NIM AND INVARIANTS IN ALGEBRAIC TOPOLOGY

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Typing "Invariant theory" into Wikipedia yields the theory of functions like $x_1x_2 + x_1x_3 + x_2x_3$ which are unaltered by permuting the variables. In algebraic topology, particularly post-1950, a different notion of "invariant" emerged. This use of invariant (e.g. Hopf invariant, Arf invariant) denotes an algebraic quantity which gives a partial answer to a topological question.

Often invariants in this sense are very technical, both in their context and their construction. However, a very simple invariant occurs in the game of Nim ([1] 36-38). In the 1960's this game was popular amongst students due to its enigmatic appearance in Alain Resnais's 1961 avant-garde movie "L'Année Dernière à Marienbad".

A set of matchsticks is divided arbitrarily into several heaps. Two players play alternately. A play consists of selecting a heap and removing from it any (non-zero) number of matchsticks. The winner is the player whose move leaves no remaining matchsticks. The question which the Nim invariant answers is: "If my opponent and I play out of our skins, will I win?"

Suppose there are k heaps of matchsticks of sizes $n_1, \ldots, n_k$ with $0 < n_i < 2^{s+1}$ for some s and all $1 \leq i \leq k$. Written dyadically

$$n_i = a_{i,0} + 2a_{i,1} + \ldots + 2^s a_{i,s}$$

for all $1 \leq i \leq k$ with each $a_{i,j} = 0$ or 1. The Nim invariant is the $(s+1)$-tuple of integers modulo 2 given by

$$(\sum_{i=1}^{k} a_{i,0}, \sum_{i=1}^{k} a_{i,1}, \sum_{i=1}^{k} a_{i,2}, \cdots, \sum_{i=1}^{k} a_{i,s}).$$

If every entry in the Nim invariant is even after a player's then the answer to the question is "yes" otherwise it is "no".

For suppose that

$$\sum_{i=1}^{k} a_{i,j} \equiv 0 \text{ (modulo 2)} \quad (1)$$

for all $0 \leq j \leq s$. If Player A removes some matchsticks from the j-th heap, suppose that the number removed is $2^{e_1} + 2^{e_2} + \ldots + 2^{e_t}$ with $0 \leq e_1 < e_2 < \ldots < e_t \leq s$. After A's turn we must have

$$\sum_{i=1}^{k} a_{i,e_r} \equiv 1 \text{ (modulo 2)}$$

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for $1 \leq r \leq t$. In particular some heap is still non-empty.

Now suppose that (1) does not hold. We show that Player B, the other player, can restore condition (1). Suppose there are k heaps of matchsticks of sizes $n_1, \ldots, n_k$ with $0 < n_i < 2^{s+1}$ for some s and all $1 \leq i \leq k$, written dyadically as before. Let $0 \leq p$ be the smallest integer such that

$$\sum_{i=1}^{k} a_{i,j} \equiv 0 \text{ (modulo 2)}$$

for all $1 + p \leq j \leq s$. Hence

$$\sum_{i=1}^{k} a_{i,p} \equiv 1 \text{ (modulo 2)}$$

and $a_{q,p} = 1$ for some q. We have

$$n_q = a_{q,0} + 2a_{i,1} + \ldots + 2^p a_{q,p} + \ldots 2^s a_{i,s}.$$

Note that

$$1 \leq a_{q,0} + 2a_{q,1} + \ldots + 2^p \leq 2^{p+1} - 1.$$

Player B removes $a_{q,0} + 2a_{q,1} + \ldots + 2^p$ matchsticks and replaces

$$a'_{q,0} + 2a'_{q,1} + \ldots + 2^{t-1} a'_{q,p-1} < 2^p$$

where

$$a'_{q,j} = \begin{cases} a_{q,j} & \text{if } \sum_{i=1}^{k} a_{i,j} \equiv 0 \text{ (modulo 2)}, \\ 0 & \text{if } \sum_{i=1}^{k} a_{i,j} \equiv 1 \text{ (modulo 2) and } a_{q,j} = 1, \\ 1 & \text{if } \sum_{i=1}^{k} a_{i,j} \equiv 1 \text{ (modulo 2) and } a_{q,j} = 0. \end{cases}$$

With this Player B has restored condition (1) and reduced the total number of matchsticks. Since Player A cannot win, playing perfectly, Player B must do so.

References

[1] W.W. Rouse Ball: *Mathematical Recreations and Essays*; Macmillan (1959).