NON-FACTORISATION OF ARF-KERVAIRE CLASSES THROUGH $\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty$

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Abstract. As an application of the upper triangular technology method of [8] it is shown that there do not exist stable homotopy classes of $\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty$ in dimension $2^{s+1}-2$ with $s \geq 2$ whose composition with the Hopf map to $\mathbb{R}P^\infty$ followed by the Kahn-Priddy map gives an element in the stable homotopy of spheres of Arf-Kervaire invariant one.

1. Introduction

1.1. For $n > 0$ let $\pi_n(\Sigma^\infty S^0)$ denote the n-th stable homotopy group of S^0, the 0-dimensional sphere. Via the Pontrjagin-Thom construction an element of this group corresponds to a framed bordism class of an n-dimensional framed manifold. The Arf-Kervaire invariant problem concerns whether or not there exists such a framed manifold possessing a Kervaire surgery invariant which is non-zero (modulo 2). In [4] it is shown that this can happen only when $n = 2^{s+1}-2$ for some $s \geq 1$. Resolving this existence problem is an important unsolved problems in homotopy theory (see [8] for a historical account of the problem together with new proofs of all that was known up to 2008). Recently important progress has made ([5]; see also [2], [3]) which shows that $n = 126$ is the only remaining possibility for existence (more details may be found in the survey article [9].

In view of the renewed interest in the Arf-Kervaire invariant problem it may be of interest to describe a related non-existence result. An equivalence formulation (see [8] §1.8) is that there exists a stable homotopy class $\Theta : \Sigma^\infty S^{2^{s+1}-2} \longrightarrow \Sigma^\infty \mathbb{R}P^\infty$ with mapping cone $\mathrm{Cone}(\Theta)$ such that the Steenrod operation

$$Sq^{2^s} : H^{2^s-1}(\mathrm{Cone}(\Theta); \mathbb{Z}/2) \cong \mathbb{Z}/2 \longrightarrow H^{2^{s+1}-1}(\mathrm{Cone}(\Theta); \mathbb{Z}/2)$$

is non-trivial. Using the upper triangular technology (UTT) of [8] we shall prove the following result:

Theorem 1.2.
Let $H : \Sigma^\infty \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \longrightarrow \Sigma^\infty \mathbb{R}P^\infty$ denote the map obtained by applying the Hopf construction to the multiplication on $\mathbb{R}P^\infty$. Then, if $s \geq 2$, there does not exist a stable homotopy class

$$\tilde{\Theta} : \Sigma^\infty S^{2^{s+1}-2} \longrightarrow \Sigma^\infty \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty$$

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such that the composition $\Theta = H \cdot \tilde{\Theta}$ is detected by a non-trivial Sq^{2^s} as in §1.1.

In §2.2 this result will be derived as a simple consequence of the UTT relations ([8] Chapter Eight). The basics of the UTT method are sketched in §2.1. Doubtless there are other ways to prove Theorem 1.2 (for example, from the results of [10]; see also [8] Chapter Two) but it provides an elegant application of UTT.

2. Upper triangular technology (UTT)

2.1. Let $F_{2n}(\Omega^2 S^3)$ denote the $2n$-th filtration of the combinatorial model for $\Omega^2 S^3 \simeq W \times S^1$. Let $F_{2n}(W)$ denote the induced filtration on W and let $B(n)$ be the Thom spectrum of the canonical bundle induced by $f_n : \Omega^2 S^3 \longrightarrow BO$, where $B(0) = S^0$ by convention. From [7] one has a 2-local, left bu-module homotopy equivalence of the form[1]

$$\bigvee_{n \geq 0} bu \wedge \Sigma^{4n} B(n) \stackrel{\simeq}{\longrightarrow} bu \wedge bo.$$

Therefore, if Θ is as in §1.1, then

$$(bu \wedge bo)_*(\mathrm{Cone}(\Theta)) \cong \bigoplus_{n \geq 0} (bu_*(\mathrm{Cone}(\Theta) \wedge \Sigma^{4n} B(n)).$$

Let $\alpha(k)$ denote the number of 1's in the dyadic expansion of the positive integer k. For $1 \leq k \leq 2^{s-1} - 1$ and $2^s \geq 4k - \alpha(k) + 1$ there are isomorphisms of the form ([8] Chapter Eight §4)

$$bu_{2^{s+1}-1}(C(\Theta) \wedge \Sigma^{4k} B(k)) \cong bu_{2^{s+1}-1}(\mathbb{R}P^\infty \wedge \Sigma^{4k} B(k)) \cong V_k \oplus \mathbb{Z}/2^{2^s - 4k + \alpha(k)}$$

where V_k is a finite-dimensional $\mathbb{F}_2$-vector space consisting of elements which are detected in mod 2 cohomology (i.e. in filtration zero, represented on the $s = 0$ line) in the mod 2 Adams spectral sequence. The map $1 \wedge \psi^3 \wedge 1$ on $bu \wedge bo \wedge C(\Theta)$ acts on the direct sum decomposition like the upper triangular matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

[1]In [8] and related papers I consistently forgot what I had written in my 1998 McMaster University notes "On $bu_*(BD_8)$". Namely, in the description of Mahowald's result I stated that $\Sigma^{4n} B(n)$ was equal to the decomposition factor F_{4n}/F_{4n-1} in the Snaith splitting of $\Omega^2 S^3$. Although this is rather embarrassing, I got the homology correct so that the results remain correct upon replacing F_{4n}/F_{4n-1} by $\Sigma^{4n} B(n)$ throughout! I have seen errors like this in the World Snooker Championship where the no.1 player misses an easy pot by concentrating on positioning the cue-ball. In mathematics such errors are inexcusable whereas in snooker they only cost one the World Championship.

In other words $(1\wedge\psi^3\wedge 1)_*$ sends the k-th summand to itself by multiplication by 9^{k-1} and sends the $(k-1)$-th summand to the $(k-2)$-th by a map

$$(\iota_{k,k-1})_* : V_k \oplus \mathbb{Z}/2^{2^s-4k+\alpha(k)} \longrightarrow V_{k-1} \oplus \mathbb{Z}/2^{2^s-4k+4+\alpha(k-1)}$$

for $2 \leq k \leq 2^{s-1}-1$ and $2^s \geq 4k-\alpha(k)+1$. The right-hand component of this map is injective on the summand $\mathbb{Z}/2^{2^s-4k+\alpha(k)}$ and annihilates V_k.

It is shown in [6] (also proved by UTT in ([8] Chapter Eight when $s \geq 2$) that Θ corresponds to a stable homotopy class of Arf-kervaire invariant one if and only if it is detected by the Adams operation ψ^3 on $\iota \in bu_{2^{s+1}-1}(\mathrm{Cone}(\Theta))$, an element of infinite order.

From these properties and the formula for $\psi^3(\iota)$ one easily obtains a series of equations ([8] §8.4.3) for the components of $(\eta\wedge 1\wedge 1)_*(\iota)$ where $\eta : S^0 \longrightarrow bu$ is the unit of bu-spectrum. Here we have used the isomorphism $bu_{2^{s+1}-1}(C(\Theta)) \cong bo_{2^{s+1}-1}(C(\Theta))$ since, strictly speaking, the latter group is the domain of $(\eta\wedge 1\wedge 1)_*$. It is shown in ([8] Theorem 8.4.7) that this series of equations implies that the $bu_{2^{s+1}-1}(\mathrm{Cone}(\Theta)\wedge\Sigma^{2^s}B(2^{s-2}))$-component of $(\eta\wedge 1\wedge 1)_*(\iota)$ is non-trivial and gives some information on the identity of this non-trivial element.

It is this information which we shall now use to prove Theorem 1.2.

2.2. *Proof of Theorem 1.2*

Suppose, for a contradiction, that Θ and $\tilde{\Theta}$ exist. We must assume that $s \geq 2$ because the UTT results of ([8] Theorem 8.4.7) are only claimed for this range.

The mod 2 cohomology of $\Sigma^{2^s}B(2^{s-2})$ is given by the $\mathbb{F}_2$-vector space with basis $\{z_{2^s+2j}, 0 \leq j \leq 2^{s-1}-2; z_{2^s+3+2k}, 0 \leq k \leq 2^{s-1}-2\}$ on which the left action by Sq^1 and $Sq^{0,1} = Sq^1Sq^2 + Sq^2Sq^1$ are given by $Sq^1(z_{2^s+2j}) = z_{2^s+3+2(j-1)}$ for $1 \leq j \leq 2^{s-1}-1$ and $Sq^{0,1}(z_{2^s+2j}) = z_{2^s+3+2j}$ for $0 \leq j \leq 2^{s-1}-2$ and $Sq^1, Sq^{0,1}$ are zero otherwise. This cohomology module is the $\mathbb{F}_2$-dual of the "lightning flash" module depicted in ([1] p.341).

Now consider the two 2-local Adams spectral sequences

$$E_2^{s,t} = \mathrm{Ext}_B^{s,t}(H^*(C(\Theta);\mathbb{Z}/2)\otimes H^*(\Sigma^{2^s}B(2^{s-2};\mathbb{Z}/2),\mathbb{Z}/2)$$

$$\Longrightarrow bu_{t-s}(C(\Theta)\wedge\Sigma^{2^s}B(2^{s-2})),$$

which collapses and

$$\tilde{E}_2^{s,t} = \mathrm{Ext}_B^{s,t}(H^*(C(\tilde{\Theta});\mathbb{Z}/2)\otimes H^*(\Sigma^{2^s}B(2^{s-2};\mathbb{Z}/2),\mathbb{Z}/2)$$

$$\Longrightarrow bu_{t-s}(C(\tilde{\Theta})\wedge\Sigma^{2^s}B(2^{s-2})),$$

where B is the exterior subalgebra of the mod 2 Steenrod algebra generated by Sq^1 and $Sq^{0,1}$.

To fit in with the notation of ([8] Theorem 8.4.7) set $s = q+2$ in Theorem 1.2. As mentioned in §2.1, it is shown in ([8] Theorem 8.4.7) that the component of $(\eta \wedge 1 \wedge 1)_*(\iota)$ lying in

$$bu_{2^{q+3}-1}(C(\Theta) \wedge \Sigma^{2^s} B(2^{s-2}))$$

$$\cong bu_{2^{q+3}-1}(\mathbb{R}P^\infty \wedge \Sigma^{2^s} B(2^{s-2}))$$

$$\cong \mathrm{Ext}_B^{0,2^{q+3}-1}(H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) \otimes H^*(\Sigma^{2^s} B(2^{s-2};\mathbb{Z}/2), \mathbb{Z}/2)$$

$$\subseteq \mathrm{Hom}(\oplus_{u+v=2^{q+3}-1} \ H^u(\mathbb{R}P^\infty;\mathbb{Z}/2) \otimes H^v(\Sigma^{2^s} B(2^{s-2};\mathbb{Z}/2), \mathbb{Z}/2)$$

corresponds to a homomorphism f such that $f(x^{2^{q+2}-1} \otimes z_{2^{q+2}})$ is non-trivial.

The factorisation $\Theta = H \cdot \tilde{\Theta}$ implies that there exists $h \in \tilde{E}_\infty^{0,2^{q+3}-1} \subseteq \tilde{E}_2^{0,2^{q+3}-1}$ such that $H_*(h) = f$. On the other hand

$$\tilde{E}_2^{0,2^{q+3}-1} \cong \mathrm{Ext}_B^{0,2^{q+3}-1}(H^*(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty;\mathbb{Z}/2) \otimes H^*(\Sigma^{2^s} B(2^{s-2};\mathbb{Z}/2), \mathbb{Z}/2).$$

Therefore the homomorphism The homomorphism

$$\mathrm{Ext}_B^{0,2^{q+3}-1}(H^*(\mathbb{R}P^\infty \wedge \mathbb{R}P^\infty;\mathbb{Z}/2) \otimes H^*(F_{2^{q+2}}/F_{2^{q+2}-1};\mathbb{Z}/2), \mathbb{Z}/2)$$

$$(H \wedge 1)_* \ \downarrow$$

$$\mathrm{Ext}_B^{0,2^{q+3}-1}(H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) \otimes H^*(F_{2^{q+2}}/F_{2^{q+2}-1};\mathbb{Z}/2), \mathbb{Z}/2)$$

satisfies $(H \wedge 1)_*(h)(x^{2^{q+2}-1} \otimes z_{2^{q+2}}) = f(x^{2^{q+2}-1} \otimes z_{2^{q+2}}) \not\equiv 0$. However

$$(H \wedge 1)_*(h)(x^{2^{q+2}-1} \otimes z_{2^{q+2}})$$

$$= h(\textstyle\sum_{a=1}^{2^{q+2}-2} \ x^a \otimes x^{2^{q+2}-a-1} \otimes z_{2^{q+2}}).$$

On the other hand

$$Sq^1(x^\alpha \otimes x^{2^{q+2}-2-\alpha} \otimes z_{2^{q+2}})$$

$$= \alpha(x^\alpha \otimes x^{2^{q+2}-1-\alpha} \otimes z_{2^{q+2}} + x^{\alpha+1} \otimes x^{2^{q+2}-2-\alpha} \otimes z_{2^{q+2}})$$

$$+ x^\alpha \otimes x^{2^{q+2}-2-\alpha} \otimes Sq^1(z_{2^{q+2}})$$

$$= \alpha(x^\alpha \otimes x^{2^{q+2}-1-\alpha} \otimes z_{2^{q+2}} + x^{\alpha+1} \otimes x^{2^{q+2}-2-\alpha} \otimes z_{2^{q+2}})$$

since $Sq^1(z_{2^{q+2}})$ is trivial. Therefore

$$f(x^{2^{q+2}-1} \otimes z_{2^{q+2}}) \in h(\mathrm{Im}(Sq^1) \equiv 0$$

because h is a B-module homomorphism and Sq^1 is trivial on $\mathbb{Z}/2$. □

Remark 2.3. When $s = 2, 3$ in the situation of Theorem 1.2 there is a map $\alpha : \Sigma^\infty \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \longrightarrow \Sigma^\infty \mathbb{R}P^\infty$ but it is just not equal to H! In the loopspace structure of $Q\mathbb{R}P^\infty$ form the product minus the two projections to give a map $\mathbb{R}P^\infty \times \mathbb{R}P^\infty \longrightarrow Q\mathbb{R}P^\infty$ which factors through the smash product. The adjoint of this factorisation is α. Then the smash product of two copies of a map of Hopf invariant one $\Sigma^\infty S^{2^s-1} \longrightarrow \Sigma^\infty \mathbb{R}P^\infty$ composed with α is detected by Sq^{2^s} on its mapping cone (see [10]).

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