REMARKS ON A PAPER OF GUY HENNIART

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Contents

Abstract. In other lectures in this series one encounters local L-functions, functional equations and local epsilon factors of admissible representations. The p-adic Galois epsilon factors are numbers lying on the unity circle and they are fundamental in the local Langlands correspondence which was proved by Mike Harris and Richard Taylor. Later part of the proof was simplified by Henniart using his "uniqueness theorem", which is characterised in terms of p-adic epsilon factors .

In this lecture, which is mainly expository, I shall outline the calculation by Deligne and Henniart of the p-adic epsilon factors of wild, homogeneous Galois representations modulo p-primary roots of unity. This formula is an important ingredient in the proofs of the uniqueness theorem. The only novel ingredients in my exposition will be the use of monomial resolutions to reduce to the one-dimensional case and an explicit formulae for the Deligne-Henniart "Gauss sum" (which seems in my opinion to contradict, in the tamely ramified case, one of the lemmas - claimed in general but used by Henniart only in the wild case - at the crux of the proof).

1. The basic ingredients

1.1. These notes are an exposition of the papers [27] and [37] which culminate in the derivation of a formula for Galois local constants (otherwise known as Galois epsilon factors) modulo p-power roots of unity for wildly ramified, homogeneous representations on the Weil group of a p-adic local field.

My account will differ from [27] in §3.3, which I shall derive using monomial resolutions. Furthermore, since it is well-known how to pass to and fro between Galois group and Weil group representations, I shall restrict the discussion to the Galois case.

I am posting this one my webpage in this unpolished form, rather than posting it on the Arxiv, because I have been allowing it to languish completely unnoticed for nearly two years. I am very grateful to Paul Buckingham and Guy Henniart for their expert assistance.

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1.2. *Ramification groups and functions*

Let us recall from [61] the properties of the ramification groups $\mathrm{Gal}(K/E)_i$, $\mathrm{Gal}(K/E)^{\alpha}$ and functions $\phi_{K/E}, \psi_{K/E}$ associated to a Galois extension K/E of local fields with residue characteristic p.

Let v_K denote the valuation on K, $\mathcal{O}_K$ the valuation ring of K and write $G = \mathrm{Gal}(K/E)$. The ramification groups form a finite chain of normal subgroups ([61] p.62 Proposition 1)

$$\{1\} = G_r \subseteq \ldots \subseteq G_{i+1} \subseteq G_i \subseteq \ldots \subseteq G_1 \subseteq G_0 \subseteq G_{-1} = G$$

defined by

$$G_i = \{g \in G \mid v_L(g(x) - x) \geq i + 1 \text{ for all } x \in \mathcal{O}_K\}.$$

The inertia group is G_0 and G_{-1}/G_0 is isomorphic to the Galois group of the residue field extension. If $H = \mathrm{Gal}(K/M) \subseteq G$ then $H_i = G_i \bigcap H$. The quotient G_0/G_1 is cyclic of order prime to p while G_1 is a p-group and each G_i/G_{i+1} with $i \geq 1$ is an elementary abelian p-group.

The function $\phi_{K/E} : [-1, \infty) \longrightarrow [-1, \infty)$ is a piecewise-linear homeomorphism given by

$$\phi_{K/E}(u) = \begin{cases} u & \text{if } -1 \leq u \leq 0, \\ \frac{u|G_1|}{|G_0|} & \text{if } 0 \leq u \leq 1, \\ \frac{|G_1| + \ldots + |G_m| + (u-m)|G_{m+1}|}{|G_0|} & \text{if } m \leq u \leq m+1, 1 \leq m \text{ an integer.} \end{cases}$$

At a positive integer $i \geq 1$ the slope of $\phi_{K/E}$ just to the left of i equals $\frac{|G_i|}{|G_0|}$ and just to the right it is $\frac{|G_{i+1}|}{|G_0|}$. Therefore the condition that $G_i = G_{i+1}$ is equivalent to $\phi_{K/E}$ being linear at i. If $G_i \neq G_{i+1}$ then $\phi_{K/E}$ is concave downward at i and i is called a "jump" value for the lower filtration. If $G_0 = \ldots = G_r \neq G_{r+1}$ then $\phi_{K/E}(x) = x$ if $-1 \leq x \leq r$ and $\phi_{K/E}(x) < x$ if $r < x$.

If $g \in G_0$ then $g \in G_i$ if and only if $g(\pi_K)/\pi_K \equiv 1$ (modulo $\mathcal{P}_K^i$).

The function $\psi_{K/E} : [-1, \infty) \longrightarrow [-1, \infty)$ is the piecewise-linear homeomorphism given by the inverse of $\phi_{K/E}$. Hence $\psi_{K/E}(i) = \alpha$ is not necessarily an integer. To accommodate this we extend the definition of the G_i's to G_u for any real number $u \geq -1$ by setting $G_u = G_j$ where j is the smallest integer satisfying $u \leq j$.

Given a chain of fields $E \subseteq M \subseteq K$ there are chain rules

$$\phi_{M/E}(\phi_{K/M}(x)) = \phi_{K/E}(x), \quad \psi_{K/M}(\psi_{M/E}(y)) = \psi_{K/E}(y).$$

The upper numbering of the ramification groups is defined by the relations

$$G^v = G_{\psi_{K/E}(v)} \text{ and } G^{\phi_{K/E}(u)} = G_u.$$

If $H = \mathrm{Gal}(K/M) \lhd G$ is a normal subgroup then $(G/H)^v = G^v H/H$.

In §1.5 we shall utilise the extension of the upper numbering filtration to the case of infinite Galois extensions such as $\overline{F}/E$. Following ([61] Remark 1, p.75) for K/E an infinite Galois extension we set

$$\mathrm{Gal}(K/E)^v = \lim_{\leftarrow} \mathrm{Gal}(K'/E)^v$$

where K' runs through the set of finite Galois extensions of E contained in K. The filtration $\mathrm{Gal}(K/E)^v$ is left continuous in the sense that

$$\mathrm{Gal}(K/E)^v = \bigcap_{w<v} \mathrm{Gal}(K/E)^w.$$

As we shall see in Proposition 1.3, the upper filtration $\mathrm{Gal}(K/E)^v$ is *not* right continuous. One says that v is a "jump" for the upper numbering filtration if $\mathrm{Gal}(K/E)^v \neq \mathrm{Gal}(K/E)^{v+\epsilon}$ for all $\epsilon > 0$. Even for finite Galois extensions an upper numbering jump need not be an integer ([61] Exercise 2, p.77).

Proposition 1.3.
(i) Let K/E be a, not necessarily finite, Galois extension of p-adic local fields[1]. Then for any $\alpha \in \mathbb{R}$ there exists $\gamma < \alpha$ such that $\mathrm{Gal}(K/E)^\gamma = \mathrm{Gal}(K/E)^\alpha$.
(ii) Let K/E be an infinite Galois extension of p-adic local fields. Then the filtration $\mathrm{Gal}(K/E)^v$ is not right continuous in the sense that, if v is a jump for the upper numbering filtration,

$$\bigcup_{v<w} \mathrm{Gal}(K/E)^w \overset{\subset}{\neq} \mathrm{Gal}(K/E)^v.$$

Proof
In (i) the set of jumps in the upper ramification filtration is discrete. Suppose that $\{\beta_n\}$ is an increasing sequence of real numbers such that $\beta_n < \alpha$ tending to α from below. Therefore the sequence G^{β_n} will eventually stabilise (i.e. becoming equal for large enough $n < \alpha$). Therefore there is one of these β_n's, say γ, such that

$$\mathrm{Gal}(K/E)^\alpha = \bigcap_{w<\alpha} \mathrm{Gal}(K/E)^w = \bigcap_{w<\gamma} \mathrm{Gal}(K/E)^w = \mathrm{Gal}(K/E)^\gamma,$$

as required.
In (ii) we consider the infimum

$$\underline{\alpha} = \inf\{\alpha \in \mathbb{R} \mid \mathrm{Gal}(K/E)^\alpha \subseteq \mathrm{Gal}(K/E)^v\}.$$

By part (i) there exists $\gamma < \underline{\alpha}$ such that $\mathrm{Gal}(K/E)^\gamma = \mathrm{Gal}(K/E)^{\underline{\alpha}}$. Suppose that $\mathrm{Gal}(K/E)^\gamma \subseteq \mathrm{Gal(K/E)}^v$ then, by definition, $\underline{\alpha} \leq \gamma$ which is a contradiction. Therefore

$$\mathrm{Gal}(K/E)^{\underline{\alpha}} = \mathrm{Gal}(K/E)^\gamma \not\subseteq \mathrm{Gal}(K/E)^v.$$

[1] I am very grateful to Paul Buckingham and Guy Henniart for explaining the proof of this proposition to me.

However, if $v < w$ then $\mathrm{Gal}(K/E)^w \subseteq \mathrm{Gal}(K/E)^v$ so that $\underline{\alpha} \leq w$ and therefore $\mathrm{Gal}(K/E)^w \subseteq \mathrm{Gal}(K/E)^{\underline{\alpha}}$ which implies that

$$\bigcup_{v<w} \mathrm{Gal}(K/E)^w \subseteq \mathrm{Gal}(K/E)^{\underline{\alpha}} \overset{\subset}{\neq} \mathrm{Gal}(K/E)^v,$$

as required. □

The proof of part (ii) of Proposition 1.3 establishes the following result.

Corollary 1.4.

Let K/E be an infinite Galois extension of p-adic local fields. Let v be a jump for the upper numbering filtration and define

$$\underline{\alpha} = \inf\{\alpha \in \mathbb{R} \mid \mathrm{Gal}(K/E)^\alpha \subseteq \mathrm{Gal}(K/E)^v\}.$$

Then $\underline{\alpha}$ is strictly smaller than v.

1.5. *Wild, homogeneous local Galois representations*

All fields are non-Archimedean local containing $F/\mathbb{Q}_p$. Let σ be a non-trivial, continuous, finite-dimensional complex representation of $\mathrm{Gal}(\overline{F}/E)$. The level $\alpha(\sigma)$ is the least α such that σ restricted to $\mathrm{Gal}(\overline{F}/E)^\alpha$ is non-trivial but σ restricted to $\mathrm{Gal}(\overline{F}/E)^{\alpha+\epsilon}$ is trivial for all $\epsilon > 0$. There exists an upper numbering ramification group with this property by left continuity of the $\mathrm{Gal}(\overline{F}/E)^\alpha$'s because there are certainly ramification groups $\mathrm{Gal}(\overline{F}/E)^w$ on which σ is non-trivial. Define $\alpha(\sigma)$ to be the supremum of the set of real numbers v such that σ is non-trivial on $\mathrm{Gal}(\overline{F}/E)^v$. Therefore

$$\mathrm{Gal}(\overline{F}/E)^{\alpha(\sigma)} = \bigcap_{v<\alpha(\sigma)} \mathrm{Gal}(\overline{F}/E)^v.$$

Then σ is wild if $\alpha(\sigma) > 0$. If σ is wild and irreducible then ([36] §3)

$$a(\sigma) = \dim(\sigma)(1 + \alpha(\sigma))$$

where $f(\sigma) = \mathcal{P}_E^{a(\sigma)}$ is the Artin conductor of σ (see §1.10 for the definition of $a(\sigma)$).

There exists a finite Galois extension K/E such that σ (faithfully) factors through the finite Galois group $\mathrm{Gal}(K/E)$. Since $\mathrm{Gal}(\overline{F}/E)^{\alpha(\sigma)}$ is the last ramification group (in the upper numbering) on which σ is non-trivial its image in $G = \mathrm{Gal}(K/E)$ is abelian and normal. The representation σ is homogeneous if

$$\mathrm{Res}^{\mathrm{Gal}(\overline{F}/E)}_{\mathrm{Gal}(\overline{F}/E)^{\alpha(\sigma)}}(\sigma) = n\chi_\sigma$$

where $n = \dim(\sigma)$ and $\chi_\sigma : \mathrm{Gal}(\overline{F}/E)^{\alpha(\sigma)} \longrightarrow \mathbb{C}^*$ is a character of finite order.

Suppose that the image of $\mathrm{Gal}(\overline{F}/E)^{\alpha(\sigma)}$ in G is

$$A = \mathrm{Gal}(K/M) = \mathrm{Gal}(K/E)^{\alpha(\sigma)} \lhd G.$$

Suppose that σ is irreducible and wild then it will not necessarily also be homogeneous but let us suppose that it is.

We remark that, if σ is not homogeneous then Clifford theory, which deals with restriction to normal abelian subgroups, implies that

$$\mathrm{Res}_A^G(\sigma) = m(\chi_1 \oplus \chi_2 \oplus \ldots \oplus \chi_t)$$

where the conjugacy G-orbit of χ_1 is $\{\chi_1, \ldots, \chi_t\}$. This means that each of the χ_i's is non-trivial on A.

Furthermore, if σ is irreducible, wild and homogeneous, then χ_σ will be fixed under the conjugation action by G. This is because for $a \in A, g \in G, v \in V$

$$\sigma(a)v = \chi_\sigma(a) \cdot v \text{ and } \sigma(gag^{-1})(v) = \sigma(g)(\chi_\sigma(a) \cdot (\sigma(g)^{-1}(v)) = \sigma(a)v$$

since $\sigma(a)$ is multiplication by a scalar.

Therefore χ_σ corresponds via class field theory to a character $\chi : M^* \longrightarrow \mathbb{C}^*$ which is invariant under the Galois action of G on M. On $1 + \mathcal{P}_M^{a(\chi)-1}$ χ has the form $\chi(1+x) = \psi_M(gx)$ for $g \in M^*$ such that $g \in M^*/1 + \mathcal{P}_M$ is well-defined. Here ψ_M is a choice of additive character, which depends on the choice of ψ_F and is then defined as $\psi_M = \psi_F \cdot \mathrm{Trace}_{M/F}$. Therefore, defining $C_E = ((E^*/1 + \mathcal{P}_E) \otimes \mathbb{Z}[1/p])$, we have a well-defined element

$$g = g_\sigma \in ((M^*/1 + \mathcal{P}_M) \otimes \mathbb{Z}[1/p])^G \cong ((E^*/1 + \mathcal{P}_E) \otimes \mathbb{Z}[1/p]) = C_E.$$

From the short exact sequence, when p is odd,

$$0 \longrightarrow (\mathcal{O}_E/\mathcal{P}_E)^* \otimes \mathbb{Z}[1/p] \longrightarrow C_E \longrightarrow \mathbb{Z}[1/p] \longrightarrow 0$$

we see that $C_E \otimes \mathbb{Z}/2$ has four elements.

Note that the element g_σ can be defined for any wild, homogeneous representation, irreducible or not.

1.6. *Varying M in §1.5*

One can vary the choice of M in the above construction. Suppose we take another subfield M' fixed by $A = \mathrm{Gal}(K/M) = \mathrm{Gal}(K/E)^{\alpha(\sigma)}$. Therefore we must have $M' \subseteq M$ and we may as well assume that $M' \neq M$. Set $A' = \mathrm{Gal}(K/M')$ so that $A \subset A'$ is a proper subgroup. Note that A' is not necessarily abelian (it would be if σ were faithful) but A is, in fact it is cyclic because χ is faithful on A.

We shall also assume that σ restricted to A' is equal to $n\chi'$ for a character χ' which must restrict to χ on A. In terms of local class field theory we have

$$\chi = \chi' \cdot N : M^* \xrightarrow{N} (M')^* \xrightarrow{\chi'} \mathbb{C}^*.$$

Consider $\underline{A} = \mathrm{Gal}(\overline{F}/M) = \mathrm{Gal}(\overline{F}/E)^{\alpha(\sigma)}$ and $\underline{A}' = \mathrm{Gal}(\overline{F}/M')$. Let $\underline{\alpha}(M/M')$ denote the real number

$$\underline{\alpha}(M/M') = \inf\{\alpha \in \mathbb{R} \mid (\underline{A}')^\alpha \subseteq \underline{A}\}.$$

Theorem 1.7.

In the notation of §1.6

$$\alpha(M/M') < \alpha(\chi') = a(\chi') - 1.$$

We begin with an intermediate result.

Proposition 1.8.
In the notation of §1.6, $\alpha(M/M') \leq \alpha(\chi')$.

Proof
Since the upper ramification index is preserved under passage to quotient Galois groups it will suffice to prove this by studying the finite extension K/E as in §1.6, where we continue to assume that σ is faithful although K/E is not necessarily abelian.

We can show that $\underline{\alpha}(M/M') \leq \alpha(\chi')$ by showing that

$$(A')^{\alpha(\chi')} \subseteq A.$$

By definition ([61] p.71, Remark 1) $\alpha(\sigma)$ is a jump because $\sigma : \mathrm{Gal}(K/E) \longrightarrow GL_n\mathbb{C}$ is one-one and σ is non-trivial on $A = \mathrm{Gal}(K/M) = \mathrm{Gal}(K/E)^{\alpha(\sigma)}$ but is trivial on $\mathrm{Gal}(K/E)^{\alpha(\sigma)+\epsilon}$ for all $\epsilon > 0$ so

$$\{1\} = \mathrm{Gal}(K/E)^{\alpha(\sigma)+\epsilon} \neq \mathrm{Gal}(K/E)^{\alpha(\sigma)}$$

for all $\epsilon > 0$.

By the theory of the Herbrand functions ϕ and ψ (§1.2; see also [61] Chapter IV §3) there exists a real number $\gamma = \psi_{K/E}(\alpha(\sigma))$ such that

$$\mathrm{Gal}(K/E)^{\alpha(\sigma)} = \mathrm{Gal}(K/E)_{\gamma}.$$

This means that $\mathrm{Gal}(K/E)_{\gamma} = \mathrm{Gal}(K/E)_i$ where i is the smallest integer satisfying $\psi_{K/E}(\alpha(\sigma)) = \gamma \leq i$. If $\gamma < i$ then $\alpha(\sigma) = \phi_{K/E}(\gamma) < \phi_{K/E}(i) = \delta$ and $\mathrm{Gal}(K/E)^{\alpha(\sigma)} = \mathrm{Gal}(K/E)^{\delta}$, which is a contradiction. Therefore $\psi_{K/E}(\alpha(\sigma)) = i$, an integer. Furthermore i must be a jump for otherwise $G_i = G_{i+1}$ and $\mathrm{Gal}(K/E)^{\alpha(\sigma)} = \mathrm{Gal}(K/E)^{\phi_{K/E}(i+1)}$. Therefore we have

$$\{1\} = \mathrm{Gal}(K/E)_{i+1} \subset \mathrm{Gal}(K/E)_i = \mathrm{Gal}(K/E)^{\alpha(\sigma)}.$$

Now consider $A' \bigcap A$ which equals, by ([61] p.62 Proposition 2) since i is an integer,

$$A' \bigcap A = A' \bigcap \mathrm{Gal}(K/E)^{\alpha(\sigma)} = A' \bigcap \mathrm{Gal}(K/E)_i = (A')_i = (A')^{\beta}$$

for $\beta = \phi_{K/M'}(i)$. Since χ' restricted to A is equal to χ, which is one-one, χ' restricted to $(A')^{\beta}$ is non-trivial.

Now let j be the largest integer for which the restriction of χ' to $(A')_j$ is non-trivial. Therefore $j \geq i$ and also, by [61] p.102 Proposition 5), $\phi_{K/M'}(j) = \alpha(\chi')$. Hence ([61] p.73 Proposition 12)

$$\beta = \phi_{K/M'}(i) \leq \phi_{K/M'}(j) = \alpha(\chi').$$

By definition of $\alpha(M/M')$ we have

$$\underline{\alpha}(M/M') \leq \beta \leq \alpha(\chi').$$

□

1.9. *Proof of Theorem 1.7*

Suppose that $\underline{\alpha}(M/M') = \alpha(\chi')$ then, in the notation of the proof of Proposition 1.8, $\underline{\alpha}(M/M') = \beta = \alpha(\chi')$ and so

$$\underline{A}' \bigcap \underline{A} = (\underline{A}')^{\beta} = (\underline{A}')^{\alpha(\chi')}.$$

Therefore

$$\underline{\alpha}(M/M') = \inf\{\alpha \in \mathbb{R} \mid (\underline{A}')^{\alpha} \subseteq (\underline{A}')^{\alpha(\chi')}\}.$$

By the proof of Proposition 1.3(ii)

$$(\underline{A}')^{\underline{\alpha}(M/M')} \overset{\subset}{\neq} (\underline{A}')^{\alpha(\chi')},$$

which contradicts the assumption that $(\underline{A}')^{\underline{\alpha}(M/M')} = (\underline{A}')^{\alpha(\chi')}$. $\square$

1.10. *Recap of abelian local root numbers*

Let us recall from ([52] p.29) the formula for the abelian local roots numbers. Let $\chi : E^* \longrightarrow \mathbb{C}^*$ be a character (i.e. with open kernel). Let $a(\chi) = 0$ if χ is trivial on $\mathcal{O}_E^*$ and otherwise let $a(\chi)$ be the least integer $n \geq 1$ such that χ is trivial on $1+\mathcal{P}_E^n$. The Artin conductor is given by the ideal $f(\chi) = \mathcal{P}_E^{a(\chi)}$. For example, if the residue field satisfies $\mathcal{O}_E/\mathcal{P}_E \cong \mathbb{F}_{p^d}$ and χ restricted to $\mathcal{O}_E^*$ has the form

$$\mathcal{O}_E^* \longrightarrow (\mathcal{O}_E/\mathcal{P}_E)^* \stackrel{\mathrm{Norm}_{\mathbb{F}_{p^d}/\mathbb{F}_p}}{\longrightarrow} \mathbb{F}_p^* \subset \mathbb{C}^*$$

then $a(\chi) = 1$.

In each of these cases the local root number is given by the formula ([52] p.29)

$$W_E(\chi) = \frac{1}{\sqrt{N\pi_E}} \sum_{w \in (\mathcal{O}_E/\mathcal{P}_E)^*} \chi(w)\chi(c)^{-1}\psi_E(w/c)$$

where c is a generator of $f(\chi)\mathcal{D}_E$.

1.11. *The Gauss sum of* [37]

Let $\mathcal{P}_E = \pi_E\mathcal{O}_E$ and let ν_E be the E-adic order of ψ_E on so that the inverse different satisfies $\mathcal{D}_E^{-1} = \mathcal{P}_E^{-\nu_E}$.

Suppose that $p \neq 2$ and that $x \in E^*$ satisfies $\nu_E(x) + \nu_E$ is odd. Therefore we have an integer b such that

$$0 = \nu_E(x) + \nu_E + 2b + 1.$$

Hence

$$x\pi_E^{2b}\mathcal{P}_E \subseteq \mathcal{P}_E^{2b-\nu_E-2b-1+1} = \mathcal{D}_K^{-1}$$

so that $\psi_E(x\pi_E^{2b}\xi) = 1$ for all $\xi \in \mathcal{P}_E$.

Consider the Gauss sum

$$\phi(x) = \sum_{\xi \in (\mathcal{O}_E/\mathcal{P}_E)^*} \psi_E(x\pi_E^{2b}\xi^2/2) \in \mathbb{C}^*.$$

Note that there is a misprint[2] in the definition of ϕ in ([37] §2)

If we replace ξ by $\xi + \pi_E u$ with $u \in \mathcal{O}_E$ we have

$$\psi_E(x\pi^{2b}(\xi+\pi_E u)^2/2) = \psi_E(x\pi^{2b}\xi^2/2)\cdot\psi_E(x\pi^{2b}(\xi\pi_E u+\pi_E^2 u^2/2) = \psi_E(x\pi^{2b}\xi^2/2)$$

so that $\phi(x)$ is well-defined.

If $v \in \mathcal{O}_E^*$ then

$$\phi(xv^2) = \sum_{\xi\in(\mathcal{O}_E/\mathcal{P}_E)^*} \psi_E(x\pi_E^{2b}(v\xi)^2/2) = \phi(x)$$

and for any a

$$\phi(x\pi_E^{2a}) = \sum_{\xi\in(\mathcal{O}_E/\mathcal{P}_E)^*} \psi_E(x\pi_E^{2a}\pi_E^{2b-2a}\xi^2/2) = \phi(x)$$

so that we have a function, which is not a homomorphism (see $E = \mathbb{Q}_p$ in §1.12 below),

$$\phi : (E^*/(1+\mathcal{P}_E)) \otimes \mathbb{Z}/2 \longrightarrow \mathbb{C}^*$$

defined by setting $\phi(x) = 1$ if $\nu_E(x) + \nu_E$ is even. By the usual argument, if q_E is the order of the residue field $\mathcal{O}_E/\mathcal{P}_E$ then $\phi(x)^2 = (-1)^{(q_E-1)/2}q_E$ if $\nu_E(x) + \nu_E$ is odd. Define a map

$$G_E : E^*/(1+\mathcal{P}_E) \longrightarrow \mu_4$$

by the formula

$$G_E(x) = \begin{cases} \frac{\phi(x)}{+\sqrt{q_E}} & \text{if } \nu_E(x)+\nu_E \text{ is odd} \\ \\ 1 & \text{if } \nu_E(x)+\nu_E \text{ is even.} \end{cases}$$

Since $\phi(x) = \phi(x^p)$ because p is odd we may extend G_E to a non-homomorphic function

$$G_E : C_E = ((E^*/1+\mathcal{P}_E) \otimes \mathbb{Z}[1/p]) \longrightarrow \mu_4.$$

When $p = 2$ set $G_E(x) = 1$ for all x.

1.12. *The case $E = \mathbb{Q}_p$ and the misprint of (*[37] §*2)*

When $E = \mathbb{Q}_p$ with $p \neq 2$ we have

$$G_{\mathbb{Q}_p} : \mathbb{Q}_p^*/(\mathbb{Q}_p^{2*}) = \mathbb{Q}_p^* \otimes \mathbb{Z}/2 \longrightarrow \mu_4.$$

Now $\mathbb{Q}_p^*/(\mathbb{Q}_p^{2*}) = \{1, u, p, up\}$ where $u \in \mathbb{Z}_p^*$ and the mod p Legendre symbol satisfies ([67] p. 267)

$$\left(\frac{u}{p}\right) = -1$$

We have $\nu_{\mathbb{Q}_p} = 0$ and $\nu_{\mathbb{Q}_p}(x) + \nu_{\mathbb{Q}_p}$ is even for $x = 1, u$ and $\nu_{\mathbb{Q}_p}(x) + \nu_{\mathbb{Q}_p} + 2(-1) + 1 = 0$ when $x = p, up$. Therefore

$$G_{\mathbb{Q}_p}(1) = 1 = G_{\mathbb{Q}_p}(u)$$

[2] In ([37] §2) the sum is taken over *all* the elements of the residue field. The error can be seen by taking $E = \mathbb{Q}_p$ (see §1.12 below).

and, if $\xi_p = e^{2\pi\sqrt{-1}/p}$,

$$
\begin{aligned}
G_{\mathbb{Q}_p}(p) &= \tfrac{1}{\sqrt{p}} \textstyle\sum_{z\in(\mathbb{Z}/p)^*} \ e^{2\pi\sqrt{-1}pp^{-2}z^2/2} \\
&= \tfrac{1}{\sqrt{p}} \textstyle\sum_{z\in(\mathbb{Z}/p)^*} \ \xi_p^{z^2/2} \\
&= \tfrac{1}{\sqrt{p}} \textstyle\sum_{z\in(\mathbb{Z}/p)^*} \ \xi_p^{z/2} + \tfrac{1}{\sqrt{p}} \textstyle\sum_{w\in(\mathbb{Z}/p)^*} \ \left(\frac{w}{p}\right) \xi_p^{w/2} \\
&= \tfrac{1}{\sqrt{p}} \textstyle\sum_{w\in(\mathbb{Z}/p)^*} \ \left(\frac{w}{p}\right) \xi_p^{w/2} \\
&= \left(\tfrac{2}{p}\right) \tfrac{1}{\sqrt{p}} \textstyle\sum_{w\in(\mathbb{Z}/p)^*} \ \left(\frac{w/2}{p}\right) \xi_p^{w/2} \\
&= \left(\tfrac{2}{p}\right) W_{\mathbb{Q}_p}(l(p)) \text{ in the notation of ([67] p.267)} \\
&= \begin{cases} -\left(\frac{2}{p}\right)\sqrt{-1} & \text{if } p \equiv 3 \pmod 4 \\ \left(\frac{2}{p}\right) & \text{if } p \equiv 1 \pmod 4. \end{cases}
\end{aligned}
$$

Hence $G_{\mathbb{Q}_p}(p)^2 = \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$.[3]

[3]The sum over all the residue field, as in ([37] §2), would add $\frac{1}{\sqrt{p}}$ to $G_{\mathbb{Q}_p}(p)$ and then its square would not be equal to $(-1)^{(p-1)/2}$ as claimed in ([37] §2)

$$
\begin{aligned}
G_{\mathbb{Q}_p}(up) &= \tfrac{1}{\sqrt{p}} \textstyle\sum_{z\in(\mathbb{Z}/p)^*} \ e^{2\pi\sqrt{-1}upp^{-2}z^2/2} \\
&= \tfrac{1}{\sqrt{p}} \textstyle\sum_{z\in(\mathbb{Z}/p)^*} \ \xi_p^{uz^2/2} \\
&= \tfrac{1}{\sqrt{p}} \textstyle\sum_{z\in(\mathbb{Z}/p)^*} \ \xi_p^{z/2} - \tfrac{1}{\sqrt{p}} \textstyle\sum_{w\in(\mathbb{Z}/p)^*} \ \left(\frac{w}{p}\right) \xi_p^{w/2} \\
&= -\tfrac{1}{\sqrt{p}} \textstyle\sum_{w\in(\mathbb{Z}/p)^*} \ \left(\frac{w}{p}\right) \xi_p^{w/2} \\
&= -\left(\tfrac{2}{p}\right) \tfrac{1}{\sqrt{p}} \textstyle\sum_{w\in(\mathbb{Z}/p)^*} \ \left(\frac{w/2}{p}\right) \xi_p^{w/2} \\
&= -\left(\tfrac{2}{p}\right) W_{\mathbb{Q}_p}(l(p)) \text{ in the notation of ([67] p.267)} \\
&= \begin{cases} \left(\frac{2}{p}\right)\sqrt{-1} & \text{if } p \equiv 3 \pmod 4 \\ -\left(\frac{2}{p}\right) & \text{if } p \equiv 1 \pmod 4. \end{cases}
\end{aligned}
$$

Therefore $G_{\mathbb{Q}_p}(up) \neq G_{\mathbb{Q}_p}(u)G_{\mathbb{Q}_p}(p)$ which confirms that $G_{\mathbb{Q}_p}$ is not a homomorphism.

In general, therefore, the formula for $G_{\mathbb{Q}_p}$ is given by ([67] p.267)

$$G_{\mathbb{Q}_p}(x) = \left(\frac{2^{\nu_{\mathbb{Q}_p}(x)}}{p}\right) W_{\mathbb{Q}_p}(l(x)).$$

1.13. *The formula for G_E in general when $p \neq 2$*

Let $N : \mathbb{F}_{q^d}^* \longrightarrow \mathbb{F}_q^*$ denote the norm. It is a surjective homomorphism, by Hilbert's Theorem 90 and element counting, so that we have a surjection

$$N : \mathbb{F}_{q^d}^*/\mathbb{F}_{q^d}^{2*} = \mathbb{F}_{q^d}^* \otimes \mathbb{Z}/2 \longrightarrow \mathbb{F}_q^*/\mathbb{F}_q^{2*} = \mathbb{F}_q^* \otimes \mathbb{Z}/2$$

which is therefore an isomorphism since both groups have only two elements.

The exact sequence

$$0 \longrightarrow \mathcal{O}_E^* \longrightarrow E^* \longrightarrow \mathbb{Z} \longrightarrow 0$$

yields a short exact sequence

$$0 \longrightarrow \mathcal{O}_E^* \otimes \mathbb{Z}/2 \longrightarrow E^*/E^{2*} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and in the short exact sequence

$$0 \longrightarrow 1 + \mathcal{P}_E \longrightarrow \mathcal{O}_E^* \longrightarrow \mathbb{F}_{q^d}^* \longrightarrow 0$$

the group $1 + \mathcal{P}_E$ is 2-divisible so that we have an isomorphism

$$\mathcal{O}_E^* \otimes \mathbb{Z}/2 \xrightarrow{\cong} \mathbb{F}_q^* \otimes \mathbb{Z}/2.$$

Therefore E^*/E^{2*} has four elements which are $\{1, u, \pi_E, u\pi_E\}$ where $u \in \mathcal{O}_E^*$ maps to a non-square in $\mathbb{F}_{q^d}^*$. If q is a power of p then the condition on u is equivalent to

$$\left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(u)}{p}\right) = -1.$$

Recall that $\mathcal{D}_E^{-1} = (\pi_E)^{-\nu_E}$.

Suppose that $\nu_E + 2b + 1 = 0$ so that

$$\nu_E(1) + \nu_E + 2b + 1 = 0 = \nu_E(u) + \nu_E + 2b + 1.$$

Therefore for $x = 1, u$ we have

$$\begin{aligned}
G_E(x) &= \tfrac{1}{\sqrt{N\pi_E}} \textstyle\sum_{z\in(\mathcal{O}_E/\mathcal{P}_E)^*} \ \psi_E(x\pi_E^{2b}z^2/2) \\
&= \tfrac{1}{\sqrt{N\pi_E}} \textstyle\sum_{z\in(\mathcal{O}_E/\mathcal{P}_E))^*} \ \psi_E(x\pi_E^{2b}z/2) \\
&\qquad + \tfrac{1}{\sqrt{N\pi_E}} \textstyle\sum_{w\in(\mathcal{O}_E/\mathcal{P}_E))^*} \ \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(w)}{p}\right) \psi_E(x\pi_E^{2b}w/2) \\
&= \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(2x)}{p}\right) \tfrac{1}{\sqrt{N\pi_E}} \textstyle\sum_{w\in(\mathcal{O}_E/\mathcal{P}_E))^*} \ \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(w)}{p}\right) \psi_E(\pi_E^{2b}w).
\end{aligned}$$

It would be nice to be able to apply the Davenport-Hasse theorem ([47] p.20) to this Gauss sum but this is only immediate in the case of $E/\mathbb{Q}_p$ being unramified because the additive character ψ_E involves the trace for $E/\mathbb{Q}_p$ rather than the trace for their residue fields. When ν_E is odd then $G_E(u\pi_E) = 1 = G_E(\pi_E)$.

Suppose that $\nu_E + 2b + 2 = 0$ so that

$$\nu_E(\pi_E) + \nu_E + 2b + 1 = 0 = \nu_E(u\pi_E) + \nu_E + 2b + 1.$$

Therefore for $x = \pi_E, u\pi_E$ we have

$$\begin{aligned}
G_E(x) &= \tfrac{1}{\sqrt{N\pi_E}} \textstyle\sum_{z\in(\mathcal{O}_E/\mathcal{P}_E)^*} \ \psi_E((x/\pi_E)\pi_E^{2b+1}z^2/2) \\
&= \tfrac{1}{\sqrt{N\pi_E}} \textstyle\sum_{z\in(\mathcal{O}_E/\mathcal{P}_E))^*} \ \psi_E((x/\pi_E)\pi_E^{2b+1}z/2) \\
&\qquad + \tfrac{1}{\sqrt{N\pi_E}} \textstyle\sum_{w\in(\mathcal{O}_E/\mathcal{P}_E))^*} \ \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(w)}{p}\right) \psi_E((x/\pi_E)\pi_E^{2b+1}w/2) \\
&= \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(2(x/\pi_E))}{p}\right) \tfrac{1}{\sqrt{N\pi_E}} \textstyle\sum_{w\in(\mathcal{O}_E/\mathcal{P}_E))^*} \ \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(w)}{p}\right) \psi_E(\pi_E^{2b+1}w).
\end{aligned}$$

1.14. *The case when $a(\chi) = 1$*

Now suppose that c is a generator of $f(\chi)\mathcal{D}_E$. In the notation of §1.13 the inverse different is given by $\mathcal{D}_E^{-1} = (\pi_E)^{-\nu_E}$. Therefore if $a(\chi) = 1$ and $\nu_E + 2b + 1 = 0$ then $f(\chi)\mathcal{D}_E = \mathcal{P}_E^{1+\nu_E} = \mathcal{P}_E^{-2b}$ and so $c^{-1} = \pi_E^{2b}$. Similarly

$a(\chi) = 1$ and $\nu_E + 2b + 2 = 0$ then $f(\chi)\mathcal{D}_E = \mathcal{P}_E^{1+\nu_E} = \mathcal{P}_E^{-2b-1}$ and so $c^{-1} = \pi_E^{2b+1}$.

Therefore, if $\chi_E : E^* \longrightarrow \mathbb{C}^*$ satisfies $\chi_E(\pi_E) = 1$ and

$$\chi_E(z) = (\frac{\mathrm{Norm}_{\mathbb{F}_{p^d}/\mathbb{F}_p}(z + \mathcal{P}_E)}{p}) \in \{\pm 1\}$$

for $z \in \mathcal{O}_E^*$ and $E^*/E^{2*} = \{1, u, \pi_E, u\pi_E\}$. the formulae of §1.13 become

$$G_E(x) = \begin{cases} \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(2x)}{p}\right) W_E(\chi_E) & x = 1, u \text{ and } \nu_E \text{ odd,} \\ \\ 1 & x = \pi_E, u\pi_E \text{ and } \nu_E \text{ odd,} \\ \\ \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(2(x/\pi_E))}{p}\right) W_E(\chi_E) & x = \pi_E, u\pi_E \text{ and } \nu_E \text{ even,} \\ \\ 1 & x = 1, u \text{ and } \nu_E \text{ even.} \end{cases}$$

2. The formula of ([37] p.123 (5)) for the biquadratic extension

2.1. Consider the formula of ([37] p.123 (5)) for an extension N/E

$$W_E(\mathrm{Ind}_{N/E}(1)) = \delta_{N/E}(g)^{-1} G_N(g)^{-1} G_E(g)^{[N:E]} \in \mu_4$$

which holds for all $g \in E^*$ but only depends upon $g \in E^*/E^{2*} = \langle 1, u, \pi_E, u\pi_E \rangle$. In this subsection we shall examine the formula in the all important case when $N = E(\sqrt{u}, \sqrt{\pi_E})$. Bear in mind that p is odd so that $E(\sqrt{u})/E$ is unramified and $E(\sqrt{\pi_E})/E$ is totally ramified. The extension N/E is the unique biquadratic extension of E.

Recall that (see §3.4)

$$\begin{aligned} W_E(\mathrm{Ind}_{N/E}(1)) &= SW_2(\mathrm{Ind}_{N/E}(1)) \cdot W_E(\mathrm{Det}(\mathrm{Ind}_{N/E}(1)) \\ &= SW_2(\mathrm{Ind}_{N/E}(1)) \cdot W_E(\delta_{N/E}). \end{aligned}$$

The trace form of N/E is represented (in the sense of Galois descent theory ([67] p.102 Example (2.31)) by the regular representation of $\mathrm{Gal}(N/E) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

$$\langle N/E \rangle = 1 + l(u) + l(\pi_E) + l(u\pi_E)$$

in the notation of [67]. Therefore the second Hasse-White invariant is given in $H^2(E; \mathbb{Z}/2) \cong \{\pm 1\}$ by

$$\begin{aligned} HW_2(\langle N/E \rangle) &= l(u)l(\pi_E) + l(u)l(u\pi_E) + l(\pi_E)l(u\pi_E) \\ &= l(u)l(\pi_E) + l(u)(l(u) + l(\pi_E)) + l(\pi_E)(l(u) + l(\pi_E)) \\ &= l(u)l(\pi_E) + l(-1)l(u) + l(-1)l(\pi_E). \end{aligned}$$

By a formula of Serre ([63]; [65]; [67] p.95 Corollary 2.8)

$$SW_2(\mathrm{Ind}_{N/E}(1)) = HW_2(\langle N/E\rangle) + l(2)\delta_{N/E} = HW_2(\langle N/E\rangle)$$

since, in cohomological notation

$$\delta_{N/E} = l(u) + l(\pi_E) + l(u\pi_E) \in H^1(E;\mathbb{Z}/2) \cong E^*/E^2$$

which is trivial because $l(u)+l(\pi_E) = l(u\pi_E)$ (c.f. [67] p.102 Example (2.31)).

Therefore the formula under discussion simplifies to the form

$$(l(u)l(\pi_E))\cdot(l(-1)l(u))\cdot(l(-1)l(\pi_E)) = W_E(\mathrm{Ind}_{N/E}(1)) = G_N(g)^{-1} \in \{\pm 1\}$$

where the products on the left hand side are cup-products in mod 2 Galois cohomology.

Next we need to verify that ν_N is even, which will follow from the transitivity formula for discriminants ([61] III §4). Let us recall how that goes.

We have the trace $\mathrm{Tr}_{L/K} : L \longrightarrow K$ for an extension of local fields L/K. The trace form $\langle L/K\rangle : (x,y) \mapsto \mathrm{Tr}_{L/K}(xy)$ is symmetric and non-singular on L. Let $\{e_i\}$ be a choice of $\mathcal{O}_K$-basis for the free module $\mathcal{O}_L$ then the discriminant of L/K is the ideal of $\mathcal{O}_L$ generated by the element $\det(\mathrm{Tr}_{L/K}(e_ie_j)) = (\det(\sigma(e_i))^2$ where σ runs through the set of K-monomorphisms of L into an algebraic closure of K. Set

$$D^{-1}_{L/K} = \{y \in L \mid \mathrm{Tr}_{L/K}(xy) \in \mathcal{O}_K \text{ for all } x \in \mathcal{O}_L\},$$

which is the inverse different of L/K (or codifferent) and it is the largest $\mathcal{O}_L$-submodule of L whose image under $\mathrm{Tr}_{L/K}$ lies in $\mathcal{O}_K$. The inverse of the codifferent is the different $D_{L/K}$ which is a non-zero ideal of $\mathcal{O}_L$. The absolute different is the case when $K = \mathbb{Q}_p$, the prime field. Transitivity for the chain of fields $\mathbb{Q}_p \subseteq E \subseteq N$ takes the form

$$D_{N/\mathbb{Q}_p} = D_{N/E}\cdot D_{E/\mathbb{Q}_p}.$$

Also $D_{E/\mathbb{Q}_p} = D_E = (\pi_E)^{\nu_E}$ and the ramification index of N/E is 2 so that the order ν_N of the discriminant of N is even unless $E/\mathbb{Q}_p$ is unramified.

Suppose that $E/\mathbb{Q}_p$ is ramified so that ν_N is even, $\pi_N = \sqrt{\pi_E}$ and

$$G_N(x) = \begin{cases} \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(2(x/\pi_N))}{p}\right) W_N(\chi_N) & x = \pi_N, u_N\pi_N \text{ and } \nu_N \text{ even,} \\ \\ 1 & x = 1, u_N \text{ and } \nu_N \text{ even.} \end{cases}$$

Therefore $G_N(g) = 1$ for $g \in E^*$.

This means that the unique biquadratic of E extends to a Q_8-extension, which is correct because each p-adic local field has a Q_8-extension and each such extension has a unique biquadratic subfield.

It remains to consider the case where E/Q$_p$ is unramified and so we may assume $\pi_E = p$. In this case, by the ramification criterion of ([61] III §5 Theorem 1) $D_{E/\mathbb{Q}_p} = \mathcal{O}_E$ and $D_{N/E} = \mathcal{O}_N$ if and only if N/E is unramified.

However $E(\sqrt{u})/E$ is unramified but for $N/E(\sqrt{u})$ in fact the inverse different is the fractional ideal generated by $\pi_N^{-1} = \sqrt{\pi_E}^{-1}$ so that $\nu_N = 1$ and

$$G_N(x) = \begin{cases} \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(2x)}{p}\right) W_N(\chi_N) & x = 1, u_N \text{ and } \nu_N \text{ odd}, \\ \\ 1 & x = \pi_N, u_N\pi_N \text{ and } \nu_N \text{ odd}. \end{cases}$$

Once again, because of the existence of Q_8 extensions of E we must have, for $g \in E^*$,

$$1 = G_N(g) = \left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(2)}{p}\right) W_N(\chi_N).$$

Now

$$\left(\frac{N_{\mathbb{F}_{q^d}/\mathbb{F}_p}(2)}{p}\right) = \left(\frac{2}{p}\right)^{\dim_{\mathbb{F}_p}(\mathcal{O}_N/\mathcal{P}_N)} = (-1)^{\frac{(p^2-1)\dim_{\mathbb{F}_p}(\mathcal{O}_N/\mathcal{P}_N)}{8}}$$

by the second subsidiary law of quadratic reciprocity.

Next we use the Davenport-Hasse theorem to compute the local roots number

$$W_N(\chi_N) = \frac{1}{\sqrt{N\pi_N}} \sum_{w \in (\mathcal{O}_N/\mathcal{P}_N)^*} \chi_N(w)\psi_N(w/c),$$

where we have used the fact that $\chi_N(c) = 1$. Also c generates $f(\chi_N)\mathcal{D}_N = \mathcal{P}_N^{1+1} = (\sqrt{p}^2) = (p)$ so that

$$W_N(\chi_N) = \frac{1}{\sqrt{N\pi_N}} \sum_{w \in (\mathcal{O}_N/\mathcal{P}_N)^*} \chi_N(w)\psi_N(w/p).$$

Now let $L = \mathbb{Q}_p(\sqrt{p})$ then

$$\begin{aligned} W_L(\chi_L) &= \tfrac{1}{\sqrt{p}} \textstyle\sum_{x \in \mathbb{F}_p^*} (\frac{x}{p})\psi_L(x/p) \\ &= (\tfrac{2}{p})\tfrac{1}{\sqrt{p}} \textstyle\sum_{x \in \mathbb{F}_p^*} (\frac{2x}{p})\psi_{\mathbb{Q}_p}(2x/p) \\ &= (\tfrac{2}{p}) W_{\mathbb{Q}_p}(l(p)) \\ &= \begin{cases} (\frac{2}{p})(-i) & \text{if } p \equiv 3 \text{ (modulo 4)} \\ \\ (\frac{2}{p}) & \text{if } p \equiv 1 \text{ (modulo 4)}, \end{cases} \end{aligned}$$

by ([67] p.266).

Since N/L is unramified the Davenport-Hasse theorem ([47] p.20) implies that

$$W_N(\chi_N) = -(-W_L(\chi_L))^{\dim_{\mathbb{F}_p}(\mathcal{O}_N/\mathcal{P}_N)}.$$

The residue degree is even so set $\dim_{\mathbb{F}_p}(\mathcal{O}_N/\mathcal{P}_N) = 2d$. Then we have[4]

$$G_N(g) = -(\tfrac{2}{p})^{2d} \begin{cases} (\tfrac{2}{p})^{2d}(-1)^d & \text{if } p \equiv 3 \text{ (modulo 4)} \\ (\tfrac{2}{p})^{2d} & \text{if } p \equiv 1 \text{ (modulo 4)} \end{cases}$$

$$= \begin{cases} (-1)^{d+1} & \text{if } p \equiv 3 \text{ (modulo 4)} \\ -1 & \text{if } p \equiv 1 \text{ (modulo 4).} \end{cases}$$

2.2. *The Davenport-Hasse theorem when $E/\mathbb{Q}_p$ is unramified*
Suppose that $E/\mathbb{Q}_p$ is unramified of degree d and that $p \neq 2$. The restriction of the trace

$$\mathrm{Trace}_{E/\mathbb{Q}_p} : \mathcal{O}_E \longrightarrow \mathbb{Z}_p$$

is surjective so that $\nu_E = 0$ and we may choose $\pi_E = p$. Then $G_E(1) = 1 = G_E(u)$ and for $x = p, up$

$$G_E(x) = \left(\frac{N_{\mathbb{F}_{p^d}/\mathbb{F}_p}(2(x/p))}{p}\right) \frac{1}{\sqrt{p^d}} \sum_{w \in (\mathcal{O}_E/\mathcal{P}_E))^*} \left(\frac{N_{\mathbb{F}_{p^d}/\mathbb{F}_p}(w)}{p}\right) \psi_E(p^{-1}w)$$

$$= -\left(\frac{N_{\mathbb{F}_{p^d}/\mathbb{F}_p}(2(x/p))}{p}\right) \frac{1}{\sqrt{p^d}}(-1) \sum_{w \in (\mathcal{O}_E/\mathcal{P}_E))^*} \left(\frac{N_{\mathbb{F}_{p^d}/\mathbb{F}_p}(w)}{p}\right) \psi_E(p^{-1}w)$$

$$= -\left(\frac{N_{\mathbb{F}_{p^d}/\mathbb{F}_p}(2(x/p))}{p}\right) \frac{1}{\sqrt{p^d}}(-\sum_{v \in \mathbb{F}_p^*} \left(\frac{v}{p}\right) \xi_p^v)^d$$

$$= (-1)^{d+1} \left(\frac{N_{\mathbb{F}_{p^d}/\mathbb{F}_p}(2(x/p))}{p}\right) W_{\mathbb{Q}_p}(l(p))^d,$$

by the Davenport-Hasse theorem ([47] p.20).

Question 2.3. Perhaps the proof of the Davenport-Hasse theorem given in ([47] p.20) would evaluate the Gauss sums of §1.13 in general?

3. p-adic Galois epsilon factors modulo p-primary roots of unity

In this section I shall give the proof of the main result of [37].

Theorem 3.1.
Let σ be a wild, homogeneous representation of $\mathrm{Gal}(\overline{F}/E)$ then

$$W_E(\sigma) \equiv \mathrm{Det}(\sigma)(g_\sigma)^{-1} G_E(_\sigma)^{\deg(\sigma)} \text{ (modulo } \mu_{p^\infty}).$$

[4]This seems to contradict the formula of ([37] p.123 (5)). However this is only a tamely ramified extension.

3.2. *How to use the strict inequality of ([37] p.121)*

We shall use the strict inequality of §1.6(•)

$$\alpha(M/M') < \alpha(\chi'),$$

assuming which I can run the following argument from ([37] p.121). In this case, providing that $\alpha(M/M') < n$, we have a commutative diagram ([27] Corollary 3.12(i))

$$\begin{array}{ccc}
\mathcal{P}_M^m/\mathcal{P}_M^{m+1} & \xrightarrow[x \ \mapsto \ 1+x]{} & (1+\mathcal{P}_M^m)/(1+\mathcal{P}_M^{m+1}) \\
\Big\downarrow \mathrm{Trace}_{M/M'} & & \Big\downarrow \mathrm{Norm}_{M/M'} \\
\mathcal{P}_{M'}^n/\mathcal{P}_{M'}^{n+1} & \xrightarrow{y \ \mapsto \ 1+y} & (1+\mathcal{P}_{M'}^n)/(1+\mathcal{P}_{M'}^{n+1})
\end{array}$$

in which the vertical maps are surjective and where $m = \psi_{M/M'}(n)$.

If we have the strict inequality $\alpha(M/M') < \alpha(\chi')$ we may take $n = \alpha(\chi') = a(\chi') - 1$ and $m = \psi_{M/M'}(\alpha(\chi'))$. By [27]

$$\alpha(\chi) = \alpha(\chi' \cdot \mathrm{Norm}_{M/M'}) \leq \psi_{M/M'}(\alpha(\chi')) = m$$

and since $\mathrm{Norm}_{M/M'}$ is surjective the composition $\chi = \chi' \cdot \mathrm{Norm}_{M/M'}$ is non-trivial on $(1+\mathcal{P}_M^m)/(1+\mathcal{P}_M^{m+1})$ so that $m \leq \alpha(\chi)$. Therefore

$$m = \alpha(\chi) \text{ and } a(\chi) = m+1.$$

Now suppose that we have $g' \in (M')^*/(1+\mathcal{P}_{M'})$ such that for all $y \in \mathcal{P}_{M'}^{\alpha(\chi')}/\mathcal{P}_{M'}^{\alpha(\chi')+1}$

$$\chi'(1+y) = \psi_{M'}(g'y)$$

and that we have $g \in M^*/(1+\mathcal{P}_M)$ such that for all $x \in \mathcal{P}_M^{\alpha(\chi)}/\mathcal{P}_M^{\alpha(\chi)+1}$

$$\chi(1+x) = \psi_M(gx).$$

Taking $y = \mathrm{Trace}_{M/M'}(x)$ we have

$$\begin{aligned} \psi_M(gx) &= \chi(1+x) \\ &= \chi'(\mathrm{Norm}_{M/M'}(1+x)) \\ &= \chi(1+y) \\ &= \psi_{M'}(g' \cdot y) \\ &= \psi_{M'}(g' \cdot \mathrm{Trace}_{M/M'}(x)) \\ &= \psi_{M'}(\mathrm{Trace}_{M/M'}(g'x)) \\ &= \psi_M(g'x) \end{aligned}$$

which shows that choosing χ' instead of χ leads to the construction of the same element

$$g_\sigma \in ((E^*/1 + \mathcal{P}_E) \otimes \mathbb{Z}[1/p]) = C(E).$$

We have

$$C(E) = (E^*/1 + \mathcal{P}_E) \otimes \mathbb{Z}[1/p] \stackrel{\cong}{\longrightarrow} (((M')^*/1 + \mathcal{P}_{(M')}) \otimes \mathbb{Z}[1/p])^G$$

$$\stackrel{\cong}{\longrightarrow} ((M^*/1 + \mathcal{P}_M) \otimes \mathbb{Z}[1/p])^G$$

and we have just shown that $g_\sigma = g_{\sigma,\chi} = g_{\sigma,\chi'}$ when considered as elements of $C(E)$. This means that, if the elements $\hat{g}_\sigma \in E^*$ and $\hat{g}_{\sigma,\chi'} \in (M')^*$ both represent $g_\sigma \in C(E)$ then, for some positive integer r,

$$(\hat{g}_\sigma/\hat{g}_{\sigma,\chi'})^{p^r} \in 1 + \mathcal{P}_{M'}.$$

Therefore, if $\rho : (M')^* \longrightarrow \mathbb{C}^*$ is a character of finite order then

$$\rho(\hat{g}_\sigma)/\rho(\hat{g}_{\sigma,\chi'}) \in \mu_{p^\infty}.$$

In addition

$$G_{M'}(\hat{g}_\sigma) = G_{M'}(\rho(\hat{g}_{\sigma,\chi'}).$$

These facts are used below in §3.4.

3.3. *The improved induction theorem*

Here I shall assume a familiarity with monomial resolutions for finite groups.

If σ is a wild and homogeneous representation of $G = \mathrm{Gal}(K/E)$ as in §1.5 then $V^{(A,\chi)} = V$ and the (A,χ)-part of the monomial resolution for V gives another monomial resolution in which every stabilising pair $(\mathrm{Gal}(K/M_j), \chi_j)$ is larger than or equal to (A,χ). Furthermore the Euler characteristic in $R_+(G)$ is well-defined because the Euler characteristic of the whole monomial resolution is well-defined. Therefore we have

$$\sigma = \sum_i n_i \mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^G(\chi_i) \in R(G)$$

where $A \subseteq \mathrm{Gal}(K/M_i)$ and $\chi_i|A = \chi$ for all i. Therefore each $\mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^{G}(\chi_i)$ restricts to $[K : M_i]\chi$ on A. This means that

$$g_\sigma = g_{\mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^{G}(\chi_i)}$$

for each i.

Note: There is a subtlety here to be careful of.

The entire representation $\mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^{G}(\chi_i)$ is wild and homogeneous (with the same associated character as σ) but this does not mean that χ_i is wild. We know that A is a ramification group for $\mathrm{Gal}(K/M_i)$ with the same lower numbering as A has for G but the definition of g_{χ_i} depends on the upper numbering which does not intersect well with subgroups!

3.4. *The proof of Theorem 3.1*

The result is already known when $p = 2$ [72] so henceforth p is an odd prime.

Write $\lambda_{N/E} = W_E(\mathrm{Ind}_{N/E}(1)) \in \mu_4$. The $\lambda_{N/E}$'s are a very subtle and important family of numbers, to the construction of which approximately 200 pages of the 400 pages essay [48] are devoted. In [26] it is shown that, for any orthogonal Galois representation,

$$W_E(\sigma) = SW_2(\sigma)W_E(\mathrm{Det}(\sigma)).$$

In ([67] p.274; see also [68]) is given a very quick construction of $W_E(\sigma)$ when σ is orthogonal, which immediately gives the existence and Deligne's formula. The formula is used in §2.1.

Suppose the we are given a wild, homogeneous representation σ as in §1.5. Define

$$\zeta(\sigma) = \mathrm{Det}(\sigma)(g_\sigma)^{-1}G_E(_\sigma)^{\deg(\sigma)} \in \mathbb{C}^*/\mu_{p^\infty}$$

so that our objective is to show that

$$W_E(\sigma) = \zeta(\sigma) \in \mathbb{C}^*/\mu_{p^\infty}.$$

Notice that if σ and τ are two wild, homogeneous representations such that $g_\sigma = g_\tau$ then $\zeta(\sigma \oplus \tau) = \zeta(\sigma)\zeta(\tau)$. Furthermore when η is one-dimensional we have $W_E(\eta) = \zeta(\eta) \in \mathbb{C}^*/\mu_{p^\infty}$ by ([34] p.4). Using the inductivity properties of $W_E(-)$ and the induction theorem of §3.3 we shall derive Theorem 3.1 from the one-dimensional case.

Consider the equation of §1.5

$$\sigma = \sum_i n_i \mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^{\mathrm{Gal}(K/E)}(\chi_i) \in R(\mathrm{Gal}(K/E)).$$

By inductivity (in relative dimension zero) of the local constants we have

$$W_E(\sigma) = \prod_i W_E(\mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^{\mathrm{Gal}(K/E)}(\chi_i))^{n_i} = \prod_i \lambda_{M_i/E}^{n_i} W_{M_i}(\chi_i)^{n_i}.$$

Since, by §3.3, for each i

$$g_\sigma = g_{\mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^{G}(\chi_i)}$$

we find that

$$\zeta(\sigma) = \prod_i \zeta(\mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^{\mathrm{Gal}(K/E)}(\chi_i))^{n_i} \in \mathbb{C}^*/\mu_{p^\infty}.$$

Therefore, if we can show that for each i

$$\frac{W_E(\mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^{\mathrm{Gal}(K/E)}(\chi_i))}{W_{M_i}(\chi_i)} = \frac{\zeta(\mathrm{Ind}_{\mathrm{Gal}(K/M_i)}^{\mathrm{Gal}(K/E)}(\chi_i))}{\zeta(\chi_i)} \in \mathbb{C}^*/\mu_{p^\infty}$$

then the result follows because $W_{M_i}(\chi_i) = \zeta(\chi_i) \in \mathbb{C}^*/\mu_{p^\infty}$.

In the situation of §1.5 we have $\mathrm{Res}_A^{\mathrm{Gal}(K/E)}(\sigma) = n\chi_\sigma$ and on $1 + \mathcal{P}_M^{a(\chi_\sigma)-1}$ we have $\chi_\sigma(1+x) = \psi_M(g_\sigma x)$ and $g_\sigma \in C(E)[1/p]$.

Now suppose (N, χ) is one of the (M_i, χ_i)'s so that

$$(A, \chi_\sigma) \leq (H = \mathrm{Gal}(K/N), \chi) \text{ and } N \leq M.$$

Therefore, if $\delta_{N/E} = \mathrm{Det}(\mathrm{Ind}_{N/E}(1))$, in $\mathbb{C}^*/\mu_{p^\infty}$ we have

$$\begin{aligned} W_E(\mathrm{Ind}_{N/E}(\chi)) &= \lambda_{N/E} W_N(\chi) \\ &= \delta_{N/E}(g)^{-1} G_N(g)^{-1} G_E(g)^{[N:E]} \chi(\hat{g})^{-1} G_N(\hat{g}), \end{aligned}$$

by Proposition 3.5, for any $g \in E^*$ and where on $1+\mathcal{P}_N^{a(\chi)-1}$ we have $\chi(1+y) = \psi_N(\hat{g}y)$. Then $\hat{g} = g_\sigma \in C(M)[1/p]$ so that $G_N(g_\sigma) = G_N(\hat{g})$.

Therefore

$$W_E(\mathrm{Ind}_{N/E}(\chi)) = \delta_{N/E}(g_\sigma)^{-1} G_E(g_\sigma)^{[N:E]} \chi(\hat{\hat{g}})^{-1} \in \mathbb{C}^*/\mu_{p^\infty}$$

but $g_\sigma/\hat{g}$ lies in a pro-p group so $\chi(\hat{g})/\chi(g_\sigma) \in \mu_{p^\infty}$. Hence, by the formula

$$\mathrm{Det}(\mathrm{Ind}_{N/E}(\chi)) = \mathrm{Res}_{E^*}^{N^*}(\chi)\delta_{N/E}$$

of ([25] Proposition 1.2)

$$W_E(\mathrm{Ind}_{N/E}(\chi)) = \mathrm{Det}(\mathrm{Ind}_{N/E}(\chi))^{-1}(g_\sigma) G_E(g_\sigma)^{[N:E]} \in \mathbb{C}^*/\mu_{p^\infty}$$

which implies the general formula for $W_E(\sigma)$ (modulo μ_{p^∞}).□

Proposition 3.5. *(*[37] *Proposition 1 p.123)*

Let K be a finite extension of E in $\overline{F}$ and let $g \in C_E \otimes \mathbb{Z}[1/p]$. Then

$$\frac{W_E(\mathrm{Ind}_{K/E}(\chi))}{W_K(\chi)} = \delta_{K/E}(g)^{-1} G_K(g)^{[K:E]} \in \mathbb{C}^*/\mu_{p^\infty}$$

for every character χ of K^*.

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