OSSA'S THEOREM VIA THE KUNNETH FORMULA

ROBERT BRUNER, KHAIRIA MIRA, LAURA STANLEY AND VICTOR SNAITH

ABSTRACT. Let p be a prime. In [5], [7] and [9], using variations of an Adams spectral sequence calculation, the spectrum $bu \wedge B\mathbb{Z}/p \wedge B\mathbb{Z}/p$ is shown to be equivalent to $\vee_{i=1}^{p-1} S^{2i} \wedge bu \wedge B\mathbb{Z}/p$ up to wedges of mod p Eilenberg-Maclane spectra. We derive the result, without the Adams spectral sequence, using a Künneth formula short exact sequence for bu. We also explain how this result easily (a) implies the bo-analogue proved in [5] and (b) highlights the errors in the bo-analogue asserted in [9].

1. Introduction

This paper arose as a result of discussions during a graduate course at the University of Sheffield during 2008. In order to introduce Frank Adams technique of constructing homology resolutions as realisations of iterated cofibrations of spectra a simpler example than the classical Adams spectral sequence was needed. We had the spectrum bu to hand but, in order to postpone the algebraic intricacies of spectral sequences, what was required was an example whose geometric resolution gave rise to a short exact sequence rather than a spectral sequence. As it happens the bu-resolution of $B\mathbb{Z}/2$ yields such an example, which was simple enough for the purposes of the course. At that point, John Greenlees mentioned the existence of [9], which prompted the writing of §2. In §2 we use the bu-resolution of §2.1 to give a proof (Theorem 2.12) of the bu-result from [9] without Adams spectral sequences.

More importantly, unlike [5], [7] and [9], we do not resort to Adams spectral sequences to construct the essential map

$$\tilde{\mu}_* : bu_*(B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \longrightarrow \vee_{i=1}^{p-1} bu_*(S^{2i} \wedge B\mathbb{Z}/p)$$

of §2.11. Since the principal ingredient of $\tilde{\mu}$ is a simple lift of the map induced by the multiplication on $B\mathbb{Z}/p$ it is virtually invariant under switching the $B\mathbb{Z}/p$-factors, which may prove useful in calculations of bu_* of other p-groups.

In [5] and [9] bo_*-analogues of the bu-result are offered when $p = 2$. Consider the cofibration

$$\Sigma bo \xrightarrow{\eta} bo \xrightarrow{c} bu,$$

discovered by Raoul Bott during the proof of his famous Periodicity Theorem. If one has a map of nice spectra $f : Y \longrightarrow Z$ such that $1 \wedge f : bu \wedge Y \longrightarrow bu \wedge Z$ is an equivalence then $1 \wedge f : bo \wedge Y \longrightarrow bo \wedge Z$ because the Bott sequence may be wrapped around into an exact couple to give a Bott spectral

Date: 20 June 2010.

sequence converging to $bo_*(-)$ from $bu_*(-)$ and f yields an isomorphism of spectral sequences. This Bott spectral sequence argument amounts to a simple diagram chase.

In §3 we sketch how the analogous diagram chase goes which derives the bo-result of [5] from the bu-result. This is slightly more difficult than the use of the Bott spectral sequence because the result of [5] features $bo\langle 1\rangle \wedge B\mathbb{Z}/2$ rather than $bo \wedge B\mathbb{Z}/2$. It is a chase around diagrams derived from that of Bott. At first sight, the diagram chase of §3 appears as if it would immediately imply, by some variation of the five lemma, from the bu result and the bo-result of [5], that there is a bo-equivalence of the type that is claimed in [9]. We shall explain the feature of the diagram chase which implies this does not happen. In fact, both ([9] Proposition 2) and ([9] Lemma 2) are simply wrong with the result that the calculation of $bo \wedge B\mathbb{Z}/2 \wedge B\mathbb{Z}/2$ claimed in [9] is false. For example, taken together ([9] Proposition 2) and ([5] Theorem 4.4) contradict the $bo\langle 1\rangle_*(\mathbb{RP}^\infty)$, $bo_*(\mathbb{RP}^\infty)$ tables given in §3. Incidentally, [6] contains an up-dated elaboration concerning the Adams spectral sequence approach of [5] to the bo_*-analogue.

2. The connective unitary case

2.1. Let bu_* denote connective unitary K-homology on the stable homotopy category of CW spectra [2] so that if X is a space without a basepoint its unreduced bu-homology is $bu_*(\Sigma^\infty X_+)$, the homology of the suspension spectrum of the disoint union of X with a base-point. In particular $bu_*(\Sigma^\infty S^0) = \mathbb{Z}[u]$ where $\deg(u) = 2$. Let p be a prime and consider the cofibration of pointed spaces

$$B\mathbb{Z}/p \xrightarrow{i} BS^1 \xrightarrow{\pi} W_p$$

where i is induced by the inclusion of the cyclic group of order p into the circle. This cofibration maps to the fibration

$$B\mathbb{Z}/p \xrightarrow{i} BS^1 \xrightarrow{Bp} BS^1$$

and the comparison for mod p and integral unreduced singular homology yields the following result:

Lemma 2.2.

For all j, $H_j(W_p;\mathbb{Z}) \cong H_j(BS^1;\mathbb{Z})$ being $\mathbb{Z}$ when $j \geq 0$ is even and zero otherwise.

From the Atiyah-Hirzebruch spectral sequence ([2] p.47) we obtain the following result, which also follows from the Thom isomorphism $bu_*(W_p) \cong bu_*(BS^1)$, since W_p is Thom complex of the p-th tensor power of the canonical complex line bundle, by §2.1.

Corollary 2.3.

Both $bu_*(\Sigma^\infty W_p)$ and $bu_*(\Sigma^\infty BS^1)$ are free modules over $bu_*(\Sigma^\infty S^0) = \mathbb{Z}[u]$
.

2.4. The Atiyah-Hirzebruch spectral sequences for for bu_* and KU_* of $\Sigma^\infty B\mathbb{Z}/p$ both collapse for dimensional reasons and the map between them is injective so that $bu_*(\Sigma^\infty B\mathbb{Z}/p)$ injects into $KU_*(\Sigma^\infty B\mathbb{Z}/p)$ which, by the universal coefficient theorem for KU-theory [3] and the calculations of [4], is given by $KU_{2j+1}(\Sigma^\infty B\mathbb{Z}/p) \cong \oplus_{j=1}^{p-1} \mathbb{Z}/p^\infty$ ([9] §2; see also [8] Chapter I, §2) and is zero in even dimensions.

When p is odd it will be convenient to replace bu by $bu\mathbb{Z}_p$, connective unitary K-theory with p-adic integers coefficients and similarly for $KU\mathbb{Z}_p$. These p-adic spectra possess Adams decompositions [1] (see also [8])

$$bu\mathbb{Z}_p \simeq \vee_{i=1}^{p-1} \Sigma^{2i-2} lu \text{ and } KU\mathbb{Z}_p \simeq \vee_{i=1}^{p-1} \Sigma^{2i-2} LU$$

where $lu_*(\Sigma^\infty S^0) \cong \mathbb{Z}_p[v]$ where $\deg(v) = 2p-2$ corresponds to u^{p-1} and multiplication by u translates the summand $\Sigma^{2i-2}lu$ to $\Sigma^{2i}lu$ for $0 \leq i \leq p-2$ and $\Sigma^{2p-4}lu$ to lu. LU-theory is obtained from lu by localising to invert v. In addition there are canonical isomorphisms

$$bu_*(\Sigma^\infty B\mathbb{Z}/p) \cong (bu\mathbb{Z}_p)_*(\Sigma^\infty B\mathbb{Z}/p) \text{ and } KU_*(\Sigma^\infty B\mathbb{Z}/p) \cong (KU\mathbb{Z}_p)_*(\Sigma^\infty B\mathbb{Z}/p)$$

with $LU_{2j+1}(\Sigma^\infty B\mathbb{Z}/p) \cong \mathbb{Z}/p^\infty$.

Corollary 2.5. Let p be a prime and let lu_* be as in §2.4 when p is odd or $lu = (bu\mathbb{Z}_2)_*$ when $p = 2$. Then, as a $\mathbb{Z}_p[v]$-module, where $\deg(v_{2i-1}) = 2i-1$,

$$lu_*(\Sigma^\infty B\mathbb{Z}/p) \cong \frac{\mathbb{Z}_p[u]\langle v_1, v_3, v_5, \ldots \rangle}{(pv_1, pv_3, \ldots, pv_{2p-3}, vv_{2i-1} - pv_{2(p-1)+2i-1})}.$$

Proof

The injection mentioned in §2.4 maps $lu_{2i-1}(\Sigma^\infty B\mathbb{Z}/p)$ into $LU_{2i-1}(\Sigma^\infty B\mathbb{Z}/p) \cong \mathbb{Z}/p^\infty$. Therefore this group must be cyclic and an order-count in the collapsed Atiyah-Hirzebruch spectral sequence shows that the non-zero groups $lu_{2k(p-1)+2i-1}(\Sigma^\infty B\mathbb{Z}/p) \cong \mathbb{Z}/p^{k+1}$ for $i = 1, \ldots, p-1$, generated by $v_{2k(p-1)+2i-1}$. In $KU_{2i+1}(\Sigma^\infty B\mathbb{Z}/p)$ the element $vv_{2k(p-1)+2i-1}$ has order p^{k+1}, by Bott periodicity, so we may choose $v_{2(k+1)(p-1)+2i-1}$ so that $vv_{2k(p-1)+2i-1} = pv_{2(k+1)(p-1)+2i-1}$. $\square$

Corollary 2.6.

The cofibration of §2.1 gives a free $\mathbb{Z}[u]$-module resolution

$$0 \longrightarrow bu_*(\Sigma^\infty BS^1) \xrightarrow{\pi_*} bu_*(\Sigma^\infty W_p) \longrightarrow bu_{*-1}(\Sigma^\infty B\mathbb{Z}/p) \longrightarrow 0$$

as well as similar resolutions for $bu\mathbb{Z}_p$ and lu.

2.7. If A is a $\mathbb{Z}$-graded group we write $A[n]$ for the graded group with $A[n]_j = A_{j+n}$ so that $bu_{*-1}(\Sigma^\infty B\mathbb{Z}/p)$ equals $bu_*(\Sigma^\infty B\mathbb{Z}/p)[-1]$. By a cell-by-cell induction, for all CW spectra of finite type X the external product gives isomorphisms

$$bu_*(\Sigma^\infty BS^1) \otimes_{\mathbb{Z}[u]} bu_*(X) \xrightarrow{\cong} bu_*(\Sigma^\infty BS^1 \wedge X),$$

$$bu_*(\Sigma^\infty W_p) \otimes_{\mathbb{Z}[u]} bu_*(X) \xrightarrow{\cong} bu_*(\Sigma^\infty W_p \wedge X).$$

Smashing the cofibration of §2.1 with X and applying the argument of [3] yields the following Künneth formula:

Theorem 2.8.
There is a natural short exact sequence

$$0 \longrightarrow bu_*(\Sigma^\infty B\mathbb{Z}/p) \otimes_{\mathbb{Z}[u]} bu_*(X) \longrightarrow bu_*(\Sigma^\infty B\mathbb{Z}/p) \wedge X)$$

$$\longrightarrow \mathrm{Tor}^1_{\mathbb{Z}[u]}(bu_*(\Sigma^\infty B\mathbb{Z}/p), bu_*(X))[1] \longrightarrow 0$$

as well as similar exact sequences for $bu\mathbb{Z}_p$ and lu.

Example 2.9. Let p be a prime. As in Corollary 2.5, let lu_* be as in §2.4 when p is odd or $lu = (bu\mathbb{Z}_2)_*$ when $p = 2$. In Theorem 2.8 set $X = \Sigma^\infty B\mathbb{Z}/p$. Then $lu_{2*}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$ comes entirely from the left-hand graded group which is generated by $\{v_{2i-1} \otimes v_{2j-1}\ i, j \geq 1\}$ but if $2i-1 > 2(p-1)$ then

$$\begin{aligned} pv_{2i-1} \otimes v_{2j-1} &= vv_{2i-1-2(p-1)} \otimes v_{2j-1} \\ &= v_{2i-1-2(p-1)} \otimes vv_{2j-1} \\ &= pv_{2i-1-2(p-1)} \otimes v_{2(p-1)+2j-1} \end{aligned}$$

which is zero by induction and similarly if $2j-1 > 2(p-1)$. Therefore $lu_{2m}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$ is the graded $\mathbb{F}_p$-vector space spanned by $v_1 \otimes v_{2m-1}, \ldots, v_{2m-1} \otimes v_1$ which are linearly independent, being detected by the canonical homomorphism to $H_{2m}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p); \mathbb{Z}/p)$. Therefore for each $m \geq 1$

$$\dim_{\mathbb{F}_p}(lu_{2m}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))) = \dim_{\mathbb{F}_p}(\pi_{2m}(\vee_{i,j>0}\ \Sigma^{2i+2j-2}H\mathbb{Z}/p)).$$

Similarly $lu_{2*+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$ comes entirely from the right-hand graded group

$$T_* = \mathrm{Tor}^1_{\mathbb{Z}_p[v]}(lu_*(\Sigma^\infty B\mathbb{Z}/p), lu_*(\Sigma^\infty B\mathbb{Z}/p))[1].$$

For $1 \leq 2i-1 \leq 2p-3$ let Y_i denote the $\mathbb{Z}_p[v]$-submodule of $lu_*(\Sigma^\infty B\mathbb{Z}/p)$ generated by $\{v_{2k(p-1)+2i-1} \mid k \geq 0\}$ so that $lu_*(\Sigma^\infty B\mathbb{Z}/p) \cong \oplus_i Y_i$ with a corresponding decomposition $T_* \cong \oplus_i T_{i,*}$. This decomposition has a well-known geometric origin ([9] §2).

A free $\mathbb{Z}_p[v]$-module resolution is given by

$$0 \longrightarrow \oplus_{j=0}^\infty\ \mathbb{Z}_p[v]\langle a_j\rangle \xrightarrow{d} \oplus_{j=0}^\infty\ \mathbb{Z}_p[v]\langle b_j\rangle \xrightarrow{\epsilon} Y_i \longrightarrow 0$$

where a_j, b_j have internal degree $2j(p-1)+2i-1$, $\epsilon(b_j) = v_{2j(p-1)+2i-1}$, $d(a_0) = pb_0$ and $d(a_j) = pb_j - vb_{j-1}$ for $j \geq 1$. Therefore

$$T_{i,*} = \mathrm{Ker}(1 \otimes d : \oplus_{j=0}^\infty\ lu_*(\Sigma^\infty B\mathbb{Z}/p)\langle a_j\rangle \longrightarrow \oplus_{j=0}^\infty\ lu_*(\Sigma^\infty B\mathbb{Z}/p)\langle b_j\rangle).$$

For $2m \geq 2i-1$ write $2m-2i+1 = 2t(p-1)+2j-1$ with $1 \leq 2j-1 \leq 2p-3$. Then

$$T_{i,2m} = \mathbb{Z}/p^{t+1}\langle v_{2t(p-1)+2j-1}a_0 + v_{2(t-1)(p-1)+2j-1}a_1 + \ldots + v_{2j-1}a_t\rangle.$$

From this, for $1 \leq i \leq p-1$, one finds that

$$\mathrm{Tor}^1_{\mathbb{Z}_p[v]}(lu_*(\Sigma^\infty B\mathbb{Z}/p), Y_i)[1] \cong lu_{2*+1}(\Sigma^{2i} B\mathbb{Z}/p)$$

and therefore

$$lu_{2*+1}(\Sigma^\infty B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \cong \oplus_{i=1}^{p-1}\ lu_{2*+1}(\Sigma^{2i} B\mathbb{Z}/p).$$

Adding together the suspensions of lu as in §2.4 yields a similar isomorphism for $bu\mathbb{Z}_p$ and therefore there is a $\mathbb{Z}[u]$-module isomorphism

$$bu_{2*+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \cong \oplus_{i=1}^{p-1}\ bu_{2*+1}(\Sigma^\infty(S^{2i} \wedge B\mathbb{Z}/p))$$

and $\oplus_{i=1}^{p-1} bu_{2*}(\Sigma^\infty(S^{2i} \wedge B\mathbb{Z}/p)) = 0$.

From Example 2.9 we have an isomorphism

$$lu_{2*}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \cong \pi_{2*}(\vee_{i,j>0}\ \Sigma^{2i+2j-2} H\mathbb{Z}/p)$$

and therefore

$$bu_{2*}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \cong \pi_{2*}(\vee_{a=0}^{p-2} \vee_{i,j>0}\ \Sigma^{2a+2i+2j-2} H\mathbb{Z}/p).$$

Lemma 2.10.

The homomorphism induced by the multiplication μ in the group $\mathbb{Z}/p$ is injective

$$\mu_* : bu_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \longrightarrow bu_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p)).$$

Proof

For simplicity we prove this only for $p = 2$. The proof, which uses KU, may be modified for odd primes but requires a more careful analysis of the splittings of §2.4 and ([9] §2) in relation to the embedding

$$KU_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \subset \mathrm{Hom}(R(\mathbb{Z}/p \times \mathbb{Z}/p), \mathbb{Z}/p^\infty)$$

By Example 2.8, multiplication by u is injective in odd dimensions so that

$$bu_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \longrightarrow KU_{2m+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$$

is injective, because it is localisation by inverting u. We shall use this observation to show that

$$\mu_* : KU_{2m+1}(\Sigma^\infty(B\mathbb{Z}/2 \wedge B\mathbb{Z}/2)) \longrightarrow KU_{2m+1}(\Sigma^\infty(B\mathbb{Z}/2))$$

is injective, which suffices to prove the result when $p = 2$. Since $B\mathbb{Z}/2 = \mathbb{RP}^\infty$ a skeletal approximation to the multiplication gives

$$\mu : \Sigma^\infty(\mathbb{RP}^{2r} \wedge \mathbb{RP}^{2v}) \longrightarrow \Sigma^\infty \mathbb{RP}^{2r+2v}.$$

Consider the effect on reduced, periodic complex K-theory

$$\mu^* : \tilde{KU}^0(\mathbb{RP}^{2r+2v}) \cong \mathbb{Z}/2^{r+v} \longrightarrow \tilde{KU}^0(\mathbb{RP}^{2r} \wedge \mathbb{RP}^{2v}) \cong \mathbb{Z}/2^{\min(r,v)}.$$

If L is the Hopf line bundle then $\mu^*(L-1) = (L-1) \otimes (L-1)$ so that μ^* is onto and, by the universal coefficient formula for KU_*, KU^*,

$$\mu_* : \tilde{KU}_{2m+1}(\mathbb{RP}^{2r} \wedge \mathbb{RP}^{2v}) \cong \mathbb{Z}/2^{\min(r,v)} \longrightarrow \tilde{KU}_{2m+1}(\mathbb{RP}^{2r+2v}) \cong \mathbb{Z}/2^{r+v}$$

is injective. Letting r, s tend to infinity yields the result. □

2.11. We have a cofibration of spectra $\Sigma^2 bu \longrightarrow bu \longrightarrow H\mathbb{Z}$. By Example 2.9 and Lemma 2.10 the composition

$$bu \wedge \Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \stackrel{1\wedge\mu}{\longrightarrow} bu \wedge \Sigma^\infty(B\mathbb{Z}/p) \longrightarrow H\mathbb{Z} \wedge \Sigma^\infty(B\mathbb{Z}/p)$$

is trivial on homotopy groups. Therefore, when $p = 2$, $(1 \wedge \mu)_*$ induces an isomorphism

$$\tilde{\mu}_* : bu_{2*+1}(\Sigma^\infty(B\mathbb{Z}/2 \wedge B\mathbb{Z}/2)) \stackrel{\cong}{\longrightarrow} bu_{2*+1}(\Sigma^\infty(S^2 \wedge B\mathbb{Z}/2)).$$

Similarly at odd primes, using the multiplication μ together with the stable homotopy splittings of $\Sigma^\infty B\mathbb{Z}/p$ [9], yields an isomorphism

$$\tilde{\mu}_* : bu_{2*+1}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \stackrel{\cong}{\longrightarrow} \oplus_{i=1}^{p-1}\ bu_{2*+1}(\Sigma^\infty(S^{2i} \wedge B\mathbb{Z}/p)).$$

By Example 2.9, the $\mathbb{F}_p$-vector space $bu_{2*}(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p))$ is detected in mod p homology and there is a map of spectra

$$h : bu \wedge \Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p) \longrightarrow (\vee_{a=0}^{p-2}\ \vee_{i,j>0}\ \Sigma^{2a+2i+2j-2} H\mathbb{Z}/p)$$

which induces an isomorphism on even dimensional homotopy. Therefore we obtain the following result:

Theorem 2.12. *([9]; see also [5] and [7])*
There is an isomorphism

$$(\tilde{\mu}_*, h_*) : bu_*(\Sigma^\infty(B\mathbb{Z}/p \wedge B\mathbb{Z}/p)) \stackrel{\cong}{\longrightarrow}$$

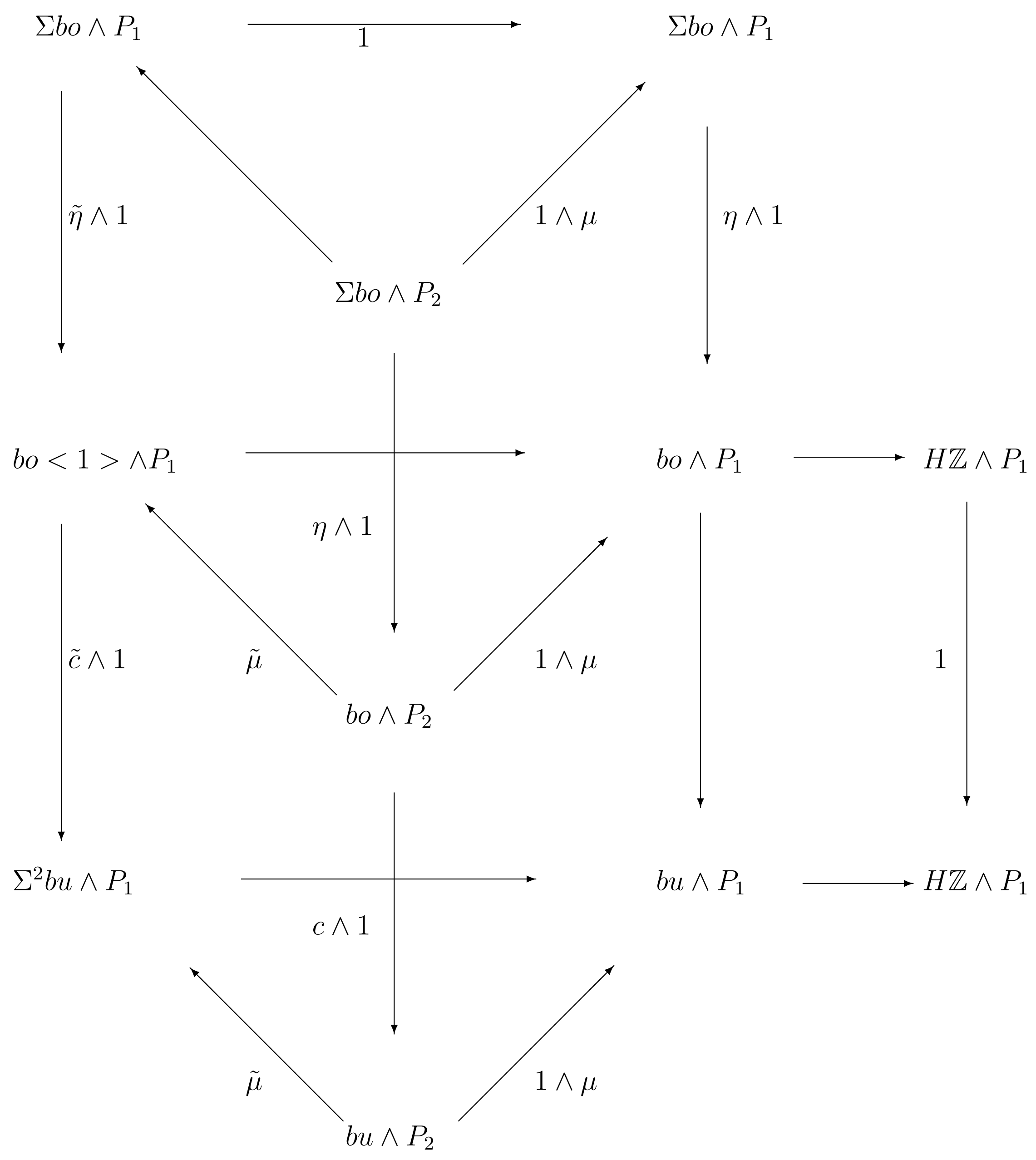
$$\oplus_{i=1}^{p-1}\ bu_*(\Sigma^\infty(S^{2i} \wedge B\mathbb{Z}/p)) \oplus \oplus_{a=0}^{p-2} \oplus_{i,j>0}\ \pi_*(\Sigma^{2a+2i+2j-2} H\mathbb{Z}/p).$$

3. The connective orthogonal case

3.1. In this section we shall concentrate on $p = 2$ and connective orthogonal K-theory bo. Consider the following commutative diagram of spectra of horizontal and vertical cofibrations in which c is complexification and η is multiplication by the generator of $\pi_1(bo)$. The notation for $bo\langle 1\rangle$ is taken from [5].

$$\begin{array}{ccccc}
\Sigma bo & \xrightarrow{1} & \Sigma bo & & \\
\downarrow \tilde{\eta} & & \downarrow \eta & & \\
bo\langle 1\rangle & \longrightarrow & bo & \longrightarrow & H\mathbb{Z} \\
\downarrow \tilde{c} & & \downarrow c & & \downarrow 1 \\
\Sigma^2 bu & \longrightarrow & bu & \longrightarrow & H\mathbb{Z}
\end{array}$$

Write P_1, P_2 for the spectra $\Sigma^\infty \mathbb{RP}^\infty, \Sigma^\infty(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty)$ respectively. We shall sketch the proof of ([5] Theorem 4.4) by chasing the homotopy groups of the following homotopy commutative diagram, in which μ and $\tilde{\mu}$ are orthogonal analogues of the maps of the same names in §2.11. The essential ingredient will be Theorem 2.12.



The following result differs from ([5] Theorem 4.4) only in the fact that we explicitly use the map $\tilde{\mu}$.

Theorem 3.2.
There is a homotopy equivalence of the form

$$\tilde{\mu} \vee h' : bo \wedge \mathbb{RP}^\infty \wedge \mathbb{RP}^\infty \xrightarrow{\simeq} (bo < 1 > \wedge \mathbb{RP}^\infty) \vee (\vee_{i,j>0}\, \Sigma^{2i+4j-2} H\mathbb{Z}/2).$$

3.3. *$bo < 1 >_* (\mathbb{RP}^\infty)$, $bo_*(\mathbb{RP}^\infty)$ tables*

We have the following table of (reduced) orthogonal connective K-theory groups, supplied (29-05-09) by Bob Bruner:

$bo_{8n}(\mathbb{RP}^\infty)$	0
$bo_{8n+1}(\mathbb{RP}^\infty)$	$\mathbb{Z}/2$
$bo_{8n+2}(\mathbb{RP}^\infty)$	$\mathbb{Z}/2$
$bo_{8n+3}(\mathbb{RP}^\infty)$	$\mathbb{Z}/2^{4n+3}$
$bo_{8n+4}(\mathbb{RP}^\infty)$	0
$bo_{8n+5}(\mathbb{RP}^\infty)$	0
$bo_{8n+6}(\mathbb{RP}^\infty)$	0
$bo_{8n+7}(\mathbb{RP}^\infty)$	$\mathbb{Z}/2^{4n+4}$

The graded group $bo_*(\mathbb{RP}^\infty)$ is a module over

$$bo_*(S^0) = \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

and multiplication by η is nontrivial from dimension $8n+1$ to $8n+2$ and from $8n+2$ to $8n+3$. Multiplication by α has kernel of order 4 from dimension $8n+3$ to $8n+7$ and is one-one from dimension $8n+7$ to $8n+11$. Multiplication by β is always one-one.

By definition of $bo < 1 >$ as the fibre of $bo \longrightarrow H\mathbb{Z}$ yields a long exact sequence of reduced homology theories

$$\ldots \longrightarrow bo < 1 >_i (\mathbb{RP}^\infty) \longrightarrow bo_i(\mathbb{RP}^\infty) \longrightarrow H_i(\mathbb{RP}^\infty; \mathbb{Z}) \longrightarrow \ldots$$

and there is a factorisation $bo \longrightarrow bu \longrightarrow H\mathbb{Z}$. Using the fact that $H_i(\mathbb{RP}^\infty; \mathbb{Z}) \cong \mathbb{Z}/2$ for odd $i > 0$ and is zero otherwise we may calculate $bo < 1 >_* (\mathbb{RP}^\infty)$. In addition we may double-check the results from the long exact homotopy sequence of the left-hand vertical fibration in the diagram of §3.1

$$\ldots \longrightarrow bo_{i-1}(\mathbb{RP}^\infty) \longrightarrow\longrightarrow bo < 1 >_i (\mathbb{RP}^\infty) \longrightarrow bu_{i-2}(\mathbb{RP}^\infty) \longrightarrow \ldots .$$

The following table is straightforward to verify by diagram chasing.

$bo < 1 >_{8n} (\mathbb{RP}^\infty)\ n \geq 0$	$\mathbb{Z}/2 \cong H_{8n+1}(\mathbb{RP}^\infty; \mathbb{Z}) \xrightarrow{\cong} bo < 1 >_{8n} (\mathbb{RP}^\infty)$
$bo < 1 >_1 (\mathbb{RP}^\infty)$	0
$bo < 1 >_{8n+1} (\mathbb{RP}^\infty)\ n \geq 1$	$\mathbb{Z}/2 \cong bo < 1 >_{8n+1} (\mathbb{RP}^\infty) \xrightarrow{\cong} bo_{8n+1}(\mathbb{RP}^\infty)$
$bo < 1 >_{8n+2} (\mathbb{RP}^\infty)$	$\mathbb{Z}/2 \cong bo < 1 >_{8n+2} (\mathbb{RP}^\infty) \xrightarrow{\cong} bo_{8n+2}(\mathbb{RP}^\infty)$
$bo < 1 >_{8n+3} (\mathbb{RP}^\infty)$	$\mathbb{Z}/2^{4n+2} \cong bo < 1 >_{8n+3} (\mathbb{RP}^\infty) \xrightarrow{2(2s+1)} bu_{8n+3}(\mathbb{RP}^\infty)$
$bo < 1 >_{8n+4} (\mathbb{RP}^\infty)$	$\mathbb{Z}/2 \cong H_{8n+5}(\mathbb{RP}^\infty; \mathbb{Z}) \xrightarrow{\cong} bo < 1 >_{8n+4} (\mathbb{RP}^\infty)$
$bo < 1 >_{8n+5} (\mathbb{RP}^\infty)$	0
$bo < 1 >_{8n+6} (\mathbb{RP}^\infty)$	0
$bo < 1 >_{8n+7} (\mathbb{RP}^\infty)$	$\mathbb{Z}/2^{4n+3} \cong bo < 1 >_{8n+7} (\mathbb{RP}^\infty) \xrightarrow{1-1} bo_{8n+7}(\mathbb{RP}^\infty)$

3.4. *The Bott sequence versus mod 2 homology*

The subalgebra $\mathcal{B}$ generated in the mod 2 Steenrod algebra by Sq^1 and Sq^2 has dimension eight and contained the exterior subalgebra $\mathcal{E} = E(Sq^1, Sq^{0,1}) = \{1, Sq^1, Sq^1Sq^2 + Sq^2Sq^1, Sq^2Sq^2\}$.

Consider the Bott sequence

$$\cdots \longrightarrow bo_i(X) \xrightarrow{c} bu_i(X) \longrightarrow bo_{i-2}(X) \xrightarrow{\eta_*} bo_{i-1}(X) \longrightarrow \cdots$$

which is isomorphic to the homotopy sequence of the cofibration

$$bo \wedge X \longrightarrow bo \wedge \Sigma^{-2}\mathbb{CP}^2 \wedge X \longrightarrow bo \wedge S^2 \wedge X,$$

where the middle spectrum is identified with $bu \wedge X$ via an equivalence due to Anderson-Wood [10].

The following commutative diagram is easy to establish.

$$\begin{array}{ccc}
bo_i(X) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(H^i(X;\mathbb{Z}/2),\mathbb{Z}/2) \\
\downarrow c & & \downarrow \tilde{\phi} \\
bu_i(X) & \longrightarrow & \mathrm{Hom}_{\mathcal{E}}(H^i(X;\mathbb{Z}/2),\mathbb{Z}/2) \\
\downarrow & & \downarrow \tilde{\lambda} \\
bo_{i-2}(X) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(H^{i-2}(X;\mathbb{Z}/2),\mathbb{Z}/2)
\end{array}$$

The horizontal maps are induced by the canonical map $bo \longrightarrow H\mathbb{Z}/2$, $\tilde{\phi}(h) = h$ and $\tilde{\lambda}(g)(x) = g(Sq^2(x))$.

By Theorem 2.12 we know that the middle horizontal map is an isomorphism when i is even. The following result is straightforward.

Proposition 3.5.
(i) When $X = \mathbb{RP}^\infty \wedge \mathbb{RP}^\infty$ the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{B}}(H^i(X;\mathbb{Z}/2),\mathbb{Z}/2) \xrightarrow{\tilde{\phi}} \mathrm{Hom}_{\mathcal{E}}(H^i(X;\mathbb{Z}/2),\mathbb{Z}/2)$$

$$\xrightarrow{\tilde{\lambda}} \mathrm{Hom}_{\mathcal{B}}(H^{i-2}(X;\mathbb{Z}/2),\mathbb{Z}/2) \longrightarrow 0.$$

is exact.
(ii) For $i \geq 2$

$$\dim_{\mathbb{F}_2}(\mathrm{Hom}_{\mathcal{B}}(H^{2i}(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty;\mathbb{Z}/2),\mathbb{Z}/2)) = \begin{cases} (i-1)/2 \text{ if } i \text{ is odd,} \\ 1 + (i/2) \text{ if } i \text{ is even} \end{cases}$$

(iii) For $i \geq 2$

$$\dim_{\mathbb{F}_2}(\mathrm{Hom}_{\mathcal{E}}(H^{2i}(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty; \mathbb{Z}/2), \mathbb{Z}/2)) = i.$$

We shall also need the following result.

Proposition 3.6.
Define

$$X_n = \#\{i, j \geq 1 \mid 2i - 1 + 4j - 1 = 2n\}.$$

Then

$$X_n = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

3.7. *Sketch proof of Theorem 3.2*
We shall assume that

$$\tilde{\mu}_* : bo_{8n+e}(P_2) \longrightarrow bo < 1 >_{8n+e} (P_1) \cong \mathbb{Z}/2$$

is non-trivial when $e = 0, 4$. We shall leave this to the reader.

The result follows by induction on dimensions using §3.1 and §§3.3-3.6. The only subtlety occurs in dimensions $8n + e$ for $e = 0, 4$. Let h_* denote the canonical map from bo_* to mod 2 homology. The range and domain of the homomorphism

$$bo_{8n+e}(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty)$$

$$\tilde{\mu}_* \vee h_* \ \downarrow$$

$$bo < 1 >_{8n+e} (\mathbb{RP}^\infty) \oplus \mathrm{Hom}_{\mathcal{B}}(H^{8n+e}(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty; \mathbb{Z}/2), \mathbb{Z}/2)$$

are both $\mathbb{F}_2$-vector spaces. The map is injective for its cokernel has dimension one. For example,

$$\dim_{\mathbb{F}_2}(\mathrm{Hom}_{\mathcal{B}}(H^{8n+4}(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty; \mathbb{Z}/2), \mathbb{Z}/2)) = 2n + 2 = 1 + X_{4n+2}.$$

Therefore $bo_{8n+e}(\mathbb{RP}^\infty \wedge \mathbb{RP}^\infty)$ is detected by a map to a wedge of $2n+2$ mod 2 Eilenberg-Maclane spectra but a one-dimensional subspace is also detected by $\tilde{\mu}_*$. Therefore we obtain the correct result for Theorem 3.2 in dimension $8n + 4$ by discarding one mod 2 Eilenberg-Maclane spectra from the wedge. A similar situation occurs in dimension $8n$. □

Remark 3.8. It is the anomolous behaviour in dimensions $8n, 8n + 4$, occurring in §3.7, which renders the five lemma inapplicable. If we delete a copy of $\mathbb{Z}/2$ from the either end of the short exact sequence of Proposition 3.5 it is no longer exact.

References

[1] J.F. Adams: Lectures on generalised cohomology; Lecture Notes in Math. #99 Springer Verlag (1969) 1-138.

[2] J.F. Adams: *Stable Homotopy and Generalised Homology*; University of Chicago Press (1974).

[3] M.F. Atiyah: Vector bundles and the Künneth formula; Topology 1 (1962) 245-248.

[4] M.F. Atiyah: *K-Theory*; Benjamin (1968).

[5] Robert R. Bruner: Ossa's theorem and Adams covers; Proc. A.M. Soc. (8) 127 (1999) 2443-2447.

[6] Robert R. Bruner: Ossa's theorem (version of 2009-04-16 15:56) private correspondence.

[7] David Copeland Johnson and W. Stephen Wilson: On a theorem of Ossa; Proc. A.M. Soc. (12) 125 (1997) 3753-3755.

[8] Richard M. Kane: Operations in connective K-theory; Mem. A.M. Soc. vol 34, #254 (1981).

[9] E. Ossa: Connective K-theory of elementary abelian groups; *transformation Groups, Osaka 1987* (ed. K. Kawakubo) Springer Verlag Lecture Notes in Math. #1375 (1989) 269-275.

[10] Victor Snaith: *Stable Homotopy Around the Arf-Kervaire Invariant*; Progress in Math. #273, Birkhäuser (2009).