SOME ANCIENT GREEKS DISCUSS THE PLATONIC SOLIDS

FROM VERY OLD MANUSCRIPTS DISCOVERED BY VICTOR SNAITH

About 2000 years ago clever Athenians used to gather in the home of the cleverest of all, whose name was Socrates. They used to discuss philosophy, democracy, morality and most often of all they discussed geometry. On this occasion Socrates had invited round Bombasticles and Sarcasticles to discuss the Platonic solids. They had scarcely begun when a latecomer called Smartacus, cousin of the legendary revolutionary slave Spartacus, turned up.

"What's the topic today, friends?" he asked.

"Platonic solids," Socrates replied grumpily, annoyed by Smartacus's lack of punctuality.

"What are they?" Smartacus is not so smart after all, thought Socrates to himself, being too polite to blurt the thought out rudely to a guest. Ancient Greeks used to put high value on politeness.

"A Platonic solid is a regular convex polyhedron made by sticking polygonal faces together along their edges two at a time, dummy! And regular means each polygon is the same size and shape."

This came from Bombasticles, who was the exception to the politeness rule - every rule has one.

Smartacus, who was far from being a dummy, retorted:"Give me an example."

"May the Gods forgive you," blurted Bombasticles. "Even those primitive pagans, the Egyptians, knew about a pyramid made from a triangular base, with 3 equal sides, and three more such triangles glued together to make a pyramid."

"Oh, a tetrahedron?"

"Yes, a tetrahedron, dummy!"

"Less of the rudeness, Bombie, you are a guest," reminded Socrates. "The next simplest Platonic solid is the cube."

"I see! Six square, flat faces each glued to the edges of four other squares."

"Right!" said Socrates."For years, probably since the time of Archimedes of Syracuse, it has been known how to make three others - in all the list to date is tetrahedron, cube, octahedron, dodecahedron and icosahedron."

"What do they look like?" asked Smartacus.

Date: March 2020.

“Children and fools should never see a problem half finished,” butted in Sarcasticles. “More importantly, we are gathered to contemplate the question: are there more than five Platonic solids?”

“Furthermore,” added Socrates,“it is two hundred years since Archimedes, during which time no others have been discovered. So we are inclined to the answer, “No!” but we shall not be convinced by a mere rumour from our cleverest ancestors[1].”

“Great! Let’s get to it!” exclaimed Smartacus enthusiastically. “So how do we make a Platonic solid in practice.”

“We could use sheets of wood,” suggested Bombasticles.

“Ever the amateur carpenter,” muttered Sarcasticles.

“Don’t be so fast to dismiss the idea. I like it,” said Socrates.“We take a number of sheets of flat wood, with straight edges - as many as you like - but each of these sheets, let’s call them faces, must have the same number of straight edges. A face with t straight edges will be called a t-gon.”

“So could one of these 4-gon faces be a long thin rectangle?” asked Smartacus.

“Don’t be crazy! Regular! Regular! Didn’t you hear regular? That means that all the edges have the same length, all the faces have the same area,” shouted Bombasticles, losing patience. “Regular, smarty-pants!”

“Smartacus.”

“O.K. Smartacus.”

“We keep at the back of our mind that t in the term t-gon means one of the whole numbers $1, 2, 3, 4, 5, \ldots, 100, 101, 102, \ldots, 213, 214, \ldots$ and so on.”

“That’s right, Smartacus. Keep all those possibilities in mind.”

“Right, to business,” said Socrates. “How are we going to find all the other Platonic solids?”

“Ever the deep questioner,” muttered Sarcasticles, attracting an annoyed glance from Socrates.

“The easiest way out,” mused Smartacus,” is that we have them all already.”

“True!” was the unanimous agreement. “So then we just need some property which is possessed only by these solids.”

“What sort of property? It couldn’t be colour, for example.”

“I would guess that it is something connected with numbers,” ventured Bombasticles, “ but it can’t be connected with size, since my carpentry model could be made twice as big as yours.”

“That’s rather obvious,” added Sarcasticles.

“Have you got a better idea?”

[1]Archimedes is considered the most important scientist who ever lived. In his Codex B he described how to calculate volumes of several curved 3-dimensional shapes, using the method of integral calculus. Differential calculus, which includes integral calculus although the principles of the two are very close, was discovered seventeen hundred years later by Isaac Newton

"Without actually doing the carpentry, let's imagine making one of these solids," whispered Smartacus to himself, thinking aloud. "But not a particular one - a sort of generic one, where for example we don't know what sort of t-gon each face is."

"O.K.", said Socrates,"suppose we start with just one face. Then glue another one to it, edge to edge."

"I get you," said Bombasticles, "Then next you glue another face on - edge to edge."

"Hang on," Smartacus butted in."You might glue in the third face by, say, glueing two adjacent edges on what you've got already to two edges of the third face - edge to edge!"

"Ever the subtleties," muttered Sarcasticles.

"Quit the snide remarks," admonished Socrates."Smartacus has a really good point. As we build up the solid we might glue any number of adjacent edges of the t-gon to the same number of adjacent edges of the model so far - edge to edge."

"O.K." agreed Bombasticles."Now how do numbers come in?"

"We've got some numbers," Smartacus pointed out."The number of faces, of edges and of the pointed corners."

"Let's call a pointed corner a vertex, so we have the number of the vertices. For example, a tetrahedron has four faces, six edges and four vertices," suggested Socrates.

"Maybe the number we are looking for is just the sum of the numbers of faces plus edges plus vertices?"

"For a tetrahedron that is fourteen but for a cube it is twenty-six - see, six plus twelve plus eight for the cube."

"Yes, adding the numbers up was too simple."

"It was just a first thought," said Smartacus defensively. "Maybe we should add some and subtract others?"

"I've got it," Bombasticles cried out," faces MINUS edges plus vertices - that is: six minus twelve plus eight equals two for the cube and four minus six plus four equals two for the tetrahedron!!"

"It's worth a try," agreed Socrates. "So, to keep the arithmetic shorter, let's called the number of faces F, the number of edges E and the number of vertices V."

"Right oh! $F - E + V$ it is," said Sarcasticles. "Let's call this the Bombasticles number[2]"

"For our first face, just on its own - a t-gon - we get $F = 1, E = t, V = t$ so $F - E + V = 1$." Smartacus pointed out.

[2]When $F - E + V$ was rediscovered by an 18th century mathematician called Leonard Euler, a Swiss working at the then new University of St Petersburg in Russia, it was named after him - the "Euler characteristic" and the name has stuck.

"But," spluttered the excited Bombasticles, "gluing another t-gon on by one edge adds 1 to F, $t-1$ to E and $t-2$ to V so $F-E+V=1$, just like before."

"Let's suppose," said Socrates, slowly and thoughtfully, "that we have glued a number of faces together to make one blob consisting of just some of the faces of our eventual Platonic solid glued together part way through the construction - and let's guess that we still get $F-E+V=1$ for this blob. Imagine taking another face and gluing k of its adjacent edges to k adjacent edges of the blob we have made so far."

"Edge to edge?"

"Of course, edge to edge."

"In that case F goes up by one, E goes up by $t-k$ and V goes up by $t-k-1$ so that we still have $F-E+V=1$."

"Except in one case," Sarcasticles pointed out smugly.

"He's right," yelled Smartacus. "If we are putting the last face into place we have add 1 to F but no new edges or vertices. So the final Platonic solid has $F-E+V=2$."

"Brilliant! That is certainly correct for the tetrahedron and the cube."

"Now for some alegbra concerning the Bombasticles number of a Platonic sold. Suppose there are n faces, each a t-gon. Let's suppose that r is the number of faces meeting at each vertex. C'mon, let's calculate the Bombasticles number" said Socrates, getting quite excited.

"That's easy," said Smartacus. "At each edge there will be exactly 2 faces with that edge in common so there are $nt/2$ edges and at each vertex r faces come together so that the number of vertices is nt/r. Therefore we have the equation $2=n-nt/2+nt/r$."

"Don't forget," Bombasticles pointed out, "that these fractions $nt/2$ and nt/r are positive whole numbers."

"Yes, yes," snapped Sarcasticles. "Ever the obvious."

"Obvious, but important."

"Right! Let's start," commanded Socrates. With that, the quartet - following the example of Archimedes and his famous "Sand Reckoner" - trooped out and went down to the beach, where for several hours they performed the following experimental calculations, scratching the numbers in the sand with a stick[3].

If $t=3$ so the faces are all identical triangles[4] . Therefore we have

$$2=n-3n/2+3n/r \text{ or } 2+n/2=3n/r$$

[3]You can check these calculations with your mobile phone!

[4]**THIS IS WHERE CHILDREN MAY NEED TO RECRUIT THEIR PARENTS' HELP. OR VICE VERSA!**

where $n/2$ and $3n/r$ are integers[5]. Trying $n = 2, 4, 6, 8, 10, \ldots$ we get

$$
\begin{array}{l}
2+2/2 = 3 \times 2/r \text{ impossible} \\
2+4/2 = 3 \times 4/r \implies r=3 \implies \text{ tetrahedron} \\
2+6/2 = 3 \times 6/r \text{ impossible} \\
2+8/2 = 3 \times 8/r \implies r=4 \implies \text{ octahedron} \\
2+10/2 = 3 \times 10/r \text{ impossible} \\
2+12/2 = 3 \times 12/r \text{ impossible} \\
2+14/2 = 3 \times 14/r \text{ impossible} \\
2+16/2 = 3 \times 16/r \text{ impossible} \\
2+18/2 = 3 \times 18/r \text{ impossible} \\
2+20/2 = 3 \times 20/r \implies r=5 \implies \text{ dodecahedron} \\
n \ \geq 22 \text{ and even is impossible!}
\end{array}
$$

If $t = 4$ so the faces are identical squares we have

$$2 = n - 4n/2 + 4n/r \text{ or } 2 + n = 4n/r$$

wherre $4n/r$ is an integer. Trying $n = 3, 4, 5, \ldots$ we obtain

$$
\begin{array}{l}
2+3 = 4 \times 3/r \text{ impossible} \\
2+4 = 4 \times 4/r \text{ impossible} \\
2+5 = 4 \times 5/r \text{ impossible} \\
2+6 = 4 \times 6/r \implies r=3 \implies \text{ cube} \\
2+7 = 4 \times 7/r \text{ impossible} \\
2+8 = 4 \times 8/r \text{ impossible} \\
n \geq 9 \text{ is impossible!}
\end{array}
$$

If $t = 5$ so the faces are identical pentagons we have

$$2 = n - 5n/2 + 5n/r \text{ or } 2 + 3n/2 = 5n/r$$

and $n/2$ and $5n/r$ are integers. Trying $n = 2, 4, 6, 8, \ldots$ we obtain

$$
\begin{array}{l}
2+3 \times 2/2 = 5 \times 2/r \text{ impossible} \\
2+3 \times 4/2 = 5 \times 4/r \text{ impossible} \\
2+3 \times 6/2 = 5 \times 6/r \ \text{ impossible} \\
2+3 \times 8/2 = 5 \times 8/r \text{ impossible} \\
2+3 \times 10/2 = 5 \times 10/r \text{ impossible} \\
2+3 \times 12/2 = 5 \times 12/r \implies r=3 \implies \text{ icosahedron} \\
2+3 \times 14/2 = 5 \times 14/r \text{ impossible} \\
n \geq 14 \text{ and even is impossible!}
\end{array}
$$

If $t = 6$ so the faces are identical hexagons we have

$$2 = n - 6n/2 + 6n/r \text{ or } 2 + 4n/2 = 6n/r$$

[5]This, you will remember, was Bombasticles' most important observation! It is the main reason why the "impossible" cases are impossible.

and $n/2$ and $5n/r$ are integers. Trying $n = 2, 4, 6, 8, \ldots$ we obtain

$$
\begin{array}{l}
2 + 4 \times 2/2 = 6 \times 2/r \text{ impossible} \\
2 + 4 \times 4/2 = 6 \times 4/r \text{ impossible} \\
2 + 4 \times 6/2 = 6 \times 6/r \text{ impossible} \\
2 + 4 \times 8/2 = 6 \times 8/r \text{ impossible} \\
2 + 4 \times 10/2 = 6 \times 10/r \text{ impossible} \\
2 + 4 \times 12/2 = 6 \times 12/r \text{ impossible} \\
2 + 4 \times 14/2 = 6 \times 14/r \text{ impossible} \\
n \geq 14 \text{ and even is impossible!}
\end{array}
$$

This last set of equations are each impossible because the right hand side is smaller than or equal to the $(t-2)n/2$ on the left so the right side is strictly smaller than $2 + (t-2)n/2$. This fact persists for any t greater than or equal to 6.

When our quartet finished convincing themselves of the arithmetic, they were delighted to have shown, logically and beyond all guesswork, that there exist only 5 Platonic solids. Such confirmation by logic is called a "theorem" in Greek. That evening was Bacchanalia-time[6] in Athens - to celebrate the first non-existence theorem in the history of the human race. □

Added in the 21st century: should anyone want to make models of the 5 Platonic solids they will find, in a nearby pdf called "Platonets" pictures - called nets - which can be printed out, cut around and then folded along the edges and glued together to make the model[7].

A mathematical puzzle for you: Imagine you have made one of these Platonic solids, hollow inside. In the middle of each face is a point which is the same distance from each of the vertices - called the centroid or centre of gravity (Archimedes' term for it) - so imagine inside the original Platonic solid there is sitting another Platonic solid whose vertices are exactly the centroids of the first one. For example the cube has 6 faces so the one inside it will have 6 vertices - which one will that be? Which solid will you get by doing this to each of the other Platonic solids?

What will you get by doing this centroid-thing to the second Platonic solid you made before - the one that is inside the first one - which solid will that be?

If you glue tetrahedra, with triangular faces, to each of the triangular faces of the icosahedron the result makes a nice three-dimensional Christmas star decoration[8]

I bet that our clever Greeks spotted this centroid business 2000 years ago!

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[6]Party-time named after Bacchus, the Greek god of partying.

[7]A kind model-maker gave me the tip that, when cutting out the nets, it is a good idea to cut some extra 'tabs' on which to spread the glue.

[8]It is called the greater stellated icosahedron.