PSH-ALGEBRAS AND THE SHINTANI CORRESPONDENCE

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Contents

1. PSH-algebras over the integers

A PSH-algebra is a connected, positive self-adjoint Hopf algebra over $\mathbb{Z}$. The notion was introduced in [11]. Let $R = \oplus_{n\geq 0} R_n$ be an augmented graded ring over $\mathbb{Z}$ with multipication

$$m : R \otimes R \longrightarrow R.$$

Suppose also that R is connected, which means that there is an augmentation ring homomorphism of the form

$$\epsilon : \mathbb{Z} \xrightarrow{\cong} R_0 \subset R.$$

These maps satisfy associativity and unit conditions.

Associativity:

$$m(m \otimes 1) = m(1 \otimes m) : R \otimes R \otimes R \longrightarrow R.$$

Unit:

$$m(1 \otimes \epsilon) = 1 = m(\epsilon \otimes 1); R \otimes \mathbb{Z} \cong R \cong \mathbb{Z} \otimes R \longrightarrow R \otimes R \ \longrightarrow R.$$

R is a Hopf algebra if, in addition, there exist comultiplication and counit homomorphisms

$$m^* : R \longrightarrow R \otimes R$$

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and

$$\epsilon^* : R \longrightarrow \mathbb{Z}$$

such that

Hopf

m^* is a ring homomorphism with respect to the product $(x \otimes y)(x' \otimes y') = xx' \otimes yy'$ on $R \otimes R$ and ϵ^* is a ring homomorphism restricting to an isomorphism on R_0. The homomorphism m is a coalgebra homomorphism with respect to m^*.

The m^* and ϵ^* also satisfy

Coassociativity:

$$(m^* \otimes 1)m^* = (1 \otimes m^*)m^* : R \longrightarrow R \otimes R \otimes R \longrightarrow R \otimes R \otimes R$$

Counit:

$$m(1 \otimes \epsilon) = 1 = m(\epsilon \otimes 1); R \otimes \mathbb{Z} \cong R \cong \mathbb{Z} \otimes R \longrightarrow R \otimes R \ \longrightarrow R.$$

R is a cocomutative if

Cocommutative:

$$m^* = T \cdot m^* : R \longrightarrow R \otimes R$$

where $T(x \otimes y) = y \otimes x$ on $R \otimes R$.

Suppose now that each R_n (and hence R by direct-sum of bases) is a free abelian group with a distinguished $\mathbb{Z}$-basis denoted by $\Omega(R_n)$. Hence $\Omega(R)$ is the disjoint union of the $\Omega(R_n)$'s. With respect to the choice of basis the positive elements R^+ of R are defined by

$$R^+ = \{r \in R \mid r = \sum \ m_\omega \omega, \ m_\omega \geq 0, \omega \in \Omega(R)\}.$$

Motivated by the representation theoretic examples the elements of $\Omega(R)$ are called the irreducible elements of R and if $r = \sum \ m_\omega \omega \in R^+$ the elements $\omega \in \Omega(R)$ with $m_\omega > 0$ are call the irreducible constituents of r.

Using the tensor products of basis elements as a basis for $R \otimes R$ we can similarly define $(R \otimes R)^+$ and irreducible constituents etc.

Positivity:

R is a positive Hopf algebra if

$$m((R \otimes R)^+) \subset R^+, m^*(R^+) \subset (R \otimes R)^+, \epsilon(\mathbb{Z}^+) \subset R^+, \epsilon^*(R^+) \subset \mathbb{Z}^+.$$

Define inner products $\langle -, - \rangle$ on R, $R \otimes R$ and $\mathbb{Z}$ by requiring the chosen basis ($\Omega(\mathbb{Z}) = \{1\}$) to be an orthonormal basis.

A positive Hopf $\mathbb{Z}$-algebra is self-adjoint if

Self-adjoint:

m and m^* are adjoint to each other and so are ϵ and ϵ^*. That is

$$\langle m(x \otimes y), z \rangle = \langle x \otimes y, m^* z \rangle$$

and similarly for ϵ, ϵ^*.

The subgroup of primitive elements $P \subset R$ is given by

$$P = \{r \in R \mid m^*(r) = r \otimes 1 + 1 \otimes r\}$$

2. The Decomposition Theorem

Let $\{R_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of PSH algebras. Define the tensor product PSH algebra

$$R = \otimes_{\alpha \in \mathcal{A}} \; R_\alpha$$

to be the inductive limit of the finite tensor products $\otimes_{\alpha \in S} \; R_\alpha$ with $S \subset \mathcal{A}$ a finite subset. Define $\Omega(R)$ to be the disjoint union over finite subsets S of $\prod_{\alpha \in S} \; \Omega(R_\alpha)$.

The following result of the PSH analogue of a structure theorem for Hopf algebras over the rationals due to Milnor-Moore [5]

Theorem 2.1.

Any PSH algebra R decomposes into the tensor product of PSH algebras with only one irreducible primitive element. Precisely, let $\mathcal{C} = \Omega \bigcap P$ denote the set of irreducible primitive elements in R. For any $\rho \in \mathcal{C}$ set

$$\Omega(\rho) = \{\omega \in \Omega \mid \langle \omega, \rho^n \rangle \neq 0 \text{ for some } n \geq 0\}$$

and

$$R(\rho) = \oplus_{\omega \in \Omega(\rho)} \; \mathbb{Z} \cdot \omega.$$

Then $R(\rho)$ is a PSH algebra with set of irreducible elements $\Omega(\rho)$, whose unique irreducible primitive is ρ and

$$R = \otimes_{\rho \in \mathcal{C}} \; R(\rho).$$

3. The PSH algebra of $\{GL_m\mathbb{F}_q, \; m \geq 0\}$

Let $R(G)$ denote the complex representation ring of a finite group G. Set $R = \oplus_{m \geq 0} \; R(GL_m\mathbb{F}_q)$ with the interpretation that $R_0 \cong \mathbb{Z}$, an isomorphism which gives both a choice of unit and counit for R.

Let $U_{k,m-k} \subset GL_m\mathbb{F}_q$ denote the subgroup of matrices of the form

$$X = \begin{pmatrix} I_k & W \\ & \\ 0 & I_{m-k} \end{pmatrix}$$

where W is an $k \times (m-k)$ matrix. Let $P_{k,m-k}$ denote the parabolic subgroup of $GL_m\mathbb{F}_q$ given by matrices obtained by replacing the identity matrices I_k and I_{m-k} in the condition for membership of $U_{k,m-k}$ by matrices from $GL_k\mathbb{F}_q$ and $GL_{m-k}\mathbb{F}_q$ respectively. Hence there is a group extension of the form

$$U_{k,m-k} \longrightarrow P_{k,m-k} \longrightarrow GL_k\mathbb{F}_q \times GL_{m-k}\mathbb{F}_q.$$

If V is a complex representation of $GL_m\mathbb{F}_q$ then the fixed points $V^{U_{k,m-k}}$ is a representation of $GL_k\mathbb{F}_q \times GL_{m-k}\mathbb{F}_q$ which gives the $(k, m-k)$ component of

$$m^* : R \longrightarrow R \otimes R.$$

Given a representation W of $GL_k\mathbb{F}_q \times GL_{m-k}\mathbb{F}_q$ so that $W \in R_k \otimes R_{m-k}$ we may form

$$\mathrm{Ind}_{P_{k,m-k}}^{GL_m\mathbb{F}_q}(\mathrm{Inf}_{GL_k\mathbb{F}_q\times GL_{m-k}\mathbb{F}_q}^{P_{k,m-k}}(W))$$

which gives the $(k, m-k)$ component of

$$m : R \otimes R \longrightarrow R.$$

We choose a basis for R_m to be the irreducible representations of $GL_m\mathbb{F}_q$ so that R^+ consists of the classes of representations (rather than virtual ones). Therefore it is clear that $m, m^*, \epsilon, \epsilon^*$ satisfy positivity. The inner product on R is given by the Schur inner product so that for two representations V, W of $GL_m\mathbb{F}_q$ we have

$$\langle V, W\rangle = \dim_{\mathbb{C}}(\mathrm{Hom}_{GL_m\mathbb{F}_q}(V, W))$$

and for $m \neq n$ R_n is orthogonal to R_m. As is well-known, with these choice of inner product, the basis of irreducible representations for R is an orthonormal basis.

The irreducible primitive elements are represented by irreducible complex representations of $GL_m\mathbb{F}_q$ which have no non-zero fixed vector for any of the subgroups $U_{k,m-k}$. These representations are usually called cuspidal.

In the remainder of this section we shall verify that R is a PSH algebra, as is shown in ([11] Chapter III). I believe, in different terminology, this structural result was known to Sandy Green at the time of writing [3] and to his research supervisor Phillip Hall.

Theorem 3.1. *(Self-adjoint)*

If X, Y, Z are complex representations of $GL_m\mathbb{F}_q, GL_n\mathbb{F}_q, GL_{m+n}\mathbb{F}_q$ respectively then

$$\langle m(X \otimes Y), Z\rangle = \langle X \otimes Y, m^*(Z)\rangle.$$

Also ϵ and ϵ^* are mutually adjoint.

Proof:

This follows from Frobenius reciprocity ([8] Theorem 1.2.39) because the Schur inner product is given by

$$\begin{aligned}\langle m(X \otimes Y), Z\rangle &= \dim_{\mathbb{C}}(\mathrm{Hom}_{GL_{m+n}\mathbb{F}_q}(m(X \otimes Y), Z)) \\ &= \dim_{\mathbb{C}}(\mathrm{Hom}_{P_{m,n}}\mathrm{Inf}_{GL_m\mathbb{F}_q\times GL_n\mathbb{F}_q}^{P_{m,n}}(X \otimes Y), Z)) \\ &= \dim_{\mathbb{C}}(\mathrm{Hom}_{P_{m,n}}\mathrm{Inf}_{GL_m\mathbb{F}_q\times GL_n\mathbb{F}_q}^{P_{m,n}}(X \otimes Y), Z^{U_{m,n}})).\end{aligned}$$

The adjointness of ϵ and ϵ^* is obvious. □

Proposition 3.2. *(Associativity and coassociativity)*
The coproduct m^* is coassociative and the product m is associative.

Proof:
Clearly m^* is coassociative because taking fixed-points $GL_a\mathbb{F}_q \times GL_b\mathbb{F}_q \times GL_c\mathbb{F}_q$ of a $GL_{a+b+c}\mathbb{F}_q$ representation is clearly associative. It follows from Theorem 3.1 that m is associative, since the Schur inner product is non-singular. $\square$

Theorem 3.3. *(Hopf condition)*
The homomorphism m^* is an algebra homomorphism with respect to m. The homomorphism m is a coalgebra homomorphism with respect to m^*.

Obviously the coalgebra homomorphism assertion follows from the algebra homomorphism assertion by the adjointness property of Theorem 3.1.

The discussion which follows will establish Theorem 3.3. It is rather delicate and involved so I am going to give it in full detail (following ([11] p.167 and p.173 with minor changes). For notational convenience I shall write $G_n = GL_n\mathbb{F}_q$ for the duration of this discussion.

Recall that we are attempting to show that for each $(\alpha, m-\alpha)$ and $(a, m-a)$ that the $R(G_a) \otimes R(G_{m-a})$-component of $m^* \cdot m$

$$R(G_\alpha) \otimes R(G_{m-\alpha}) \stackrel{m}{\longrightarrow} R(G_m) \stackrel{m^*}{\longrightarrow} R(G_a) \otimes R(G_{m-a})$$

is equal to the $R(G_a) \otimes R(G_{m-a})$-component

$$R(G_\alpha) \otimes R(G_{m-\alpha}) \stackrel{m^*\otimes m^*}{\longrightarrow} R\otimes R\otimes R\otimes R \stackrel{1\otimes T\otimes 1}{\longrightarrow} R\otimes R\otimes R\otimes R \stackrel{m\otimes m}{\longrightarrow} R\otimes R.$$

Let Z be a complex representation of G_m then the $(a, m-a)$-component of $m^*(Z)$ is given by

$$Z^{U_{a,m-a}} \in R(G_a \times G_{m-a}) \cong R(G_a) \otimes R(G_{m-a})$$

with the group action given by the induced $P_{a,m-a}/U_{a,m-a}$-action.

If $Z = m(X \otimes Y)$ with X, Y representations of $G_\alpha, G_{m-\alpha}$ respectively then

$$Z = m(X \otimes Y) = \mathrm{Ind}_{P_{\alpha,m-\alpha}}^{G_m}(\mathrm{Inf}_{G_\alpha\times G_{m-\alpha}}^{P_{\alpha,m-\alpha}}(X \otimes Y)).$$

Therefore we must study the restriction

$$\mathrm{Res}_{P_{a,m-a}}^{G_m}(\mathrm{Ind}_{P_{\alpha,m-\alpha}}^{G_m}(\mathrm{Inf}_{G_\alpha\times G_{m-\alpha}}^{P_{\alpha,m-\alpha}}(X \otimes Y)))$$

by means of the Double Coset Formula ([8] Theorem 1.2.40; [9] Chapter 7, §1). Explicitly the Double Coset Formula in this case gives

$$\sum_{g\in P_{a,m-a}\backslash G_m/P_{\alpha,m-\alpha}} \mathrm{Ind}_{P_{a,m-a}\cap gP_{\alpha,m-\alpha}g^{-1}}^{P_{a,m-a}}((g^{-1})^*\mathrm{Inf}_{G_\alpha\times G_{m-\alpha}}^{P_{\alpha,m-\alpha}}(X \otimes Y))$$

where the $(g^{-1})^*$-action is given by $(ghg^{-1})(w) = hw$.

The Double Coset Formula isomorphism (downwards) is given by

$$z \otimes_{P_{\alpha,m-\alpha}} w \mapsto j \otimes_{P_{a,m-a}\cap gP_{\alpha,m-\alpha}g^{-1}} hw$$

where $z = jgh$ with $j \in P_{a,m-a}, h \in P_{\alpha,m-\alpha}$ with inverse (upwards) given by

$$j \otimes_{P_{a,m-a} \cap g P_{\alpha,m-\alpha} g^{-1}} w \mapsto jg \otimes_{P_{\alpha,m-\alpha}} w.$$

Next let $\Sigma_m \subset GL_m \mathbb{F}_q$ denote the symmetric group on m letters embedded as the subgroup of permutation matrices (i.e. precisely one non-zero entry on each row and column which is equal to 1).

The following result is proved in ([11] p.173; see also [1] Chapter IV, §2)

Theorem 3.4. *(Bruhat Decomposition)*

The inclusion of Σ_m into $GL_m \mathbb{F}_q$ induces a bijection

$$\Sigma_a \times \Sigma_{m-a} \backslash \Sigma_m / \Sigma_\alpha \times \Sigma_{m-\alpha} \stackrel{\cong}{\longrightarrow} P_{a,m-a} \backslash G_m / P_{\alpha,m-\alpha}$$

Now we shall construct a convenient set of double coset representations for the left-hande side of Theorem 3.4.

Consider the double cosets

$$\Sigma_a \times \Sigma_{m-a} \backslash \Sigma_m / \Sigma_\alpha \times \Sigma_{m-\alpha}.$$

On page 171 of [11] one finds the assertion that the double cosets in the title of this section are in bijection with the matrices of non-negative integers

$$\begin{pmatrix} k_{1,1} & k_{1,2} \\ \\ k_{2,1} & k_{2,2} \end{pmatrix}$$

which satisfy

$$k_{1,1} + k_{1,2} = \alpha,\ k_{2,1} + k_{2,2} = m - \alpha,\ k_{1,1} + k_{2,1} = a,\ k_{1,2} + k_{2,2} = m - a.$$

Let $w \in \Sigma_m$ be a permutation of $\{1, \ldots, m\}$. Set $I_1 = \{1, 2, \ldots, a\}$, $I_2 = \{a+1, a+2, \ldots, m\}$, $J_1 = \{1, 2, \ldots, \alpha\}$ and $J_2 = \{\alpha+1, \alpha+2, \ldots, m\}$. Therefore if $g \in \Sigma_a \times \Sigma_{m-a}$ and $g' \in \Sigma_\alpha \times \Sigma_{m-\alpha}$ we have, for $t = 1, 2$ and $v = 1, 2$,

$$gwg'(J_t) \bigcap I_v = gw(J_t) \bigcap I_v = g(w(J_t) \bigcap g^{-1}(I_v)) = g(w(J_t) \bigcap I_v).$$

Therefore if we set

$$k_{t,v} = \#(w(J_t) \bigcap I_v)$$

we have a well-defined map of sets from the double cosets to the 2×2 matrices of the form described above because

$$k_{1,v} + k_{2,v} = \#(I_v) = \begin{cases} a & \text{if } v = 1, \\ \\ m - a & \text{if } v = 2 \end{cases}$$

and

$$k_{t,1} + k_{t,2,} = \#(J_t) = \begin{cases} \alpha & \text{if } t = 1, \\ \\ m - \alpha & \text{if } t = 2. \end{cases}$$

Next we consider the passage from the matrix of $k_{i,j}$'s to a double coset. Write

$$J_1 = J(k_{1,1}) \bigcup J(k_{1,2}), \; J_2 = J(k_{2,1}) \bigcup J(k_{2,2})$$

where $J(k_{1,1}) = \{1, \ldots, k_{1,1}\}$ and $J(k_{2,1}) = \{\alpha + 1, \ldots, \alpha + k_{2,1}\}$. Similarly write

$$I_1 = I(k_{1,1}) \bigcup I(k_{2,1}), \; I_2 = I(k_{1,2}) \bigcup I(k_{2,2})$$

where $I(k_{1,1}) = \{1, \ldots, k_{1,1}\}$ and $I(k_{1,2}) = \{a+1, \ldots, a+k_{1,2}\}$. Since the orders of $I(k_{i,j})$ and $J(k_{i,j})$ are both equal to $k_{i,j}$ there is a permutation, denoted by $w(k_{*,*})$ which sends $J_1 \bigcup J_2$ to $I_1 \bigcup I_2$ by the identity on $J(k_{1,1}) = I(k_{1,1})$ and $J(k_{2,2}) = I(k_{2,2})$ and interchanges $J(k_{1,2}), J(k_{2,1})$ with $I(k_{1,2}), I(k_{2,1})$ in an order-preserving manner.

Given the permutation $w(k_{*,*})$ we have

$$\#(w(k_{*,*})(J_1) \bigcap I_1) = \#(J(k_{1,1}) \bigcap I(k_{1,1})) = k_{1,1},$$

$$\#(w(k_{*,*})(J_1) \bigcap I_2) = \#(w(k_{*,*})(J(k_{2,1})) \bigcap I(k_{1,2})) = k_{1,2},$$

$$\#(w(k_{*,*})(J_2) \bigcap I_1) = \#(w(k_{*,*})(J(k_{1,2})) \bigcap I(k_{2,1})) = k_{2,1},$$

$$\#(w(k_{*,*})(J_2) \bigcap I_2) = \#(J(k_{2,2}) \bigcap I(k_{2,2})) = k_{2,2}$$

so that the map $k_{*,*} \mapsto \Sigma_a \times \Sigma_{m-a} w(k_{*,*}) \Sigma_\alpha \times \Sigma_{m-\alpha}$ is a split injection. In addition it is straightforward to verify that any permutation whose $k_{*,*}$-matrix equals that of $w(k_{*,*})$ belongs to the same double coset as $w(k_{*,*})$. Hence the map is a bijection.

For example when $a = 3, \alpha = 4, k_{11} = 1 = k_{22}, k_{21} = 2, k_{12} = 3$

$$w(k_{*,*})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad w(k_{*,*}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This permutation arises in another way as a permutation of the basis elements of tensor products of four vector spaces. Let

$$V_1 = \mathbb{F}_q^{k_{11}} \oplus \mathbb{F}_q^{k_{12}} \oplus \mathbb{F}_q^{k_{21}} \oplus \mathbb{F}_q^{k_{22}} \text{ and } V_2 = \mathbb{F}_q^{k_{11}} \oplus \mathbb{F}_q^{k_{21}} \oplus \mathbb{F}_q^{k_{12}} \oplus \mathbb{F}_q^{k_{22}}.$$

We have the linear map

$$1 \oplus T(k_{*,*}) \oplus 1 : V_1 \longrightarrow V_2$$

which interchanges the order of the two central direct sum factors. The basis for V_1 is made in the usual manner from ordered bases $\{e_1, \ldots\ e_{k_{11}}\}$, $\{e_{k_{11}+1}, \ldots\ e_{k_{11}+k_{12}}\}$, $\{e_{k_{11}+k_{12}+1}, \ldots\ e_{k_{11}+k_{12}+k_{21}}\}$ and $\{e_{k_{11}+k_{12}+k_{21}+1}, \ldots\ e_m\}$ of $\mathbb{F}_q^{k_{11}}$, $\mathbb{F}_q^{k_{12}}$, $\mathbb{F}_q^{k_{21}}$ and $\mathbb{F}_q^{k_{22}}$ respectively. Similarly the basis for V_2 is made in the usual manner from ordered bases $\{v_1, \ldots\ v_{k_{11}}\}$, $\{v_{k_{11}+1}, \ldots\ v_{k_{11}+k_{21}}\}$,

$\{v_{k_{11}+k_{21}+1}, \ldots\ v_{k_{11}+k_{21}+k_{12}}\}$ and $\{v_{k_{11}+k_{21}+k_{12}+1}, \ldots\ v_m\}$ of $\mathbb{F}_q^{k_{11}}$, $\mathbb{F}_q^{k_{21}}$, $\mathbb{F}_q^{k_{12}}$ and $\mathbb{F}_q^{k_{22}}$ respectively.

The linear map $1 \oplus T(k_{*,*}) \oplus 1$ sends the ordered set $\{e_1, \ldots, e_m\}$ to the order set $\{v_1, \ldots, v_m\}$ by $e_j \mapsto v_{w(k_{*,*})(j)}$.

Clearly

$$w(k_{*,*})G_{k_{11}} \times G_{k_{12}} \times G_{k_{21}} \times G_{k_{22}} w(k_{*,*})^{-1} = G_{k_{11}} \times G_{k_{21}} \times G_{k_{12}} \times G_{k_{22}}$$

from which is it easy to see that

$$w(k_{*,*})G_\alpha \times G_{m-\alpha} w(k_{*,*})^{-1} \bigcap G_a \times G_{m-a} = G_{k_{11}} \times G_{k_{21}} \times G_{k_{12}} \times G_{k_{22}}.$$

So far we have shown that the $(a, m-a)$-component of $m^*(m(X \otimes Y))$ is the sum of terms, one for each $w(k_{*,*})$, given by the induced $G_a \times G_{m-a}$-action on

$$\mathrm{Ind}_{P_{a,m-a} \cap w(k_{*,*} P_{\alpha,m-\alpha} w(k_{*,*})^{-1}}^{P_{a,m-a}}((w(k_{*,*})^{-1})^* \mathrm{Inf}_{G_\alpha \times G_{m-\alpha}}^{P_{\alpha,m-\alpha}}(X \otimes Y)).$$

On the other hand, for each $w(k_{*,*})$ there is a $(a, m-a)$-component of the other composition we are studying given by

$$R(G_\alpha \times G_{m-\alpha}) \stackrel{(-)^{U_{k_{11},k_{12}} \times U_{k_{21},k_{22}}}}{\longrightarrow} R(G_{k_{11}} \times G_{k_{12}} \times G_{k_{21}} \times G_{k_{22}})$$

$$\stackrel{1 \otimes T(k_{*,*}) \otimes 1}{\longrightarrow} R(G_{k_{11}} \times G_{k_{21}} \times G_{k_{12}} \times G_{k_{22}})$$

$$\stackrel{\mathrm{IndInf} \times \mathrm{IndInf}}{\longrightarrow} R(G_a \times G_{m-a}).$$

Composing this second route with the split surjection

$$R(G_a \times G_{m-a}) \stackrel{\mathrm{Inf}}{\longrightarrow} R(P_{a,m-a})$$

is equal to the composition

$$R(G_\alpha \times G_{m-\alpha}) \stackrel{(-)^{U_{k_{11},k_{12}} \times U_{k_{21},k_{22}}}}{\longrightarrow} R(G_{k_{11}} \times G_{k_{12}} \times G_{k_{21}} \times G_{k_{22}})$$

$$\stackrel{1 \otimes T(k_{*,*}) \otimes 1}{\longrightarrow} R(G_{k_{11}} \times G_{k_{21}} \times G_{k_{12}} \times G_{k_{22}}) \stackrel{\mathrm{Inf}}{\longrightarrow}$$

$$R(P_{k_{11},k_{21},k_{12},k_{22}}) \stackrel{\mathrm{Ind}}{\longrightarrow} R(P_{a,m-a})$$

because the kernels of the quotient maps $P_{k_{11},k_{21},k_{12},k_{22}} \longrightarrow P_{k_{11},k_{21}}$ and $P_{a,m-a} \longrightarrow G_a \times G_{m-a}$ are both equal to $U_{a,m-a}$.

This composition takes the $U_{k_{11},k_{12}} \times U_{k_{21},k_{22}}$-fixed points of $X \otimes Y$ with the $G_{k_{11}} \times G_{k_{12}} \times G_{k_{21}} \times G_{k_{22}}$-action and then conjugates it by $w(k_{*,*})$. Alternatively it takes the $w(k_{*,*})U_{k_{11},k_{12}} \times U_{k_{21},k_{22}} w(k_{*,*})^{-1}$-fixed points of $(w(k_{*,*})^{-1})^*(X \otimes Y)$ with the $G_{k_{11}} \times G_{k_{21}} \times G_{k_{12}} \times G_{k_{22}}$-action. Now

$$w(k_{*,*})U_{k_{11},k_{12}} \times U_{k_{21},k_{22}} w(k_{*,*})^{-1} \subset U_{a,m-a}.$$

For example, in the small example given in the Appendix, $U_{k_{11},k_{12}} \times U_{k_{21},k_{22}}$ consists of matrices of the form

$$D = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & a_{57} \\ 0 & 0 & 0 & 0 & 0 & 1 & a_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

so that $w(k_{*,*})U_{k_{11},k_{12}} \times U_{k_{21},k_{22}} w(k_{*,*})^{-1}$ consists of matrices

$$w(k_{*,*})Dw(k_{*,*})^{-1} = \begin{pmatrix} 1 & 0 & 0 & a_{12} & a_{13} & a_{14} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & a_{57} \\ 0 & 0 & 1 & 0 & 0 & 0 & a_{67} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $w(k_{*,*})Dw(k_{*,*})^{-1}$'s act trivially we may inflate the representation to $P_{k_{11},k_{21},k_{12},k_{22}}$ (i.e. extending the action trivially on $U_{k_{11},k_{21},k_{12},k_{22}}$) and then induce up to a representation of $P_{a,m-a}$.

Now let us describe the isomorphism between the result of sending $X \otimes Y$ via the second route and the $U_{a,m-a}$-fixed subspace of

$$\mathrm{Ind}_{P_{a,m-a} \cap w(k_{*,*} P_{\alpha,m-\alpha} w(k_{*,*})^{-1}}^{P_{a,m-a}}((w(k_{*,*})^{-1})^* \mathrm{Inf}_{G_\alpha \times G_{m-\alpha}}^{P_{\alpha,m-\alpha}}(X \otimes Y)).$$

There are inclusions

$$w(k_{*,*})P_{\alpha,m-\alpha})w(k_{*,*})^{-1} \bigcap P_{a,m-a} \subset P_{k_{11},k_{21},k_{12},k_{22}} \subset P_{a,m-a}.$$

For example, $w(k_{*,*})P_{\alpha,m-\alpha}w(k_{*,*})^{-1} \bigcap P_{3,4}$, in the small example of the Appendix, consists of the matrices of the form

$$E' = \begin{pmatrix} a_{11} & a_{15} & a_{16} & a_{12} & a_{13} & a_{14} & a_{17} \\ 0 & a_{55} & a_{56} & 0 & 0 & 0 & a_{57} \\ 0 & a_{65} & a_{66} & 0 & 0 & 0 & a_{67} \\ 0 & 0 & 0 & a_{22} & a_{23} & a_{24} & a_{27} \\ 0 & 0 & 0 & a_{32} & a_{33} & a_{34} & a_{37} \\ 0 & 0 & 0 & a_{42} & a_{43} & a_{44} & a_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} \end{pmatrix}$$

and, as we noted above, $w(k_{*,*})Dw(k_{*,*})^{-1}$ consists of the matrices

$$\{(b_{ij}) \in w(k_{*,*})P_{\alpha,m-\alpha}w(k_{*,*})^{-1} \bigcap U_{3,4} \mid b_{17} = 0\}.$$

There is a bijection of cosets

$$P_{k_{11},k_{21},k_{12},k_{22}}/w(k_{*,*})P_{\alpha,m-\alpha})w(k_{*,*})^{-1}\bigcap P_{a,m-a}$$

$$\cong U_{a,m-a}/w(k_{*,*})P_{\alpha,m-\alpha})w(k_{*,*})^{-1}\bigcap U_{3,4}.$$

Therefore we may take the coset representations X_α to lie in the abelian group $U_{a,m-a}$. In the small example the X_α's may be taken to be of the form

$$X_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & b & c & d & 0 \\ 0 & 0 & 1 & e & f & g & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The isomorphism from the image of $X\otimes Y$ via the second route to the $U_{a,m-a}$-fixed subspace of

$$\mathrm{Ind}^{P_{a,m-a}}_{P_{a,m-a}\cap w(k_{*,*}P_{\alpha,m-\alpha}w(k_{*,*})^{-1}}((w(k_{*,*})^{-1})^*\mathrm{Inf}^{P_{\alpha,m-\alpha}}_{G_\alpha\times G_{m-\alpha}}(X\otimes Y))$$

is given by

$$g\otimes_{P_{k_{11},k_{21},k_{12},k_{22}}} v\mapsto \sum_{X_\alpha} gX_\alpha\otimes_{w(k_{*,*})P_{\alpha,m-\alpha})w(k_{*,*})^{-1}\bigcap P_{a,m-a}} v.$$

This concludes the proof of Theorem 3.3. The remainder of the Hopf condition is given by the following result, which is proved in a similar manner to Theorem 3.3 (see [11] p.175).

Theorem 3.5.
In the notation of §1, ϵ^* is a ring homomorphism restricting to an isomorphism on R_0.

4. Semi-direct products $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)\propto GL_t\mathbb{F}_{q^n}$

Let V be an irreducible representation of $GL_t\mathbb{F}_{q^n}$ and let $\Sigma\in\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ denote the Frobenius substitution. Hence the representation $\Sigma^i(V)$ given by transporting the $GL_t\mathbb{F}_{q^n}$-action by the i-th power of Σ is another irreducible representation. Suppose that $n=sd$ and that

$$V,\Sigma(V),\Sigma^2(V),\ldots,\Sigma^{s-1}(V)$$

are inequivalent $GL_t\mathbb{F}_{q^n}$-irreducibles but that V and $\Sigma^s(V)$ are equivalent $GL_t\mathbb{F}_{q^n}$-irreducibles.

Therefore V (c.f. [9] Chapter Two, §6; see also Chapter 8 and Chapter 9, §3) extends to an irreducible representation $\tilde{V}$ of the semi-direct product $\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s})\propto GL_t\mathbb{F}_{q^n}$ for some $b\geq 1$, where $\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s})$ acts via first projecting onto $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_{q^s})$.

In discussions of the semi-direct product I shall attempt to follow the notational conventions of ([7] p.36) and ([9] Chapter 9, §3.1) as opposed to those of [6]. Explicitly, if C acts on G via $\lambda : C \longrightarrow \mathrm{Aut}(G)$ then the semi-direct product $C \propto G$ is the group whose underlying set is $C \times G$ with multiplication given by

$$(c_1, g_1) \cdot (c_2, g_2) = (c_1 c_2, g_1 \lambda(c_1)(g_2)), \qquad c_i \in C, g_i \in G.$$

We may form the induced representation

$$\hat{V} = \mathrm{Ind}_{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s}) \propto GL_t\mathbb{F}_{q^n}}^{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_q) \propto GL_t\mathbb{F}_{q^n}}(\tilde{V})$$

which restricts to give

$$\oplus_{i=0}^{s-1} \Sigma^i(V) \in R(GL_t\mathbb{F}_{q^n}).$$

Also

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_q) \propto GL_t\mathbb{F}_{q^n}}(\hat{V}, \hat{V}) \\ &\cong \mathrm{Hom}_{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s}) \propto GL_t\mathbb{F}_{q^n}}(\tilde{V}, \mathrm{Ind}_{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s}) \propto GL_t\mathbb{F}_{q^n}}^{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_q) \propto GL_t\mathbb{F}_{q^n}}(\tilde{V})) \\ &\cong \oplus_{i=0}^{s-1} \mathrm{Hom}_{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s}) \propto GL_t\mathbb{F}_{q^n}}(\tilde{V}, \Sigma^i(\tilde{V})) \\ &= \mathrm{Hom}_{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s}) \propto GL_t\mathbb{F}_{q^n}}(\tilde{V}, \tilde{V}) \end{aligned}$$

as is seen by restricting representations to $GL_t\mathbb{F}_{q^n}$. Since $\tilde{V}$ is irreducible its endomorphism ring is 1-dimensional and so therefore is that of $\hat{V}$.

In the terminology of [6] when $s > 1$ $\hat{V}$ is called an irreducible representation of the second kind and when $s = 1$ it is called an irreducible representation of the first kind..

Let $\theta : \mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_q) \longrightarrow \mathbb{C}^*$ be a character which is trivial on $\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s})$. Then

$$\theta \cdot \hat{V} = \mathrm{Ind}_{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s}) \propto GL_t\mathbb{F}_{q^n}}^{\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_q) \propto GL_t\mathbb{F}_{q^n}}(\theta \cdot \tilde{V}) = \hat{V}.$$

All the irreducibles of $\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_q) \propto GL_t\mathbb{F}_{q^n}$ are of the form $\hat{V}$ for some s dividing n. For if W is an irreducible of $\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_q) \propto GL_t\mathbb{F}_{q^n}$ then its restriction to $GL_t\mathbb{F}_{q^n}$ must have the form

$$mV_1 \oplus m\Sigma(V_1) \oplus \ldots \oplus m\Sigma^{s-1}(V_1)$$

with V_1 irreducible and $\Sigma^s(V_1) = V_1$. Therefore V_1 extends to an irreducible $\tilde{V}_1$ and there is a non-zero map of representations $\hat{V}_1 \longrightarrow W$ which must be an isomorphism (and so $m = 1$).

Twisting $\hat{V}$ by a character θ which is non-trivial on $\mathrm{Gal}(\mathbb{F}_{q^{bn}}/\mathbb{F}_{q^s})$ gives a distinct irreducible. There are bn/s cosets of such θ's so we have bn/s distinct irreducibles

$$\theta_1\hat{V}, \theta_2\hat{V}, \ldots, \theta_{bn/s}\hat{V}$$

each restricting to

$$\oplus_{i=0}^{s-1} \Sigma^i(V) \in R(GL_t\mathbb{F}_{q^n}).$$

By Shintani base change for finite general linear groups ([6]; see also [9] Chapter 8 and Chapter 9, §3) there is a bijection between $GL_t\mathbb{F}_{q^n}$-irreducibles V such that $\Sigma^s(V) = V$ and the irreducibles of $GL_t\mathbb{F}_{q^s}$. The V's in the construction of $\hat{V}$ are those which are fixed by Σ^s but by no Σ^u with u a proper divisor of s.

En route to the base change result one finds ([6] Theorem 1) that $b = 1$ or $b = 2$ suffices for the extension to the semi-direct product which was discussed in this section. This is explained in §6 just after the statement of Theorem 6.1.

5. $\tilde{R}$ and R''

Let $K = R(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q))$ which is the ring of integral linear combinations of characters $\chi : \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \longrightarrow \mathbb{C}^*$. Suppose that G is a subgroup of $GL_t\mathbb{F}_{q^n}$ which is preserved by the $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-action. Let $\mathcal{S}(G)$ denote the subset of the irreducibles $\mathrm{Irr}(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \propto G)$ of the first kind (i.e. representations which are irreducible when restricted to G). Tensoring with a Galois character χ permutes the set $\mathcal{S}(G)$ making $\mathbb{Z}[S(G)]$ into a free K-module.

Define $\tilde{R} = \oplus_{t\geq 0}\ \tilde{R}_t$ where $\tilde{R}_t = R(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \propto GL_t\mathbb{F}_{q^n})$ for a fixed choice of n. Define $R'' = \oplus_{t\geq 0}\ \mathcal{S}(GL_t\mathbb{F}_{q^n}) \subset \tilde{R}$.

For each $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-invariant irreducible $V \in \mathrm{Irr}(GL_t\mathbb{F}_{q^n})^{\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)}$ choose an irreducible $\tilde{V} \in \tilde{R}_t$ which restricts to V. The set of irreducibles which restrict to V are given by $\{\chi \otimes \tilde{V}\}$ as χ varies through Galois characters. Therefore there is an isomorphism of K-modules, depending on the choice of $\tilde{V}$'s, of the form

$$\lambda_{GL_t\mathbb{F}_{q^n}} : K[\mathrm{Irr}(GL_t\mathbb{F}_{q^n})^{\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)}] \stackrel{\cong}{\longrightarrow} R''_t = \mathcal{S}(GL_t\mathbb{F}_{q^n})$$

given by sending $\chi \otimes V$ to $\chi \otimes \tilde{V}$. Hence $\tilde{R}$ is a K-module of which $R'' = \oplus_{t\geq 0}\ \mathcal{S}(GL_t\mathbb{F}_{q^n})$ is a free K-submodule.

The first objective of this section is to make $\tilde{R}$ into a connected, graded K-algebra[1] of which R'' is a connected, graded K-subalgebra. Clearly we have an isomorphism $\epsilon : K \longrightarrow \tilde{R}_0 = R''_0$ which shows that these are connected K-algebras.

Multiplication:

Let $P_{a,b}$ be the usual parabolic subgroup of $GL_{a+b}\mathbb{F}_{q^n}$. Then inflation induces a K-module homomorphism of representation rings

$$\mathrm{Inf}_{a,b} : R(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \propto (GL_a\mathbb{F}_{q^n} \times GL_b\mathbb{F}_{q^n})) \longrightarrow R(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \propto P_{a,b}).$$

We also have induction maps

$$\mathrm{Ind}_{a,b} : R(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \propto P_{a,b}) \longrightarrow R(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \propto GL_{a+b}\mathbb{F}_{q^n}).$$

[1] The structure map giving the multiplication in this algebra first appeared in ([6] Definition 2.4)

Let V and W be representations of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \propto GL_a\mathbb{F}_{q^n})$ and $\mathrm{Gal}\mathbb{F}_{q^n}/\mathbb{F}_q) \propto GL_b\mathbb{F}_{q^n})$ respectively. Define a representation $V \otimes' W$ of $\mathbb{F}_{q^n}/\mathbb{F}_q) \propto (GL_a\mathbb{F}_{q^n} \times GL_b\mathbb{F}_{q^n})$ on the underlying vector space of $V \otimes W$ by

$$(g, X, Y)(v \otimes w) = (g, X)(v) \otimes (g, Y)(w).$$

The multiplication on $\tilde{R}$ is defined, following the $GL\mathbb{F}_{q^n}$-case, by

$$m(V \otimes W) = \mathrm{Ind}_{a,b}(\mathrm{Inf}_{a,b}(V \otimes' W)) \in \tilde{R}_{a+b}.$$

If χ belongs to the character group of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ then

$$m(\chi V \otimes W) = m(V \otimes \chi W) = \chi m(V \otimes W) \in \tilde{R}_{a+b}.$$

If V and W are representations in R''_a and R''_b respectively we shall show that $m(V \otimes W) \in R''_{a+b}$.

By additivity it suffices to assume that V, W are irreducibles (of the first kind). Then, by the construction of all the irreducibles of the finite general linear groups which first appears in [3] and is reiterated in ([4] §1) and ([9] Chapter 11, §2), $m(V \otimes W)$ is irreducible when restricted to $GL_{a+b}\mathbb{F}_{q^n}$ unless $W = \chi \otimes V$. In that case $m(V \otimes (\chi \otimes W)) = \chi \otimes m(V \otimes V)$. Restricted to $GL_{a+b}\mathbb{F}_{q^n}$ the latter is known to be the sum of two irreducibles picked out by the idempotents of the symmetric group on two letters[2]. However these idempotents also decompose $m(V \otimes V)$ into two irreducibles of the first kind, in the same way.

Note that m factorises through

$$m : \tilde{R} \otimes_K \tilde{R} \longrightarrow \tilde{R}$$

which is a K-module homomorphism. Also m is associative.

This discussion established the following result.

Theorem 5.1.

With the notation introduced above $\tilde{R}$ is a graded K-algebra of which R'' is a graded K-subalgebra.

Let V be an irreducible of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \propto GL_{a+b}\mathbb{F}_{q^n}$. We are not going to define a comultiplication on $\tilde{R}$. However, we close this section with the observation that sending V to its $U_{a,b}$-fixed points yields a homomorphism

$$m^* : \tilde{R}_{a+b} \longrightarrow R(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \propto (GL_a\mathbb{F}_{q^n} \times GL_b\mathbb{F}_{q^n}))$$

which covers (via the restriction to general linear groups) the comultiplication defined in §3 on the PSH algebra $\oplus_{t \geq 0} R(GL_t\mathbb{F}_{q^n})$.

[2]These idempotents will show up again in §6.

6. Shintani base change

Let us recall the main result of [6] which, for our notation for the semi-direct product, is stated in the following form:

Theorem 6.1. *([6] Theorem 1; see also Lemmas 2.7 and 2.11)*

(i) Let ρ be a finite-dimensional complex irreducible representation of $GL_n\mathbb{F}_q$. Then there exists an irreducible representation $\tilde{\rho}$ of the semi-direct product $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \propto GL_n\mathbb{F}_{q^m}$ which satisfies, for all $g \in GL_n\mathbb{F}_{q^m}$,

$$\chi_{\tilde{\rho}}(\Sigma, g) = \epsilon\chi_\rho([g\Sigma(g)\ldots\Sigma^{m-1}(g)])$$

where $\epsilon = \pm 1$ is independent of g. Here $[g\Sigma(g)\ldots\Sigma^{m-1}(g)]$ denotes the unique conjugacy class in $GL_n\mathbb{F}_q$ given by the intersection of the conjugacy class of $g\Sigma(g)\ldots\Sigma^{m-1}(g)$ in $GL_n\mathbb{F}_{q^m}$ with $GL_n\mathbb{F}_q$.

(ii) The Shintani base change correspondence (see [9] Appendix I, §4)

$$Sh : \mathrm{Irr}(GL_n\mathbb{F}_{q^m})^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)} \xrightarrow{\cong} \mathrm{Irr}(GL_n\mathbb{F}_q)$$

is given by

$$Sh(\mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_n\mathbb{F}_{q^m}}_{GL_n\mathbb{F}_{q^m}}(\tilde{\rho})) = \rho.$$

In this Theorem χ_ρ denotes the character function of ρ. In part (ii) of the theorem it should be noted that $\tilde{\rho}$ is an irreducible of the first kind because the $\chi_{\tilde{\rho}}(\Sigma, g)$'s are not identically zero ([6] Lemma 1.1(i)) and therefore $\mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_n\mathbb{F}_{q^m}}_{GL_n\mathbb{F}_{q^m}}(\tilde{\rho})$ is an irreducible representation.

Given $\tilde{\rho}$ as in part (i) of the theorem write $\tilde{\rho}(z, 1) = X_z$ for $z \in \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$ and $\tilde{\rho}(1, g) = \hat{\rho}(g)$ for $g \in GL_n\mathbb{F}_{q^m}$. Since $(1, g)(z, 1) = (z, g)$ we have

$$X_z\hat{\rho}(g) = \hat{\rho}(z(g))X_z$$

so that $\chi_{\tilde{\rho}}(\Sigma, g) = \mathrm{Trace}(\tilde{\rho}(1, g)X_\Sigma)$ (see [6] Theorem 1)[3].

For $\tilde{\rho}$ and $\hat{\rho}$ as in Theorem 6.1 the matrix X_Σ will satisfy $X_\Sigma^m = 1$. However, as mentioned in the statement of ([6] Theorem 1), for a general Galois invariant $\hat{\rho}$ there exists a choice satisfying $X_\Sigma^m = \pm 1$. When $X_\Sigma^m = 1$ the extension $\tilde{\rho}$ of $\hat{\rho}$ may be constructed as in Theorem 6.1 but when $X_\Sigma^m = -1$ the extension of $\hat{\rho}$ must be a representation of $\mathrm{Gal}(\mathbb{F}_{q^{2m}}/\mathbb{F}_q) \propto GL_n\mathbb{F}_{q^m}$.

Given a choice of $\hat{\rho}$ the irreducible extension $\tilde{\rho}$ to the semi-direct product, which we may take to be $\mathrm{Gal}(\mathbb{F}_{q^{2m}}/\mathbb{F}_q) \propto GL_n\mathbb{F}_{q^m}$ in general, is unique up to twists by Galois characters.

Next we shall examine the multiplicative property of the Shintani correspondence.

Suppose that $\rho_1 \in \mathrm{Irr}(GL_a\mathbb{F}_q)$ and $\rho_2 \in \mathrm{Irr}(GL_b\mathbb{F}_q)$. By Theorem 6.1 there exist $\tilde{\rho}_1 \in \mathrm{Irr}(\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \propto GL_a\mathbb{F}_q)$ and $\tilde{\rho}_2 \in \mathrm{Irr}(\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \propto GL_b\mathbb{F}_q)$ such that for $i = 1, 2$

$$\chi_{\tilde{\rho}_i}(\Sigma, g) = \epsilon_i\chi_{\rho_i}([g\Sigma(g)\ldots\Sigma^{m-1}(g)])$$

[3]The formula of [6] differs from mine because we have used different formulae for the multiplication in a semi-direct product.

where $\epsilon_i = \pm 1$ is independent of g.

Therefore, by ([6] Definition 2.4 and Lemma 2.9),

$$\chi_{m(\tilde{\rho}_1,\tilde{\rho}_2)}(\Sigma, g) = \epsilon_1\epsilon_2\chi_{m(\rho_1,\rho_2)}([g\Sigma(g)\ldots\Sigma^{m-1}(g)])$$

where on the left-hand side m denotes the multiplication in $\tilde{R}$ of §5 and on the right-hand side the multiplication in R of §3.

If $\rho_1 \neq \rho_2$ then by the Shintani correspondence the restrictions of $\tilde{\rho}_1$ and $\tilde{\rho}_2$ to the general linear groups are distinct so that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are distinct irreducible representations. Therefore $m(\tilde{\rho}_1, \tilde{\rho}_2)$ and $m(\rho_1, \rho_2)$ are both irreducible and

$$Sh(\mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_{a+b}\mathbb{F}_{q^m}}_{GL_{a+b}\mathbb{F}_{q^m}}(m(\tilde{\rho}_1,\tilde{\rho}_2))) = m(\rho_1,\rho_2).$$

However

$$\mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_{a+b}\mathbb{F}_{q^m}}_{GL_{a+b}\mathbb{F}_{q^m}}(m(\tilde{\rho}_1,\tilde{\rho}_2))$$

$$= m(\mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_{a}\mathbb{F}_{q^m}}_{GL_{a}\mathbb{F}_{q^m}}(\tilde{\rho}_1), \mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_{b}\mathbb{F}_{q^m}}_{GL_{b}\mathbb{F}_{q^m}}(\tilde{\rho}_2)).$$

Therefore, if $\hat{\rho}_1 = \mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_{a}\mathbb{F}_{q^m}}_{GL_{a}\mathbb{F}_{q^m}}(\tilde{\rho}_1)$ and $\hat{\rho}_2 = \mathrm{Res}^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\propto GL_{b}\mathbb{F}_{q^m}}_{GL_{b}\mathbb{F}_{q^m}}(\tilde{\rho}_2)$, then

$$Sh(m(\hat{\rho}_1,\hat{\rho}_2)) = m(Sh(\hat{\rho}_1), Sh(\hat{\rho}_2)).$$

If $a = b$ and $\rho_1 = \rho_2$ and (see §5 on "multiplication") $m(\rho_1, \rho_1)$ is not irreducible but there is an idempotent e of the symmetric group on two letters such that

$$m(\rho_1,\rho_1) = em(\rho_1,\rho_1) + (1-e)m(\rho_1,\rho_1)$$

and the two summands on the right are irreducible. Similarly

$$m(\tilde{\rho}_1,\tilde{\rho}_1) = em(\tilde{\rho}_1,\tilde{\rho}_1) + (1-e)m(\tilde{\rho}_1,\tilde{\rho}_1)$$

where the two summands on the right are irreducible. In addition

$$\chi_{em(\tilde{\rho}_1,\tilde{\rho}_1)}(\Sigma, g) = \epsilon_1\epsilon_1\chi_{em(\rho_1,\rho_1)}([g\Sigma(g)\ldots\Sigma^{m-1}(g)])$$

and

$$\chi_{(1-e)m(\tilde{\rho}_1,\tilde{\rho}_1)}(\Sigma, g) = \epsilon_1\epsilon_1\chi_{(1-e)m(\rho_1,\rho_1)}([g\Sigma(g)\ldots\Sigma^{m-1}(g)]).$$

Therefore

$$Sh(em(\hat{\rho}_1,\hat{\rho}_1)) = em(Sh(\hat{\rho}_1), Sh(\hat{\rho}_1))$$

and

$$Sh((1-e)m(\hat{\rho}_1,\hat{\rho}_1)) = (1-e)m(Sh(\hat{\rho}_1), Sh(\hat{\rho}_1))$$

and adding these relations yields

$$Sh(m(\hat{\rho}_1,\hat{\rho}_1)) = m(Sh(\hat{\rho}_1), Sh(\hat{\rho}_1)).$$

Set $R' = \oplus_{t\geq 0} R'_t$ where $R'_t = \mathbb{Z}[\mathrm{Irr}(GL_t\mathbb{F}_{q^m})^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)}]$, the subgroup of $R(GL_t\mathbb{F}_{q^m})$ spanned by $\mathbb{Z}$-linear combination of irreducible representations which are invariant under the Galois action. In Theorem 5.1 we saw that R'' is a subalgebra of $\tilde{R}$ and a similar argument shows that R' is a subalgebra of $\oplus_{t\geq 0} R(GL_t\mathbb{F}_{q^m})$.

Theorem 6.2.
With the notation introduced above R' is a graded subalgebra of the algebra $\oplus_{t\geq 0}\ R(GL_t\mathbb{F}_{q^m})$. Furthermore the restriction map

$$\tilde{R} = \oplus_{t\geq 0}\ R(\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \propto GL_t\mathbb{F}_{q^m}) \longrightarrow \oplus_{t\geq 0}\ R(GL_t\mathbb{F}_{q^m})$$

restricts to a surjective algebra homomorphism of the form $R'' \longrightarrow R'$.

The Shintani correspondence of Theorem 6.1 is a bijection of set of irreducible representations. Extending it by additivity yields an isomorphism of abelian groups

$$Sh : R' \stackrel{\cong}{\longrightarrow} R.$$

The preceding discussion concerning the multiplicativity of the Shintani correspondence establishes the following result.

Theorem 6.3.
The $\mathbb{Z}$-linear extension of the Shintani correspondence of Theorem 6.1 yields an algebra isomorphism

$$Sh : R' \stackrel{\cong}{\longrightarrow} R$$

between the Hopf algebras R' introduced above and R of §3.

The algebra isomorphism Sh^{-1} of Theorem 6.3 yields an injective algebra homomorphism

$$R \stackrel{\cong}{\longrightarrow} R' \subset \oplus_{t\geq 0}\ R(GL_t\mathbb{F}_{q^m})$$

between two Hopf algebras is *not* a Hopf algebra homomorphism. This is illustrated by the following $GL_2\mathbb{F}_{q^{2p}}$ example of ([6] p.412; see also [9] Chapter Eight, §1.3).

Suppose that $m = 2p$. Consider the Galois extension $\mathbb{F}_{q^{2p}}/\mathbb{F}_q$ and the irreducible representation of $GL_2\mathbb{F}_{q^{2p}}$ given by $m(\chi_1, \chi_2)$ (in [9] this is denoted by is $R(\chi_1, \chi_2)$) with $\chi_i : \mathbb{F}_{q^{2p}}^* \longrightarrow \mathbb{C}^*$ and Frobenius action $\Sigma(\chi_1) = \chi_2, \Sigma(\chi_2) = \chi_1$ so that

$$\Sigma^* R(\chi_1, \chi_2) = R(\chi_1, \chi_2).$$

This is decomposable in the Hopf algebra $\oplus_{t\geq 0}\ R(GL_t\mathbb{F}_{q^{2p}})$ and therefore is not primitive and therefore it is not primitive in R.

Hence $\mathrm{Gal}(\mathbb{F}_{q^{2p}}/\mathbb{F}_{q^2}) = \langle \Sigma^2 \rangle$ fixes χ_1 and χ_2 so that, by Hilbert's Theorem 90,

$$\chi_1 = \Theta \cdot \mathrm{Norm} : \mathbb{F}_{q^{2p}}^* \longrightarrow \mathbb{F}_{q^2}^* \longrightarrow \mathbb{C}^*$$

and

$$\chi_2 = \Sigma^*(\Theta) \cdot \mathrm{Norm} : \mathbb{F}_{q^{2p}}^* \longrightarrow \mathbb{F}_{q^2}^* \longrightarrow \mathbb{C}^*.$$

Therefore $\Theta \neq \Sigma^*(\Theta)$.

From ([6]; [8], Chapter Two) $Sh(R(\chi_1, \chi_2)) = R(\Theta)$, the Weil representation associated to Θ, which is an irreducible representation of $GL_2\mathbb{F}_q$. However the Weil representation is an example of a irreducible cuspidal representation of $GL_2\mathbb{F}_q$ and, as explained in [11], these are the same as the positive primitive irreducibles in the PSH-algebra for $GL\mathbb{F}_q$.

The character Θ is an example of a regular character of the multiplicative group of a finite field. In fact, as a consequence of the Shintani correspondence together with ([10] Theorem 8-6), the character functions of all the cuspidal representations of the $GL_t\mathbb{F}_q$'s are calculated in ([6] Theorem 2) and, in addition, these cuspidals are shown to be in one-one correspondence with regular characters[4].

7. Counting cuspidals irreducibles of $GL_n\mathbb{F}_q$

This comes from ([6] §3) which culminates in the proof of ([6] Theorem 2).

Let $B_t \subset GL_t\mathbb{F}_{q^n}$ denote the Borel subgroup of upper triangular matrices so that $B_t = D_tU_t$, the semi-direct product of the diagonal and the unitriangular matrices, D_t and U_t, respectively.

Suppose that $\mathbb{F}_q \subset \mathbb{F}_{q^n} \subset \mathbb{F}_{q^m}$ where $m = nd$. A character $\chi : \mathbb{F}_{q^n}^* \longrightarrow \mathbb{C}^*$ is regular if $g \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ and $g(\chi) = \chi$ implies that $g = 1$ (i.e. $\chi \neq \Sigma^l(\chi)$ for $l = 1, \ldots, n-1$).

Define $\tilde{\chi}$ to be the composition $\tilde{\chi} = \chi \cdot \mathrm{Norm}_{\mathbb{F}_{q^m}/\mathbb{F}_{q^n}} : \mathbb{F}_{q^m}^* \longrightarrow \mathbb{C}^*$.

Define a character $\phi_\chi : B_n\mathbb{F}_{q^m} \longrightarrow \mathbb{C}^*$ by the formula

$$\phi_\chi(X_{i,j}) = \prod_{i=1}^{n} \Sigma^{i-1(}\chi) \cdot \mathrm{Norm}_{\mathbb{F}_{q^m}/\mathbb{F}_{q^n}}(X_{i,i}).$$

Therefore, by the regularity of χ, the character ϕ_χ is regular in the sense that $\phi_\chi \neq w^*(\phi_\chi)$, the conjugate of ϕ_χ by a permutation matrix w.

Define a function ψ_χ on $GL_n\mathbb{F}_{q^m}$ by the formula

$$\psi_\chi(g) = \begin{cases} \phi_\chi(X_{i,j}) & \text{if } g = (U_{i,j})w(X_{i,j}) \\ \\ 0 & \text{otherwise} \end{cases}$$

where $(U_{i,j}) \in U_n, (X_{i,j}) \in B$ and w is the permutation matrix given by

$$w^{-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \ldots & 1 \\ 1 & 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}.$$

[4]There are other ways to prove ([6] Theorem 2). For example, in ([3] Theorem 13 p.439) the cuspidals are classified and denoted by g^λ's. Also the result can be derived from ([2] Theorem 9.3.2) which asserts that the irreducible Deligne-Lusztig characters $\pm R_T^G(\theta)$, for regular θ will be cuspidal if and only if the torus T does not lie in any proper Frobenius-stable Levi subgroup of G. I am grateful to Alexander Stasinski for explaining the latter argument to me.

Theorem 7.1. *([6] Theorem 2)*[5]

(i) If χ is a regular character of $\mathbb{F}^*_{q^n}$ there exists an m-th root of unity ξ_m and an irreducible cuspidal representation ρ_χ of $GL_n\mathbb{F}_q$ such that

$$\xi_m \rho_\chi(N_{\mathbb{F}_{q^n}/\mathbb{F}_q}(g)) = \frac{q^{-m(n-1)/2}}{|B_n\mathbb{F}_{q^m}|} \sum_{X \in GL_n\mathbb{F}_{q^m}} \psi_\chi(Xg\Sigma(X)^{-1}).$$

(ii) For two regular characters χ_1, χ_2 $\rho_{\chi_1} = \rho_{\chi_2}$ if and only if $\chi_1 = \Sigma^l(\chi_2)$ for some l. Moreover any cuspidal of $GL_n\mathbb{F}_q$ is equal to ρ_χ for some regular character χ of $\mathbb{F}^*_{q^n}$.

Sketch proof of Theorem 7.1

By Mackey's irreducibility criterion (proved by Frobenius reciprocity and the Double Coset Formula for the restriction of an induced representation) $\mathrm{Ind}_{B_n\mathbb{F}_{q^m}}^{GL_n\mathbb{F}_{q^m}}(\phi_\chi)$ is irreducible. This uses the fact that the permutation matrices are the double coset representatives of $B_n\mathbb{F}_{q^m}\backslash GL_n\mathbb{F}_{q^m}/B_n\mathbb{F}_{q^m}$.

In the tensor product notation for this induced representation as a left $GL_n\mathbb{F}_{q^m}$ we have $g \otimes_{B_n\mathbb{F}_{q^m}} 1 = gb \otimes_{B_n\mathbb{F}_{q^m}} \phi_\chi(b^{-1})$ so that we may think of $g \otimes_{B_n\mathbb{F}_{q^m}} 1$ as the complex-valued function f_g which is defined by $f(x) = 0$ unless $x \in gB_n\mathbb{F}_{q^m}$ and if $x = gb$ with $b \in B_n\mathbb{F}_{q^m}$ then $f_g(x) = \phi_\chi(x^{-1}g) = \phi_\chi(b^{-1})$.

This makes sense because

$$x \otimes_{B_n\mathbb{F}_{q^m}} f_g(x) = gb \otimes_{B_n\mathbb{F}_{q^m}} \phi_\chi(b^{-1}) = g \otimes_{B_n\mathbb{F}_{q^m}} 1.$$

Note that if $b' \in B_n\mathbb{F}_{q^m}$ then $f_g(xb') = f_g(gbb') = \phi_\chi((b')^{-1}b^{-1}g^{-1}g) = \phi_\chi(b')^{-1}f_g(x)$.

In the tensor-product notation for the induced representation the function f_g, transforming as above, corresponds to

$$g \otimes_{B_n\mathbb{F}_{q^m}} 1 = \sum_{h \in GL_n\mathbb{F}_{q^m}/B_n\mathbb{F}_{q^m}} h \otimes_{B_n\mathbb{F}_{q^m}} f_g(h) \in \mathbb{C}[GL_n\mathbb{F}_{q^m}] \otimes_{B_n\mathbb{F}_{q^m}} \mathbb{C}_{\phi_\chi}.$$

To switch from Shintani's conventions to mine we need to define a function $f^{sh}_{g^{-1}}$ by $f^{sh}_{g^{-1}}(x) = f_g(x^{-1})$. Then, if $b, b' \in B_n\mathbb{F}_{q^m}$, $f^{sh}_{g^{-1}}(bx) = f_g(x^{-1}b^{-1})$ is zero unless $x^{-1}b^{-1} = gb'$ and in the latter case

$$f^{sh}_{g^{-1}}(bx) = f_g(x^{-1}b^{-1}) = \phi_\chi(bxg) = \phi_\chi(b)f_g(x^{-1}) = \phi_\chi(b)f^{sh}_{g^{-1}}(x).$$

Following Shintani if w is a permutation matrix write $U^-_w = U \bigcap w^{-1}U^-w$ where $U = U_n\mathbb{F}_{q^m}$ and U^- is the transpose of U (i.e. the lower unitriangular

[5]This result is stated in the conventions for semi-direct products, GL-norms etc of [9] rather than those of [6].

matrices). For the permutation matrix introduced above one finds that

$$U^-_{w^{-1}} = \{u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \dots & \dots & u_{1,n} \\ 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 1 \end{pmatrix}\}.$$

Now consider the product of a matrix in $U^-_{w^{-1}}$ and an unitriangular matrix

$$\begin{pmatrix} 1 & \underline{\alpha} \\ 0 & I_{n-1} \end{pmatrix}\begin{pmatrix} 1 & \underline{\beta} \\ 0 & B \end{pmatrix} = \begin{pmatrix} 1 & \underline{\beta}+\underline{\alpha}B \\ 0 & B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 1 & \underline{\beta}+\underline{\alpha}B \\ 0 & I_{n-1} \end{pmatrix}$$

where $\underline{\beta}, \underline{\alpha}$ are row vectors and B is upper unitriangular $(n-1)\times(n-1)$ matrix. Also we have the matrix relation

$$w^{-1}\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} w = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose that b is the upper triangular matrix

$$b = D\begin{pmatrix} 1 & \underline{\beta} \\ 0 & B \end{pmatrix}$$

where D is the diagonal matrix $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$. Then there exist matrices $u, u', u'' \in U^-_{w^{-1}}$ where

$$u = \begin{pmatrix} 1 & \underline{\alpha} \\ 0 & I_{n-1} \end{pmatrix}$$

such that

$$w^{-1}u\Sigma(b) = w^{-1}\Sigma(D)ww^{-1}u'\begin{pmatrix} 1 & \Sigma(\underline{\beta}) \\ 0 & \Sigma(B) \end{pmatrix} = w^{-1}\Sigma(D)w\begin{pmatrix} \Sigma(B) & 0 \\ 0 & 1 \end{pmatrix} w^{-1}u''.$$

Notice that

$$w^{-1}\Sigma(D)w\begin{pmatrix} \Sigma(B) & 0 \\ 0 & 1 \end{pmatrix} \in B_n\mathbb{F}_{q^m}$$

and that

$$\phi_\chi(w^{-1}\Sigma(D)w\begin{pmatrix} \Sigma(B) & 0 \\ 0 & 1 \end{pmatrix}) = \phi_\chi(b).$$

In addition, as u runs through $U^-_{w^{-1}}$ so does u''.

Then Shintani defines I_Σ by the formula

$$(I_\Sigma f_g^{sh})(x) = q^{-m(n-1)/2}\sum_{u\in U^-_{w^{-1}}} f_g^{sh}(w^{-1}u\Sigma(x)).$$

and the above discussion explains why $(I_\Sigma f_g^{sh})(bx) = \phi_\chi(b)(I_\Sigma f_g^{sh})(x)$.

Therefore, in my conventions, the right hand side of the above equation is

$$(I_\Sigma f_{g^{-1}})(x^{-1}) = q^{-m(n-1)/2}\sum_{u\in U^-_{w^{-1}}} f_{g^{-1}}(\Sigma(x^{-1})uw).$$

In the tensor product notation this is equivalent to

$$I_\Sigma(g \otimes_{B_n\mathbb{F}_{q^m}} 1) = \sum_{h\in GL_n\mathbb{F}_{q^m}/B_n\mathbb{F}_{q^m}} \sum_{u\in U^-_{w^{-1}}} h \otimes_{B_n\mathbb{F}_{q^m}} f_g(\Sigma(h)uw).$$

Therefore

$$g' I_\Sigma(g \otimes_{B_n\mathbb{F}_{q^m}} 1)$$

$$= \sum_{h\in GL_n\mathbb{F}_{q^m}/B_n\mathbb{F}_{q^m}} \sum_{u\in U^-_{w^{-1}}} g'h \otimes_{B_n\mathbb{F}_{q^m}} f_g(\Sigma(h)uw)$$

$$= \sum_{h'\in GL_n\mathbb{F}_{q^m}/B_n\mathbb{F}_{q^m}} \sum_{u\in U^-_{w^{-1}}} h' \otimes_{B_n\mathbb{F}_{q^m}} f_g(\Sigma((g')^{-1})\Sigma(h')uw)$$

$$= \sum_{h'\in GL_n\mathbb{F}_{q^m}/B_n\mathbb{F}_{q^m}} \sum_{u\in U^-_{w^{-1}}} h' \otimes_{B_n\mathbb{F}_{q^m}} f_{\Sigma(g')g}(\Sigma(h')uw)$$

$$= I_\Sigma(\Sigma(g')g \otimes_{B_n\mathbb{F}_{q^m}} 1).$$

Set $\rho = \mathrm{Ind}_{B_n\mathbb{F}_{q^m}}^{GL_n\mathbb{F}_{q^m}}(\phi_\chi)$. By ([6] Lemma 3.2) $I_\Sigma^m = 1$ and

$$\rho(\Sigma(g')) \cdot I_\Sigma^{-1} = I_\Sigma^{-1} \cdot \rho(g').$$

The multiplication in my convention for semi-direct products is given by $(c, g)(c', g') = (cc', gc(g'))$. With this convention

$$(\Sigma, 1)(1, g) = (\Sigma, \Sigma(g)) = (1, \Sigma(g))(\Sigma, 1)$$

so the irreducible representation ρ extends to an irreducible $\tilde{\rho}$ on $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \propto GL_n\mathbb{F}_{q^m}$ in which (Σ, g) acts via $I_\Sigma^{-1} \cdot \rho(g) = I_{\Sigma^{-1}} \cdot \rho(g)$. Therefore, by ([6] p.409),

$$\mathrm{Trace}(\tilde{\rho}(\Sigma, g) = \frac{q^{-m(n-1)/2}}{|B_n\mathbb{F}_{q^m}|} \sum_{X\in GL_n\mathbb{F}_{q^m}} \psi_\chi(Xg\Sigma(X)^{-1}).$$

To show that $Sh(\rho) = \rho_\chi$ is a cuspidal irreducible of $GL_n\mathbb{F}_q$ it suffices to show that for any pair of irreducibles $\rho_1 \in \mathrm{Irr}(GL_a\mathbb{F}_q)$ and $\rho_2 \in \mathrm{Irr}(GL_{n-a}\mathbb{F}_q)$ that ρ_χ is not an irreducible constituent of $m(\rho_1 \otimes \rho_2)$. By Theorem 6.3, applying the Shintani correspondence up to $\mathbb{F}_{q^m}$, $Sh^{-1}(\rho_1)$ must be equivalent to the PSH algebra product of $\Sigma^{i_j}(\chi)$'s as i_j ranges over some a proper subset of $1, 2, \ldots, n$. However this is impossible because any such product is not Galois invariant.

Similarly, applying the Shintani correspondence up to $\mathbb{F}_{q^m}$, $Sh^{-1}(\rho_{\chi_1}) = Sh^{-1}(\rho_{\chi_2})$ implies that the Galois orbits of χ_1 and χ_2 coincide.

Finally, the discussion shows that the number of distinct regular characters of $\mathbb{F}^*_{q^n}$ is less than or equal to the number of inequivalent irreducible cuspidal representations of $GL_n\mathbb{F}_q$. The fact that these numbers are in fact equal follows from a counting argument given in ([10] Theorem 8.6). □

8. Appendix: An example of $w(k_{*,*})P_{\alpha,m-\alpha})w(k_{*,*})^{-1}\bigcap P_{a,m-a}$

In the notation of the discussion of Double Cosets in §3 let

$$m=7, a=3, \alpha=4, k_{11}=1=k_{22}, k_{21}=2, k_{12}=3$$

and consider the double coset representative

$$w(k_{*,*})=\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix}.$$

If $(a_{ij})\in G_7$ then the conjugate by $w=w(k_{*,*})$ takes the form

$$w(a_{ij})=\begin{pmatrix} a_{11}&a_{15}&a_{16}&a_{12}&a_{13}&a_{14}&a_{17}\\ a_{51}&a_{55}&a_{56}&a_{52}&a_{53}&a_{54}&a_{57}\\ a_{61}&a_{65}&a_{66}&a_{62}&a_{63}&a_{64}&a_{67}\\ a_{21}&a_{25}&a_{26}&a_{22}&a_{23}&a_{24}&a_{27}\\ a_{31}&a_{35}&a_{36}&a_{32}&a_{33}&a_{34}&a_{37}\\ a_{41}&a_{45}&a_{46}&a_{42}&a_{43}&a_{44}&a_{47}\\ a_{71}&a_{75}&a_{76}&a_{72}&a_{73}&a_{74}&a_{77} \end{pmatrix}.$$

In order that $w(a_{ij})=w(a_{ij})w^{-1}$ lies in $wG_4\times G_3w^{-1}$ it must have the form

$$w(a_{ij})=\begin{pmatrix} a_{11}&0&0&a_{12}&a_{13}&a_{14}&0\\ 0&a_{55}&a_{56}&0&0&0&a_{57}\\ 0&a_{65}&a_{66}&0&0&0&a_{67}\\ a_{21}&0&0&a_{22}&a_{23}&a_{24}&0\\ a_{31}&0&0&a_{32}&a_{33}&a_{34}&0\\ a_{41}&0&0&a_{42}&a_{43}&a_{44}&0\\ 0&a_{75}&a_{76}&0&0&0&a_{77} \end{pmatrix}$$

and so to lie in the intersection $w(G_4\times G_3)\bigcap G_3\times G_4$ it must have the form

$$w(a_{ij})=\begin{pmatrix} a_{11}&0&0&0&0&0&0\\ 0&a_{55}&a_{56}&0&0&0&0\\ 0&a_{65}&a_{66}&0&0&0&0\\ 0&0&0&a_{22}&a_{23}&a_{24}&0\\ 0&0&0&a_{32}&a_{33}&a_{34}&0\\ 0&0&0&a_{42}&a_{43}&a_{44}&0\\ 0&0&0&0&0&0&a_{77} \end{pmatrix}=A''.$$

Therefore in this example

$$wG_4\times G_3w^{-1}\bigcap G_3\times G_4=G_{k_{11}}\times G_{k_{21}}\times G_{k_{12}}\times G_{k_{22}}.$$

In order that $w(a_{ij}) = w(a_{ij})w^{-1}$ lies in $wP_{4,3}w^{-1}$ it must have the form

$$w(a_{ij}) = \begin{pmatrix} a_{11} & a_{15} & a_{16} & a_{12} & a_{13} & a_{14} & a_{17} \\ 0 & a_{55} & a_{56} & 0 & 0 & 0 & a_{57} \\ 0 & a_{65} & a_{66} & 0 & 0 & 0 & a_{67} \\ a_{21} & a_{25} & a_{26} & a_{22} & a_{23} & a_{24} & a_{27} \\ a_{31} & a_{35} & a_{36} & a_{32} & a_{33} & a_{34} & a_{37} \\ a_{41} & a_{45} & a_{46} & a_{42} & a_{43} & a_{44} & a_{47} \\ 0 & a_{75} & a_{76} & 0 & 0 & 0 & a_{77} \end{pmatrix}$$

and so to lie in the intersection $wP_{4,3}w^{-1} \bigcap P_{3,4}$ it must have the form

$$w(a_{ij}) = \begin{pmatrix} a_{11} & a_{15} & a_{16} & a_{12} & a_{13} & a_{14} & a_{17} \\ 0 & a_{55} & a_{56} & 0 & 0 & 0 & a_{57} \\ 0 & a_{65} & a_{66} & 0 & 0 & 0 & a_{67} \\ 0 & 0 & 0 & a_{22} & a_{23} & a_{24} & a_{27} \\ 0 & 0 & 0 & a_{32} & a_{33} & a_{34} & a_{37} \\ 0 & 0 & 0 & a_{42} & a_{43} & a_{44} & a_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} \end{pmatrix} = C.$$

A matrix in $wP_{4,3}w^{-1} \bigcap G_3 \times G_4$ has the form

$$w(a_{ij}) = \begin{pmatrix} a_{11} & a_{15} & a_{16} & 0 & 0 & 0 & 0 \\ 0 & a_{55} & a_{56} & 0 & 0 & 0 & 0 \\ 0 & a_{65} & a_{66} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{22} & a_{23} & a_{24} & a_{27} \\ 0 & 0 & 0 & a_{32} & a_{33} & a_{34} & a_{37} \\ 0 & 0 & 0 & a_{42} & a_{43} & a_{44} & a_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} \end{pmatrix} = A$$

and a matrix in $wP_{4,3}w^{-1} \bigcap U_{3,4}$ has the form

$$w(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 & b_{12} & b_{13} & b_{14} & b_{17} \\ 0 & 1 & 0 & 0 & 0 & 0 & b_{57} \\ 0 & 0 & 1 & 0 & 0 & 0 & b_{67} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = B.$$

Choosing

$$b12 = \frac{a12}{a11}, b13 = \frac{a13}{a11}, b14 = \frac{a14}{a11}$$

$$X = \begin{pmatrix} a11 & a15 & a16 \\ 0 & a55 & a56 \\ 0 & a65 & a66 \end{pmatrix}, \; X \begin{pmatrix} b17 \\ b57 \\ b67 \end{pmatrix} = \begin{pmatrix} a17 \\ a57 \\ a67 \end{pmatrix}$$

shows that $AB = C$ and therefore

$$P_{3,4} \bigcap wP_{4,3}w^{-1} = ((G_3 \times G_4) \bigcap wP_{4,3}w^{-1} \cdot (U_{3,4} \bigcap wP_{4,3}w^{-1}).$$

In order that $w(a_{ij}) = w(a_{ij})w^{-1}$ lies in $wU_{4,3}w^{-1}$ it must have the form

$$w(a_{ij}) = \begin{pmatrix} 1 & a_{15} & a_{16} & 0 & 0 & 0 & a_{17} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_{25} & a_{26} & 1 & 0 & 0 & a_{27} \\ 0 & a_{35} & a_{36} & 0 & 1 & 0 & a_{37} \\ 0 & a_{45} & a_{46} & 0 & 0 & 1 & a_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and to lie in $wU_{4,3}w^{-1} \bigcap G_3 \times G_4$ it must have the form

$$w(a_{ij}) = \begin{pmatrix} 1 & a_{15} & a_{16} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & a_{27} \\ 0 & 0 & 0 & 0 & 1 & 0 & a_{37} \\ 0 & 0 & 0 & 0 & 0 & 1 & a_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = A'.$$

To lie in $wU_{4,3}w^{-1} \bigcap U_{3,4}$ a matrix must have the form

$$w(b_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & b_{17} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = B'$$

and to lie in $wU_{4,3}w^{-1} \bigcap P_{3,4}$ it must have the form

$$w(a_{ij}) = \begin{pmatrix} 1 & a_{15} & a_{16} & 0 & 0 & 0 & a_{17} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & a_{27} \\ 0 & 0 & 0 & 0 & 1 & 0 & a_{37} \\ 0 & 0 & 0 & 0 & 0 & 1 & a_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = C'.$$

Therefore, choosing A', B', C' in a similar manner to the case of A, B, C shows that

$$P_{3,4} \bigcap wU_{4,3}w^{-1} = ((G_3 \times G_4) \bigcap wU_{4,3}w^{-1}) \cdot (U_{3,4} \bigcap wU_{4,3}w^{-1}).$$

From the matrix immediately preceding A'' in order that a matrix lies in $wG_4 \times G_3 w^{-1} \bigcap U_{3,4}$ it must have the form

$$w(b_{ij}) = \begin{pmatrix} 1 & 0 & 0 & b_{12} & b_{13} & b_{14} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & b_{57} \\ 0 & 0 & 1 & 0 & 0 & 0 & b_{67} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = B''$$

and to lie in to lie in $wG_4 \times G_3 w^{-1} \bigcap P_{3,4}$ it must have the form the form

$$w(a_{ij}) = \begin{pmatrix} a_{11} & 0 & 0 & a_{12} & a_{13} & a_{14} & 0 \\ 0 & a_{55} & a_{56} & 0 & 0 & 0 & a_{57} \\ 0 & a_{65} & a_{66} & 0 & 0 & 0 & a_{67} \\ 0 & 0 & 0 & a_{22} & a_{23} & a_{24} & 0 \\ 0 & 0 & 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & 0 & a_{42} & a_{43} & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} \end{pmatrix} = C''.$$

Therefore, choosing A'', B'', C'' in a similar manner to the case of A, B, C shows that

$$P_{3,4} \bigcap wG_4 \times G_3 w^{-1} = ((G_3 \times G_4) \bigcap wG_4 \times G_3 w^{-1}) \cdot (U_{3,4} \bigcap wG_4 \times G_3 w^{-1}).$$

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