0^{((H,\phi))} \stackrel{\epsilon}{\longrightarrow} V^{(H,\phi)} \longrightarrow 0$$

is an exact sequence of k-vector spaces.

8. Relation of the quasi-monomial resolution with π_K-adic levels

8.1. For $n \geq 1$ consider the compact open modulo the centre subgroup $J_n = K^* \cdot U_n$ where $U_n = 1 + \pi_K^n M_2 \mathcal{O}_K$. In this section we shall assume that n is large enough such that the restriction of the central character $\underline{\phi}$ to $K^* \bigcap U_n$ is trivial. In this case we shall denote by

$$\underline{\phi} : K^* \cdot U_n \longrightarrow \mathbb{C}^*$$

the character which is given by $\underline{\phi}$ on K^* is trivial on U_n.

Theorem 8.2.

In the situation of §8.1

$$\underline{M}_*^{((K^* \cdot U_n, \underline{\phi}))} \longrightarrow V^{(K^* \cdot U_n, \underline{\phi})}$$

is a $K^* \cdot U_n / U_n$-monomial resolution, whose chain homotopy class contains a finitely generated $K^* \cdot U_n / U_n$-monomial resolution of finite length.

Proof

This follows from the fact that $\underline{M}_* \longrightarrow V \longrightarrow 0$ is a monomial resolution of V. $\square$

8.3. *Epsilon factors and L-functions*

If V is an admissible representation of GL_2K and $\underline{M}_* \longrightarrow V$ is a monomial resolution as in Theorem 7.5 one may construct epsilon factors for V by applying an integral to each Line given by an integral for character values which in the finite finite case specialises to the Kondo Gauss sums. These integrals respect induction from one compact, open modulo the centre subgroup to another.

I am **assuming** an analogue of the result concerning wild epsilon factors modulo p-power roots of unity [42] holds for all but a finite set of Lines with the result that a well-defined epsilon factor modulo p-power roots of unity is defined by a finite product of Kondo-style Gauss sums. Here I ought to mention that I slightly disagree with a fundamental result in [42] (see [73]) so the epsilon factor I propose may only be well defined up to ± 1 times a p-power root of unity.

I have yet to develop the approach of Tate's thesis to each Line to get the L-functions.

These methods would apply to GL_nK if Conjecture 7.6 holds.

9. Galois descent for GL_2K

9.1. Suppose that K is a p-adic local field and $\rho : GL_2K \longrightarrow GL(V)$ is a complex, irreducible admissible representation. Let K/F be a Galois extension and suppose that $z^*()$ is equivalent to ρ for each $z \in \mathrm{Gal}(K/F)$. Therefore for $z \in \mathrm{Gal}(K/F)$ there exists $X_z \in GL(V)$ such that

$$X_z \rho(g) X_z^{-1} = \rho(z(g))$$

for all $g \in GL_2K$. Therefore if $z, z_1 \in \mathrm{Gal}(K/F)$ replacing g by $z_1(g)$ gives

$$X_z \rho(z_1(g)) X_z^{-1} = \rho(zz_1(g))$$

and so

$$X_z \rho(z_1(g)) X_z^{-1} = X_z X_{z_1} \rho(g) X_{z_1}^{-1} X_z^{-1} = X_{zz_1} \rho(g) X_{zz_1}^{-1}.$$

By Schur's Lemma $X_{z_1}^{-1} X_z^{-1} X_{zz_1}$ is a scalar matrix and so

$$f(z, z_1) = X_{z_1}^{-1} X_z^{-1} X_{zz_1}$$

is a function from $\mathrm{Gal}(K/F) \times \mathrm{Gal}(K/F)$ to $\mathbb{C}^*$. In fact, f is a 2-cocycle.

By a result of Tate $H^2(F; \mathbb{C}^*) = 0$ if K is local or global. Therefore there exists a finite Galois extension E/F such that the 2-cocycle induced by f

$$f' : \mathrm{Gal}(E/F) \times \mathrm{Gal}(E/F) \longrightarrow \mathbb{C}^*$$

is a coboundary $f' = dF$. Then $z \mapsto X_z F(z)$ is a homomorphism from $\mathrm{Gal}(E/F) \longrightarrow GL(V)$.

Recall that the semi-direct product $\mathrm{Gal}(E/F) \propto GL_2K \propto G$ is given by the set $\mathrm{Gal}(E/F) \times GL_2K$ with the product defined by

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1h_2, g_1h_1(g_2)).$$

The map

$$\tilde{\rho} : \mathrm{Gal}(E/F) \propto GL_n K \longrightarrow GL(V)$$

which sends $z, g)$ to $\rho(g)X_z F(z)$ is an irreducible admissible representation extending ρ. Any two such extensions differ by twisting via a homomorphism $\mathrm{Gal}(E/F) \longrightarrow \mathbb{C}^*$ for some E.

The action of $\mathrm{Gal}(K/F)$ on $K \oplus K$ preserves the lattice $L = \mathcal{O}_K \oplus \mathcal{O}_K$ and $L' = \mathcal{O}_K \oplus \pi_K \mathcal{O}_K$ and their stabilisers under the tree-action H_1 and H_2 Therefore the Galois action fixes the canonical fundamental domain on the tree and the semi-direct product acts on the tree of $GL_2 K$, extending the action of $GL_2 K$.

Replacing H_1 and H_2 by $\mathrm{Gal}(E/K) \propto H_1$ and $\mathrm{Gal}(E/K) \propto H_2$ yields the following result.

Theorem 9.2.

There exists a monomial resolution of $\tilde{\rho}$ which is unique up to chain homotopy and satisfies the analogue of Theorem 8.2.

9.3. *The Galois descent yoga*

Take ρ and form the monomial resolution of $\tilde{rho}$ as in Theorem 9.2. Quotient out the monomial complex by the Lines whose stabiliser group is not sub-conjugate in the semi-direct product to $\mathrm{Gal}(E/F) \times GL_n F$. This is a monomial complex for the semi-direct product which originates, via induction, with $\mathrm{Gal}(E/F) \times GL_n F$.

In one case of finite general linear groups this yoga is equivalent to Shintani descent. See [74].

I conjecture (on the basis of only one case!!) that Galois base change for admissible representations of GL_n of local fields can be described in terms of the above yoga with monomial resolutions.

10. Restricted tensor products and global automorphic representations

10.1. Automorphic representations of $GL_2 F$ where F is a number field are constructed by the tensor product theorem. For convenience let $F = \mathbb{Q}$, the rationals, so that we can just refer to [36]. Here is the tensor product theorem in that case - the reader is referred to [36] for details. Suffice to say tha the key to the construction is the fact that $V^{(GL_2 \mathbb{Z}_p, 1)}$ is one-dimensional for almost all primes p.

Theorem 10.2. *Tensor product theorem (*[36] *Vol. I Theorem 10.8.2 pp. 407)*

Let (π, V) denote an irreducible admissible $(\mathcal{U}(gl_2\mathbb{C}), K_\infty) \times GL_2 \mathbb{A}_{fin}$-module. Let $\{q_1, \ldots, q_m\}$ be the finite set of primes where π is ramified. Let $S = \{\infty, q_1, \ldots, q_m$. Then there exists

(i) an irreducible admissible $(\mathcal{U}(gl_2\mathbb{C}), K_\infty)$-module (π_∞, V_∞),

(ii) an irreducible admissible representation (π_p, V_p) of $GL_2\mathbb{Q}_p$ for each finite prime p,

(iii) a non-zero vector $v_p^0 \in V_p^{GL_2\mathbb{Z}_p}$ for each prime $\notin S$ such that

$$\pi \cong \bigotimes_{v \leq \infty}{}' \pi_v.$$

The factors are unique ([36] Vol. I Theorem 10.8.12 pp. 412).

10.3. *Restricted tensor products of monomial resolutions*

The restricted tensor product construction of Theorem 10.1 makes sense when applied to monomial resolutions of local admissible irreducible representation of $GL_2\mathbb{Q}_p$. One just restricts the tensor product of the monomial resolutions to lie in $M_*^{((\mathbb{Q}_p^p GL_2 Z_p, \underline{\phi}))}$ for primes for which $V^{(GL_2\mathbb{Z}_p, 1)}$ is one-dimensional.

This constructs a global monomial resolution, unique up to chain homotopy, for each global automorphic representation of $GL_2\mathbb{Q}$ or , more generally, for GL_2F for any number field, F.

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