Computing Borel's Regulator I

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Abstract

We expand Hamida's integral for the Borel regulator in odd dimension as an infinite series and give a computer algorithm for dimension 3.

1 Introduction

Let $\zeta_F(s)$ denote the Dedekind zeta function of a number field F. The analytic class number formula ([17] p.21) gives the residue at $s = 1$ in terms of the order of the class-group of $\mathcal{O}_F$, the algebraic integers of F, and the Dirichlet regulator $R_0(F)$. Let d_F denote the discriminant of F. In terms of algebraic K-groups of $\mathcal{O}_F$ the class-group is equal to the torsion subgroup $\mathrm{Tors}\, K_0(\mathcal{O}_F)$ of $K_0(\mathcal{O}_F)$ and the formula has the form

$$res_{s=1}\ \zeta_F(s) \ = \ \frac{2^{r_1+r_2}\pi^{r_2}R_0(F)|\mathrm{Tors}\, K_0(\mathcal{O}_F)|}{|\mathrm{Tors}\, K_1(\mathcal{O}_F)|\sqrt{d_F}}.$$

The Dirichlet regulator $R_0(F)$, which is a real number, is the covolume of the lattice given by the image of the Dirichlet regulator homomorphism ([17] p.25)

$$R_F^0 : \mathcal{O}_F^* = K_1(\mathcal{O}_F) \longrightarrow \mathbb{R}^{r_1+r_2-1}.$$

Here r_1 and $2r_2$ denote the number of real or complex embeddings of F respectively. Equivalently, by Hecke's functional equation ([10], [17] p.18), $\zeta_F(s)$ has a zero of order $r_1 + r_2 - 1$ at $s = 0$. Let $\zeta_F^*(s_0)$ denote the first non-zero coefficient in the Taylor series for ζ_F at $s = s_0$. Therefore at $s = 0$ the functional equation yields

$$\zeta_F^*(0) = \lim_{s\to 0}\ \frac{\zeta_F(s)}{s^{r_1+r_2-1}} = -\frac{R_0(F)|\mathrm{Tors}\, K_0(\mathcal{O}_F)|}{|\mathrm{Tors}\, K_1(\mathcal{O}_F)|}.$$

This form of the analytic class number formula prompted Lichtenbaum [9] to ask: Which number fields F satisfy the analogous equation for higher-dimensional algebraic K-groups

$$\zeta_F^*(r) = \pm 2^{\epsilon}\frac{R_r(F)|K_{-2r}(\mathcal{O}_F)|}{|\mathrm{Tors}\, K_{1-2r}(\mathcal{O}_F)|}$$

for $r = -1, -2, -3, \ldots$ and some integer ϵ? Here $R_r(F)$ is the covolume of the Borel regulator homomorphism defined on $K_{1-2r}(\mathcal{O}_F)$ and to which we shall return in §2. This identity has become known as the Lichtenbaum conjecture and is known to be true in many cases; for example, there are several proofs for abelian extensions of the rationals ([8], [13], [14], [15]).

In the special case of the integers the conjecture takes the form

$$\frac{\zeta^*(r)}{R_r(\mathbb{Q})} = \pm 2^{\epsilon}\frac{|K_{-2r}(\mathbb{Z})|}{|\mathrm{Tors}\, K_{1-2r}(\mathbb{Z})|}$$

for $r = -1, -2, -3, \ldots$ and some integer ϵ. Here $\zeta^*(r)$ is the leading coefficient in the Taylor series for the Riemann zeta function $\zeta(z)$ at $z = r$. As mentioned above, this and the analogous relation in the cyclotomic case, is known to be true. Moreover when $r = -1$ the value of ϵ is known in the cyclotomic case by the main result of Ion Rada's thesis [12]. In addition the value of $|\mathrm{Tors}\, K_{1-2r}(\mathcal{O}_F)|$ (for example, see [14], [15]). The nett result is that, in order to compute $|K_2(\mathcal{O}_F)|$ in the cyclotomic case it suffices to have merely *approximate values* for $\zeta_F^*(-1)$ and of the Borel regulator $R_{-1}(F)$.

This discussion provides the motivation for this paper. For example [13] the Kummer-Vandiver conjecture, which originated in an 1849 letter from Kummer to Kronecker, would be implied by the vanishing of $|K_{-2r}(\mathbb{Z})|$ for $r = -2, -4, \ldots$. Although we are only concerned here to design an algorithm to calculate an approximation to the Borel regulator in dimension three in the long run we hope that our ideas may be capable of implementation in higher dimensions to calculate orders of even dimensional K-groups of rings of algebraic integers.

2 The Borel regulator map

Let F be a number field and $\mathcal{O}_F$ its ring of integers. F have $[F\colon \mathbb{Q}]$ different embeddings σ_j into $\mathbb{C}$. Some of them, say $\sigma_1, \ldots, \sigma_{r_1}$, are real and the rest come in conjugate pairs $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}, \overline{\sigma}_{r_1+1}, \ldots, \overline{\sigma}_{r_1+r_2}$. The *Borel regulator maps* are homomorphisms

$$K_{2n+1}(\mathcal{O}_F) \longrightarrow \mathbb{R}^{d_n} \qquad (n \geq 1) \tag{1}$$

from the odd algebraic K-theory groups of $\mathcal{O}_F$ to $\mathbb{R}^{d_n}$, where d_n is $r_1 + r_2$ if n is odd and r_2 if n is even. Given an embedding $\sigma\colon F \to \mathbb{C}$, write σ_* for the induced map in K-theory. Borel defined homomorphisms

$$B_n\colon K_{2n+1}(\mathbb{C}) \longrightarrow \mathbb{C} \tag{2}$$

such that the map (1) is the composition

$$K_{2n+1}(\mathcal{O}_F) \xrightarrow{\oplus(\sigma_j)_*} \bigoplus K_{2n+1}(\mathbb{C}) \xrightarrow{\oplus B_n} \bigoplus \mathbb{C}\,.$$

The map B_n is in turn defined as the composition of the following homomorphisms. The algebraic K-theory groups can be defined as the homotopy groups

$$K_{2n+1}(\mathbb{C}) = \pi_{2n+1}(BGL(\mathbb{C})^+)\,,$$

and hence there is a Hurewicz map into homology with integral coefficients

$$K_{2n+1}(\mathbb{C}) \xrightarrow{Hur} H_{2n+1}(BGL(\mathbb{C})^+) \cong H_{2n+1}(GL(\mathbb{C}))\,.$$

To obtain the isomorphism above, recall that the plus construction does not change the homology and the homology of a discrete group G is the homology of its classifying space BG. Via Suslin's stability results [7], if $N \geq 2n+1$ then

$$H_{2n+1}(GL(\mathbb{C})) \cong H_{2n+1}(GL_N(\mathbb{C}))\,.$$

Therefore, it suffices to define a map

$$b_n : H_{2n+1}(GL_N(\mathbb{C})) \longrightarrow \mathbb{C} \tag{3}$$

and precompose with the homomorphisms above. The map (3) is called the *universal Borel class* and it is actually given by a cocycle in the continuous cohomology group of $GL_N(\mathbb{C})$. The image of b_n lands in $(2\pi i)^{n-1}\mathbb{R}$ which we identify with $\mathbb{R}$. The original construction, due to Borel and using continuous cohomology, is described in [3]. Borel also proved that the image of the map (1) is a lattice in $\mathbb{R}^{d_n}$, whose (co)volume is called the *Borel regulator*, written $R_n(F)$, as mentioned in §1.

3 Hamida's integral

By a result of Hamida [5], the universal Borel class has the following description as an integral of differential forms. Fix $n \geq 1$ and $X_0, \ldots, X_n \in GL_N(\mathbb{C})$ with $N \geq 2n+1$ (ie. in stable range). Let Δ_n be the standard n-simplex

$$\Delta_n = \Big\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\Big|\; x_i \geq 0, \sum_i x_i = 1\Big\}.$$

Define for every point $\mathbf{x} = (x_0, \ldots, x_n) \in \Delta_n$

$$\nu(\mathbf{x}) = x_0 X_0^* X_0 + \ldots + x_n X_n^* X_n\,, \tag{4}$$

where X^* denotes the conjugate transpose of X. Thus ν is a matrix of 0-forms (complex functions) on the n-manifold Δ_n, and $\nu(\mathbf{x})$ is positive definite hermitian and in particular invertible. Consider the matrix of differential n-forms $(\nu^{-1}d\nu)^{\wedge n}$ and define

$$\varphi(X_0, X_1, \ldots, X_n) = \mathrm{Tr} \int_{\Delta^n} (\nu^{-1}d\nu)^{\wedge n} \tag{5}$$

the trace of a matrix of integrals of differential forms.

Theorem 1 (Hamida [5]) *Let $n \geq 0$. The map*

$$(X_0, \ldots, X_{2n+1}) \mapsto \frac{(-1)^{n+1}}{2^{3n+1}(\pi i)^n} \varphi(X_0, X_1, \ldots, X_{2n+1}) \tag{6}$$

induces a cocycle in the continuous cohomology of $\mathrm{GL}_N(\mathbb{C})$ which is the universal Borel class b_n.

Remark 2 *The cocycle (6) is homogeneous and unitarily normalised, that is,*

$$\begin{aligned} \varphi(X_0 g, \ldots, X_n g) &= \varphi(X_0, \ldots, X_n) \quad \text{for all } g \in GL_N(\mathbb{C}), && (7) \\ \varphi(u_0 X_0, \ldots, u_n X_n) &= \varphi(X_0, \ldots, X_n) \quad \text{for all } u_i \in U_N(\mathbb{C}). && (8) \end{aligned}$$

In particular, we can assume $X_n = Id$ by (7), and all the X_i to be positive definite hermitian matrices by (8), via the polar decomposition: every invertible matrix X can be written as $X = UP$ where U is unitary and P is positive definite hermitian. Indeed, $X^*X = P^*P$; compare with (4).

4 The Infinite Series Formula

Our goal is to make the computation of Hamida's integral (5) explicit. Namely, we will transform the integral into an infinite series which we can arbitrarily approximate.

In order to integrate an n-form on an n-dimensional manifold (with boundary in this case) we need to express the integrand in terms of n cordinates rather than $n+1$. We have a homeomorphism from the n-simplex given in $\mathbb{R}^n$ by

$$\Delta^n = \{(y_1, y_2, \ldots, y_n) \mid y_i \geq 0 \text{ for all } i\ , \sum_{j=1}^{n} y_j \leq 1\}$$

to the n-simplex in $\mathbb{R}^{n+1}$ given by the map

$$(y_1, \ldots, y_n) \mapsto (1 - \sum_{j=1}^{n} y_j, y_1, y_2, \ldots, y_n).$$

Therefore, in terms of the y-coordinates, we have

$$\begin{aligned} \nu &= X_0^* X_0 + y_1(X_1^* X_1 - X_0^* X_0) + \ldots + y_n(X_n^* X_n - X_0^* X_0) \\ &= X_n^*((X_n^*)^{-1} X_0^* X_0 X_n^{-1} + y_1((X_n^*)^{-1} X_1^* X_1 X_n^{-1} - (X_n^*)^{-1} X_0^* X_0 X_n^{-1}) \\ &\qquad + \ldots + y_n(1 - (X_n^*)^{-1} X_0^* X_0 X_n^{-1}))X_n \end{aligned}$$

and

$$\begin{aligned} d\nu &= X_n^*(dy_1((X_n^*)^{-1} X_1^* X_1 X_n^{-1} - (X_n^*)^{-1} X_0^* X_0 X_n^{-1}) \\ &\qquad + \ldots + dy_n(1 - (X_n^*)^{-1} X_0^* X_0 X_n^{-1}))X_n\,. \end{aligned}$$

Therefore $\nu = X_n^* \nu' X_n, d\nu = X_n^* d\nu' X_n$ and

$$\mathrm{Tr} \int_{\Delta^n} (\nu^{-1} d\nu)^n = \mathrm{Tr} \int_{\Delta^n} ((\nu')^{-1} d\nu')^n.$$

Next, we perform a change of variable by means of the map

$$T : [0,1] \times \Delta^{n-1} \longrightarrow \Delta^n$$

given by

$$T(t, s_1, s_2, \ldots, s_{n-1}) = (s_1 t, s_2 t, \ldots, s_{n-1} t, 1-t).$$

Since $s_1t+s_2t+\ldots+s_{n-1}t+1-t = 1-t(1-s_1+s_2+\ldots+s_{n-1})$ lies between zero and one, the image of T lies in the n-simplex. Furthermore for each non-zero value of t the corresponding horizontal $(n-1)$-simplex is mapped homeomorphically onto its image while $\{0\}\times\Delta^{n-1}$ is collapsed to the vertex $(0,0,\ldots,0,1)$. The Jacobian of T is equal to $(-1)^n t^{n-1}$. Therefore when $t\neq 0$ we have $J(T)>0$ if n is even and $J(T)<0$ if n is odd. For any compact n-manifold M with boundary in $[0,1]\times\Delta^{n-1}$ having image $T(M)$ in the n-simplex we have ([1] p.28)

$$\int_M (f\cdot T)(t,s_1,\ldots,s_{n-1})(-1)^n t^{n-1}dtds_1ds_2\ldots ds_{n-1}$$

$$=\int_{T(M)} f(y_1,\ldots,y_n)dy_1\ldots dy_n.$$

We shall be interested in the integral Trace $\int_{\Delta^n}((\nu')^{-1}d\nu')^n$ which may be written as the limit of integrals over manifolds of the form $T(M)$ as they tend towards the n-simplex. Therefore we can compute this integral as a limit of corresponding integrals over $M\subset[0,1]\times\Delta^{n-1}$, provided that this limit exists. However the integral over M, involving the Jacobian of T, is merely the integral over M where $\nu\cdot T$ is written in terms of $t,s_1,\ldots,s_{n-1}$ and $d\nu$ is also computed in these coordinates. Thus we have

$$\begin{aligned}\nu'\cdot T &= (X_n^*)^{-1}X_0^*X_0X_n^{-1}+ts_1((X_n^*)^{-1}X_1^*X_1X_n^{-1}-(X_n^*)^{-1}X_0^*X_0X_n^{-1})\\ &\qquad +\ldots+(1-t)(1-(X_n^*)^{-1}X_0^*X_0X_n^{-1})\\ &= (X_n^*)^{-1}X_0^*X_0X_n^{-1}+ts_1((X_n^*)^{-1}X_1^*X_1X_n^{-1}-(X_n^*)^{-1}X_0^*X_0X_n^{-1})\\ &\qquad +\ldots+1-t-(X_n^*)^{-1}X_0^*X_0X_n^{-1}+t(X_n^*)^{-1}X_0^*X_0X_n^{-1}\\ &= ts_1((X_n^*)^{-1}X_1^*X_1X_n^{-1}-(X_n^*)^{-1}X_0^*X_0X_n^{-1})\\ &\qquad +\ldots+1-t+t(X_n^*)^{-1}X_0^*X_0X_n^{-1}\\ &= 1+tA(s_1,s_2,\ldots,s_{n-1})\end{aligned}$$

where

$$\begin{aligned}&A(s_1,s_2,\ldots,s_{n-1})\\ &=((X_n^*)^{-1}X_0^*X_0X_n^{-1}-1)+\sum_{j=1}^{n-1} s_j((X_n^*)^{-1}X_j^*X_jX_n^{-1}-(X_n^*)^{-1}X_0^*X_0X_n^{-1}).\end{aligned}\tag{9}$$

Observe that, if we *assume* that the X_i's are close together, then for all points of Δ^{n-1} the matrix $A(s_1,s_2,\ldots,s_{n-1})$ is close to zero. Under this assumption we may express the integral as a convergent infinite series in the following manner.

As M varies, we must examine the integral

$$\mathrm{Tr}\int_M((1+tA)^{-1}(dtA+tdA))^n$$

where $dA=\sum_{j=1}^{n-1} ds_j((X_n^*)^{-1}X_j^*X_jX_n^{-1}-(X_n^*)^{-1}X_0^*X_0X_n^{-1})$. Hence we obtain

$$\begin{aligned}&\mathrm{Tr}\int_M((1+tA)^{-1}(dtA+tdA))^n\\ &=\mathrm{Tr}\int_M(1+tA)^{-1}dtA((1+tA)^{-1}tdA)^{n-1}\\ &\quad+\mathrm{Tr}\int_M(1+tA)^{-1}tdA(1+tA)^{-1}dtA((1+tA)^{-1}tdA)^{n-2}\\ &\quad+\mathrm{Tr}\int_M((1+tA)^{-1}tdA)^2(1+tA)^{-1}dtA((1+tA)^{-1}tdA)^{n-3}\\ &\qquad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\\ &\qquad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\\ &\quad+\mathrm{Tr}\int_M((1+tA)^{-1}tdA)^{n-1}(1+tA)^{-1}dtA.\end{aligned}$$

since the integral is unchanged by cyclic permutation of factors and because $A(\underline{s})$ and $1 + tA(\underline{s})$ commute. Next we observe that if $W_1, \ldots, W_n$ are $m \times m$ matrix-valued functions then

$$\mathrm{Trace}(W_1 dx_1 W_2 dx_2 \ldots W_n dx_n) = (-1)^{n-1} \mathrm{Trace}(W_2 dx_2 \ldots W_n dx_n W_1 dx_1)$$

because the trace of a product of n matrices is invariant under cyclic permutations but the n-form $dx_1 \ldots dx_n$ changes by the sign of the n-cycle, which is $(-1)^{n-1}$. Accordingly each integral in the formula for

$$\mathrm{Trace} \int_M ((1+tA)^{-1}(dtA + tdA))^n$$

is equal to $(-1)^{n-1}$ times the one below it. Together with the barycentric subdivision results of the next section this observation implies that

$$\varphi(X_0, X_1, \ldots, X_{2n}) = \mathrm{Tr} \int_{\Delta^{2n}} (\nu^{-1} d\nu)^{2n} = 0.$$

For the rest of this section we shall assume that n is odd. In this case

$$\begin{aligned}
&\mathrm{Trace} \textstyle\int_{[0,1]\times\Delta^{n-1}} ((1+tA)^{-1}(dtA + tdA))^n \\
&= n\ \mathrm{Trace} \textstyle\int_{[0,1]\times\Delta^{n-1}} (1+tA)^{-1} dtA((1+tA)^{-1} tdA)^{n-1} \\
&= n\ \mathrm{Trace} \textstyle\int_{[0,1]\times\Delta^{n-1}} t^{n-1} dtA(1+tA)^{-1}((1+tA)^{-1} dA)^{n-1} \\
&= n\ \mathrm{Trace} \textstyle\int_{[0,1]\times\Delta^{n-1}} t^{n-1} dtA(1+tA)^{-2} dA(1+tA)^{-1} dA \ldots (1+tA)^{-1} dA.
\end{aligned}$$

Also we observe that the assumption that the X_i's are close together means that through the domain of integration the matrix A is very close to zero. Recall the geometric series formula for a matrix A [6, 5.6.16]: if $\|\cdot\|$ is a matrix norm and $\|A\| < 1$ then $I - A$ is invertible and $\sum_{k=0}^{\infty} A^k = (I - A)^{-1}$. (A *matrix norm* on $M_N(\mathbb{C})$ is a vector norm which satisfies $\|AB\| \leq \|A\| \|B\|$.) Therefore we may expand the integral as a series, taking care not to commute matrix factors which may not commute, to obtain the following formula:

$$\begin{aligned}
&\mathrm{Trace} \textstyle\int_{[0,1]\times\Delta^{n-1}} ((1+tA)^{-1}(dtA + tdA))^n \\
&= n\ \mathrm{Trace} \textstyle\int_{[0,1]\times\Delta^{n-1}} t^{n-1} dtA(1+tA)^{-2} dA(1+tA)^{-1} dA \ldots (1+tA)^{-1} dA \\
&= n \mathrm{Trace} \textstyle\int_{[0,1]\times\Delta^{n-1}} \sum_{0\leq m_i} (-1)^{m_1+\ldots+m_{n-1}} (m_1+1) t^{n-1} dtA(tA)^{m_1} dA \ldots (tA)^{m_{n-1}} dA \\
&= n \mathrm{Trace} \textstyle\int_{[0,1]\times\Delta^{n-1}} \sum_{m_i \geq 0} (-1)^{\underline{m}(m_1,\ldots,m_{n-1})-1} m_1 t^{\underline{m}(m_1,\ldots,m_{n-1})+n-1} dtA^{m_1} dA \ldots A^{m_{n-1}} dA
\end{aligned}$$

where $\underline{m}(m_1, \ldots, m_{n-1}) = m_1 + \ldots + m_{n-1}$.

In §5 we shall develop a barycentric subdivision algorithm to reduce to the case where the X_i's are close together so that $\|A\| < 1$ and then we shall be able to evaluate the infinite series expression

$$n\,\mathrm{Tr} \int_{[0,1]\times\Delta^{n-1}} \sum_{m_i \geq 0} (-1)^{\underline{m}(m_1,\ldots,m_{n-1})-1} m_1 t^{\underline{m}(m_1,\ldots,m_{n-1})+n-1} dtA^{m_1} dA \ldots A^{m_{n-1}} dA\,, \tag{10}$$

where $\underline{m}(m_1, \ldots, m_{n-1}) = m_1 + \ldots + m_{n-1}$.

4.1 Case $n = 1$

In this example the map T map be taken as the identity map and $M = [0, 1]$. We obtain

$$\begin{aligned}
\mathrm{Tr} \textstyle\int_{\Delta^1} (\nu^{-1} d\nu) &= \mathrm{Tr} \textstyle\int_0^1 (1+tA)^{-1} A dt \\
&= \mathrm{Tr} \textstyle\sum_{n\geq 0} \int_0^1 (-1)^n t^n A^n A dt \\
&= \mathrm{Tr}(\log(1+A)).
\end{aligned}$$

4.2 Case $n = 3$

For $n = 3$ we can simplify (10) further. We have

$$\text{Trace} \int_M ((1+tA)^{-1}(dtA + tdA))^3$$

$$= 3\text{Trace} \int_M \sum_{m_1,m_2 \geq 0} (-1)^{m_1+m_2-1} m_1 t^{m_1+m_2+2} dt A^{m_1} dA A^{m_2} dA$$

and because $t^{m_1+m_2+2}dt$ is a scalar 1-form the effect of switching $A^{m_1}dA$ with $A^{m_2}dA$ is simply a change of sign

$$\text{Trace} \int_M ((1+tA)^{-1}(dtA + tdA))^3$$

$$= -3\text{Trace} \int_M \sum_{m_1,m_2 \geq 0} (-1)^{m_1+m_2-1} m_1 t^{m_1+m_2+2} dt A^{m_2} dA A^{m_1} dA$$

$$= -3\text{Trace} \int_M \sum_{m_1,m_2 \geq 0} (-1)^{m_1+m_2-1} m_2 t^{m_1+m_2+2} dt A^{m_1} dA A^{m_2} dA.$$

This yields a symmetrical formula

$$\text{Trace} \int_M ((1+tA)^{-1}(dtA + tdA))^3$$

$$= \tfrac{3}{2}\text{Trace} \int_M \sum_{m_1,m_2 \geq 0} (-1)^{m_1+m_2-1} (m_1 - m_2) t^{m_1+m_2+2} dt A^{m_1} dA A^{m_2} dA.$$

Now let us integrate the formula

$$\text{Trace} \int_M ((1+tA)^{-1}(dtA + tdA))^3$$

$$= \tfrac{3}{2}\text{Trace} \int_M \sum_{m_1,m_2 \geq 0} (-1)^{m_1+m_2-1} (m_1 - m_2) t^{m_1+m_2+2} dt A^{m_1} dA A^{m_2} dA$$

$$= 3\text{Trace} \int_M \sum_{m_1 > m_2 \geq 0} (-1)^{m_1+m_2-1} (m_1 - m_2) t^{m_1+m_2+2} dt A^{m_1} dA A^{m_2} dA$$

$$= 3\text{Trace} \int_{\Delta^2} \sum_{m_1 > m_2 \geq 0} (-1)^{m_1+m_2-1} \tfrac{(m_1 - m_2)}{m_1+m_2+3} A^{m_1} dA A^{m_2} dA.$$

Hence the intregral for $n = 3$ and $\|A\| < 1$ is

$$3\,\text{Tr} \int_{\Delta^2} \sum_{m_1 > m_2 \geq 0} (-1)^{m_1+m_2-1} \frac{m_1 - m_2}{m_1 + m_2 + 3} A^{m_1} dA A^{m_2} dA\,. \tag{11}$$

5 Barycentric subdivision

In order to use the infinite series formula in Section 4, we need $\|A\| < 1$ for a matrix norm $\|\cdot\|$. The idea is to apply the iterated barycentric subdivision operation to the simplex represented by $(X_0, \ldots, X_n)$ until the associated matrix A of each subsimplex satisfies $\|A\| < 1$.

5.1 Definitions and main properties

Suppose that X is a normed real vector space. Define the *simplex spanned by* $x_0, x_1, \ldots, x_n \in X$ as the subset

$$\Delta[x_0, \ldots, x_n] = \{t_0 x_0 + \ldots + t_n x_n \mid t_i \in \mathbb{R}, t_i \geq 0, \sum_i t_i = 1\}\,.$$

The *barycenter* of $\Delta[x_0, \ldots, x_n]$ is the point

$$b = \frac{1}{n+1}(x_0 + \ldots + x_n)\,.$$

The *barycentric subdivision* of $\Delta[x_0, \ldots, x_n]$ (cf. [16]) is the following decomposition into $(n+1)!$ n-simplices. For each permutation $\sigma \in \Sigma_{n+1}$ we have the n-simplex

$$s_\sigma = \Delta[x_{\sigma(0)}, x_{\sigma(0)\sigma(1)}, x_{\sigma(0)\sigma(1)\sigma(2)}, \ldots, x_{\sigma(0)\sigma(1)\ldots\sigma(n)}]\,,$$

where we write

$$x_{i_1\ldots i_m} = \frac{1}{m}(x_{i_1} + \ldots + x_{i_m})\,,$$

the barycenter of $\Delta[x_{i_1}, \ldots, x_{i_m}]$. We define the *sign* of s_σ as $\text{sgn}(\sigma)$, the signature of the permutation.

Remark 3 *For these definitions to make sense, we only need X to be a convex cone in a real vector space. For example, the set of all positive definite hermitian matrices.*

The barycentric subdivision reduces the diameter by a factor of $\frac{n}{n+1}$.

Proposition 4 *Suppose that Δ is an n-simplex in a normed real vector space spanned by $x_0, \ldots, x_n$. Consider its diameter $\operatorname{diam}(\Delta) = \sup\{\|x - y\| : x, y \in \Delta\}$. Then*

(1) $\operatorname{diam}(\Delta) = \max_{i,j} \|x_i - x_j\|$;

(2) if Δ' is any n-simplex in the barycentric subdivision of Δ then

$$\operatorname{diam}(\Delta') \leq \frac{n}{n+1} \operatorname{diam}(\Delta).$$

Proof: Given $a, b \in \Delta$, suppose $b = \sum t_i x_i$ as above. Then

$$\|a - b\| = \|(\sum t_i)a - \sum t_i x_i\| = \|\sum t_i(a - x_i)\| \leq \sum t_i \|a - x_i\| \leq \\ \leq \max_i \|a - x_i\|.$$

A similar argument shows $\|a - x_i\| \leq \max_j \|x_i - x_j\|$ and the first claim follows. For (2) see, for instance, [16, pp. 124-5].

Remark 5 *The barycentric subdivision procedure can be* iterated*, by applying it to each of the resulting n-simplices. Hence we can talk of the kth barycentric subdivision of a simplex, for each $k \geq 0$.*

The second key property is that barycentric subdivision can be applied and induces the identity in group cohomology. Let G be a group, R a commutative ring and M an RG-module. Consider (F_n, ∂_n) the standard resolution of RG [2]. Each F_n is the free RG-module generated by $(n+1)$-tuples $(g_0, \ldots, g_n)$ with G-action $g(g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$ and

$$\partial_n(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \widehat{g_i}, \ldots, g_n).$$

If we define $C_n = F_n \otimes_{RG} M$, then the homology groups $H_n(G; M)$ are the homology groups of $(C_*, \partial_* \otimes 1_M)$. On the other hand, if we let $C^n = \operatorname{Hom}_{RG}(F_n, M)$, then the cohomology groups $H^n(G; M)$ are the cohomology groups of (C^*, δ^*) where

$$\delta^n(\alpha)(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \alpha(g_0, \ldots, \widehat{g_i}, \ldots, g_{n+1}) \quad \text{for all } \alpha \in C^n.$$

The *barycentric subdivision* is a chain map $S\colon F_n \to F_n$ defined on generators as

$$S(g_0, \ldots, g_n) = \sum_{\sigma \in \Sigma_{n+1}} \operatorname{sgn}(\sigma)(g_{\sigma(0)}, \ldots, g_{\sigma(0),\ldots,\sigma(n+1)}),$$

where

$$g_{i_1 \ldots i_m} = \frac{1}{m}(g_{i_1} + \ldots + g_{i_m}) \in RG.$$

(From now on we assume $\operatorname{char}(R) = 0$ so that $1/m \in R$ for every $m \geq 1$.)

S extends to (co)chain maps $S\colon C_n \to C_n$ and $S\colon C^n \to C^n$ (we use the same letter S). For example,

$$S(\alpha)(g_0, \ldots, g_n) = \sum_{\sigma \in \Sigma_{n+1}} \operatorname{sgn}(\sigma)\alpha(g_{\sigma(0)}, \ldots, g_{\sigma(0),\ldots,\sigma(n+1)}) \quad \text{for all } \alpha \in C^n.$$

Remark 6 *The barycentric subdivision can be* iterated*, simply by $S^k = S \circ \ldots \circ S$, $k \geq 1$ times.*

It is standard and not difficult to show that these chain maps induce the identity in (co)homology. In particular, if $\alpha \in C^n$ is a cocycle, then so is $S(\alpha)$ and the represent the same element in $H^n(G; M)$. Recall that Hamida's function φ is a cocyle representing (up to a coefficient) the universal Borel class in $H^n(GL_N(\mathbb{C}); \mathbb{C})$. Hence $S(\varphi)$ also represents the universal Borel class. The same holds if we further subdivide.

Proposition 7 *Let $k \geq 1$. The cocycles φ, as defined in (5), and $S^k(\varphi)$ represent the same element in $H^n_{cont}(GL_N(\mathbb{C}); \mathbb{C})$.*

5.2 On matrix norms

Let $\|\cdot\|$ a *matrix norm*, that is, a vector norm in $M_N(\mathbb{C})$ which satisfies $\|AB\| \leq \|A\|\|B\|$. Our main example will be the *spectral norm* [6, 5.6.6]

$$\|A\|_2 = \max\left\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\right\}.$$

This norm satisfies $\|A^*\|_2 = \|A\|_2$ and $\|UAV\|_2 = \|A\|_2$ if U and V are unitary. If A is hermitian then $\|A\|_2 = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$. In this case, all the eigenvalues are real and we write them in increasing order as

$$\lambda_{min}(A) = \lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A) = \lambda_{max}(A).$$

If A is positive definite hermitian (if and only if $\lambda_{min}(A) > 0$) then $\|A\|_2 = \lambda_{max}(A)$ and $\|A^{-1}\|_2 = 1/\lambda_{min}(A)$.

Theorem 8 (Weyl) *Let A and B be hermitian matrices of dimension n. For each $k = 1, \ldots, n$ we have*

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_n(B).$$

For a proof, see 4.3.1 in [6].

In particular, if $A_1, \ldots, A_n$ are hermitian matrices then

$$\lambda_{min}(A_1 + \cdots + A_n) \geq \lambda_{min}(A_1) + \cdots + \lambda_{min}(A_n) \text{ and} \quad (12)$$
$$\lambda_{max}(A_1 + \cdots + A_n) \leq \lambda_{max}(A_1) + \cdots + \lambda_{max}(A_n). \quad (13)$$

5.3 Upper bounds

Let $X_0, \ldots, X_n \in GL_N(\mathbb{C})$ and $s_1, \ldots, s_{n-1} \in \mathbb{R}$ with $s_i \geq 0$ and $\sum s_i \leq 1$. Define, as in §4,

$$A = \left((X_n^*)^{-1}X_0^*X_0X_n^{-1} - I\right) + \sum_{j=1}^{n-1} s_j \left((X_n^*)^{-1}X_j^*X_jX_n^{-1} - (X_n^*)^{-1}X_0^*X_0X_n^{-1}\right). \quad (14)$$

Lemma 9 *Let $\|\cdot\|$ be any matrix norm such that $\|A^*\| = \|A\|$ for all $A \in M_n(\mathbb{C})$. Then, for any matrices $A, B, C \in M_n(\mathbb{C})$, we have*

(a) $\|A^*A - B^*B\| \leq \|A - B\| \cdot (\|A\| + \|B\|)$,

(b) $\|C^*A^*AC - C^*B^*BC\| \leq \|C\|^2 \cdot \|A - B\| \cdot (\|A\| + \|B\|)$,

(c) $\|C^*A^*AC - I\| \leq \|C\|^2 \cdot \|A - C\| \cdot (\|A\| + \|C\|)$.

Proof:

(a) $$\|A^*A - B^*B\| = \|(A^* - B^*)A + B^*(A - B)\| \leq \|A - B\| \cdot \|A\| + \|A - B\| \cdot \|B\| \leq \|A - B\| \cdot (\|A\| + \|B\|).$$

(b) $\|C^*A^*AC - C^*B^*BC\| = \|C^*(A^*A - B^*B)C\| \leq \|C\|^2 \cdot \|A^*A - B^*B\|$ and use part (a).

Part (c) follows from (b) when $B = C^{-1}$.

Using Lemma 9 we obtain

$$\begin{aligned}\|A\| &\leq \|X_n^{-1}\|^2 \cdot \left(\|X_n - X_0\| \cdot (\|X_n\| + \|X_0\|) + \sum_{j=1}^{n-1} s_j\|X_j - X_0\| \cdot (\|X_j\| + \|X_0\|)\right) \\ &\leq \|X_n^{-1}\|^2 \cdot \max_{1\leq i\leq n} \|X_i - X_0\| \cdot \left(\|X_n\| + \|X_0\| + \sum_{j=1}^{n-1} s_j (\|X_j\| + \|X_0\|)\right) \\ &\leq \|X_n^{-1}\|^2 \cdot \max_{1\leq i\leq n} \|X_i - X_0\| \cdot \left(\|X_n\| + \max_{1\leq j\leq n-1} \|X_j\| + 2\|X_0\|\right).\end{aligned}$$

Proposition 10 *For any matrix A as in (14) and any matrix norm satisfying $\|X^*\| = \|X\|$ for all $X \in M_n(\mathbb{C})$, we have*

$$\|A\| \leq \|X_n^{-1}\|^2 \cdot \max_{1\leq i\leq n} \|X_i - X_0\| \cdot \left(\|X_n\| + \max_{1\leq j\leq n-1} \|X_j\| + 2\|X_0\|\right). \quad (15)$$

Corollary 11 *Let* $d = \text{diam}(\Delta[X_0, \dots, X_n])$*,* $M = \max_i \|X_i\|$ *and* $m = \max_i \|X_i^{-1}\|$*. Then*

$$\|A\| \leq 4dMm^2.$$

If the matrices X_i are positive definite hermitian, we can consider iterated barycentric subdivisions of $\Delta[X_0, \dots, X_n]$ (cf. Remarks 3 and 2). Then the bound above, with respect to the spectral norm, decreases arbitrarily.

Proposition 12 *Suppose that* $X_0, \dots, X_n$ *are positive definite hermitian matrices spanning a n-simplex* $\Delta = \Delta[X_0, \dots, X_n]$*. Let* $\|\cdot\|$ *be the spectral norm and define*

$$\begin{aligned} d &= \text{diam}(\Delta) = \max_{i,j} \|X_i - X_j\|, \\ M &= \max_i \{\|X_i\|\} = \max_i \{\lambda_{max}(X_i)\}, \\ m &= \max_i \{\|X_i^{-1}\|\} = \max_i \{1/\lambda_{min}(X_i)\}. \end{aligned}$$

Let $\Delta[\widetilde{X}_0, \dots, \widetilde{X}_n]$ *be any n-simplex of the k-th iterated barycentric subdivision of* $\Delta[X_0, \dots, X_n]$*. For* $s_1, \dots, s_n \geq 0$ *and* $\sum s_i \leq 1$*, define*

$$\widetilde{A} = \left((\widetilde{X}_n^*)^{-1} \widetilde{X}_0^* \widetilde{X}_0 \widetilde{X}_n^{-1} - I\right) + \sum_{j=1}^{n-1} s_j \left((\widetilde{X}_n^*)^{-1} \widetilde{X}_j^* \widetilde{X}_j \widetilde{X}_n^{-1} - (\widetilde{X}_n^*)^{-1} \widetilde{X}_0^* \widetilde{X}_0 \widetilde{X}_n^{-1}\right).$$

Then

$$\|\widetilde{A}\| \leq \left(\frac{n}{n+1}\right)^k \cdot 4dMm^2.$$

This guarantees that, for k large enough, *all* the matrices $\widetilde{A}$ as above will satisfy $\|\widetilde{A}\| < 1$ (the bounds d, M and m only depend on the initial matrices $X_0, \dots, X_n$).

To prove the proposition, we need first a bound for $\|X\|$ and $\|X^{-1}\|$ for an arbitrary point in $\Delta[X_1, \dots, X_n]$.

Lemma 13 *Let* $X_0, \dots, X_n$ *be positive definite hermitian matrices and* $X \in \Delta[X_0, \dots, X_n]$*. Suppose that* $X = \sum_{i=0}^n t_i X_i$ *with* $t_i \geq 0$*,* $\sum_{i=0}^n t_i = 1$*. Then*

$$\begin{aligned} \|X\| &\leq \sum_{i=0}^n t_i \|X_i\| \quad \textit{and} \\ \|X^{-1}\| &\leq \sum_{i=0}^n t_i \|X_i^{-1}\|. \end{aligned}$$

In particular, $\|X\| \leq \max_i \|X_i\|$ *and* $\|X^{-1}\| \leq \max_i \|X_i^{-1}\|$*.*

Proof of Lemma: The first inequality holds for any norm. For the second inequality, note that the matrix X must be positive definite hermitian and, in particular, nonsingular and $\|X^{-1}\| = 1/\lambda_{min}(X)$. By (13),

$$\lambda_{min}(X) \geq \sum \lambda_{min}(t_i X_i) = \sum t_i \lambda_{min}(X_i).$$

Hence

$$\|X^{-1}\| = \frac{1}{\lambda_{min}(X)} \leq \frac{1}{\sum t_i \lambda_{min}(X_i)}.$$

Call $\lambda_i = \lambda_{min}(X_i) > 0$. Then

$$\frac{1}{\sum t_i \lambda_i} \leq \sum_i t_i \frac{1}{\lambda_i} \;\Leftrightarrow\; \sum_{i,j} t_i t_j \frac{\lambda_i}{\lambda_j} \geq 1.$$

The right-hand side inequality holds true: $\lambda + \lambda^{-1} \geq 2$ for all $\lambda > 0$ and therefore

$$\sum_{i,j} t_i t_j \frac{\lambda_i}{\lambda_j} \geq \sum_i t_i^2 + \sum_{i<j} 2t_i t_j = \left(\sum_i t_i\right)^2 = 1.$$

Proof of Proposition: We have, as in (15),

$$\|\widetilde{A}\| \leq \|\widetilde{X}_n^{-1}\|^2 \cdot \max_{1 \leq i \leq n} \|\widetilde{X}_i - \widetilde{X}_0\| \cdot \left(\|\widetilde{X}_n\| + \max_{1 \leq j \leq n-1} \|\widetilde{X}_j\| + 2\|\widetilde{X}_0\|\right).$$

From the previous Lemma we know that $\|\widetilde{X}_i\| \leq M$ and $\|\widetilde{X}_i^{-1}\| \leq m$ for all $0 \leq i \leq n$. From Proposition 4 we have

$$\max_{i,j} \|\widetilde{X}_i - \widetilde{X}_j\| = \operatorname{diam}\left(\Delta[\widetilde{X}_0, \ldots, \widetilde{X}_n]\right) \leq \left(\frac{n}{n+1}\right)^k \cdot d.$$

Therefore

$$\|\widetilde{A}\| \leq m^2 \operatorname{diam}\big(\Delta[\widetilde{X}_0, \ldots, \widetilde{X}_n]\big)\ (M + M + 2M) \leq \left(\frac{n}{n+1}\right)^k \cdot 4dMm^2.$$

6 The infinite series for $n = 3$

In this section, we expand Hamida's integral for $n = 3$ into a explicit infinite series with no integral operator, which can be directly implemented as a computer algorithm. Recall equation (11),

$$\begin{aligned} &\operatorname{Tr} \int_M \big((1-tA)^{-1}(dtA + tdA)\big)^3 = \\ &= \ 3 \operatorname{Tr} \int_{\Delta^2} \sum_{m_1 > m_2 \geq 0} (-1)^{m_1+m_2-1} \frac{m_1 - m_2}{m_1 + m_2 + 3} A^{m_1} dA A^{m_2} dA. \end{aligned} \tag{16}$$

Write $A = U_0 + s_1 U_1 + s_2 U_2$ as in (9). Fix $m_1 > m_2 \geq 0$. We have

$$\begin{aligned} &\operatorname{Tr} \int_{\Delta^2} A^{m_1} dA A^{m_2} dA \\ &= \ \operatorname{Tr} \int_{\Delta^2} (U_0 + s_1U_1 + s_2U_2)^{m_1} dA (U_0 + s_1U_1 + s_2U_2)^{m_2} dA \\ &= \ \operatorname{Tr} \int_{\Delta^2} \sum_{\substack{0 \leq u_1+u_2 \leq m_1 \\ u_1,u_2 \geq 0}} dA \sum_{\substack{0 \leq v_1+v_2 \leq m_2 \\ v_1,v_2 \geq 0}} s_1^{v_1} s_2^{v_2} B^{m_2}_{v_1,v_2}\,. \end{aligned}$$

where $B^r_{p,q}$ ($r \geq p+q$) is the sum of all matrices of the form $U_{i_1} \ldots U_{i_r}$, $i_j \in \{0,1,2\}$, containing p times U_1 and q times U_2 (and hence $r-p-q$ times U_0). Reordering and using that $dA = ds_1 U_1 + ds_2 U_2$ we get

$$\begin{aligned} &= \ \operatorname{Tr} \int_{\Delta^2} \sum_{\substack{0 \leq u_1+u_2 \leq m_1 \\ 0 \leq v_1+v_2 \leq m_2}} s_1^{u_1+v_1} s_2^{u_2+v_2} B^{m_1}_{u_1,u_2}\ dA\ B^{m_2}_{v_1,v_2}\ dA \\ &= \ \operatorname{Tr} \int_{\Delta^2} \Bigg(\sum_{\substack{0 \leq u_1+u_2 \leq m_1 \\ 0 \leq v_1+v_2 \leq m_2}} s_1^{u_1+v_1} s_2^{u_2+v_2} B^{m_1}_{u_1,u_2} U_1 B^{m_2}_{v_1,v_2} U_2\ ds_1 ds_2 \\ &- \sum_{\substack{0 \leq u_1+u_2 \leq m_1 \\ 0 \leq v_1+v_2 \leq m_2}} s_1^{u_1+v_1} s_2^{u_2+v_2} B^{m_1}_{u_1,u_2} U_2 B^{m_2}_{v_1,v_2} U_1\ ds_1 ds_2 \Bigg)\,. \end{aligned} \tag{17}$$

The second term can be rewritten, by a cyclic permutation, to have the form $B^{m_2}_{v_1,v_2} U_1 B^{m_1}_{u_1,u_2} U_2\ ds_1 ds_2$. That is, if we write

$$E_{m,n} = \sum_{\substack{0 \leq u_1+u_2 \leq m \\ 0 \leq v_1+v_2 \leq n}} s_1^{u_1+v_1} s_2^{u_2+v_2} B^{m}_{u_1,u_2} U_1 B^{n}_{v_1,v_2} U_2\ ds_1 ds_2$$

then (17) equals $\operatorname{Tr} \int_{\Delta^2} (E_{m_1,m_2} - E_{m_2,m_1})$, for each $m_1 > m_2 \geq 0$. Hence for each pair of arbitrary integers $m_1, m_2 \geq 0$ we have E_{m_1,m_2} when $m_1 > m_2$ and $-E_{m_1,m_2}$ when $m_1 < m_2$. Consequently, if we substitute $(m_1 - m_2)$ by $|m_1 - m_2|$, we can write (16) as

$$3 \operatorname{Tr} \int_{\Delta^2} \sum_{m_1,m_2 \geq 0} (-1)^{m_1+m_2-1} \frac{|m_1 - m_2|}{m_1 + m_2 + 3}\ E_{m_1,m_2}\,. \tag{18}$$

The trace and the integral operator are linear, and

$$\operatorname{Tr} \int_{\Delta^2} E_{m_1,m_2} = \sum_{\substack{0 \leq u_1+u_2 \leq m_1 \\ 0 \leq v_1+v_2 \leq m_2}} \operatorname{Tr}\left(B^{m_1}_{u_1,u_2} U_1 B^{m_2}_{v_1,v_2} U_2\right) \int_{\Delta^2} s_1^{u_1+v_1} s_2^{u_2+v_2}\ ds_1 ds_2\,. \tag{19}$$

The integral can be calculated using integration by parts. If $n, m > 0$,

$$\int_{\Delta^2} s_1^n s_2^m ds_1 ds_2 = \int_0^1 s_1^n \left(\int_0^{1-s_1} s_2^m ds_2 \right) ds_1 = \int_0^1 \frac{s_1^n (1-s_1)^{m+1}}{m+1} ds_1$$
$$= \int_0^1 \frac{n s_1^{n-1} (1-s_1)^{m+2}}{(m+1)(m+2)} ds_1 = \ldots = \int_0^1 \frac{n!(1-s_1)^{m+n+1}}{(m+1)\ldots(m+n+1)} ds_1$$
$$= \frac{n!}{(m+1)\ldots(m+n+2)} = \frac{n!m!}{(n+m+2)!} .$$

The calculation is still valid when $n = 0$ and, by symmetry, when $m = 0$.

On the other hand, each term $B^{m_1}_{u_1,u_2} U_1 B^{m_2}_{v_1,v_2} U_2$ amounts to the sum of matrices in the form $U_{i_1} \ldots U_{i_{m_1}} U_1 U_{j_1} \ldots U_{j_{m_2}} U_2$ with u_1 respectively u_2 matrices U_1 resp. U_2 among the first m_1, and v_1 resp. v_2 matrices U_1 resp. U_2 among the last m_2. By cyclic permutation, we can write it as

$$U_1 U_{j_1} \ldots U_{j_{m_2}} U_2 U_{i_1} \ldots U_{i_{m_1}} . \tag{20}$$

Given $k \geq 1$ and an array $U_1 U_{i_1} \ldots U_{i_k}$, how many times (if any) does it appear in (18) and with which coefficient? Compare the expression $U_1 U_{i_1} \ldots U_{i_k}$ to (20). For every j with $i_j = 2$ we have a product as in (20) with

$$\begin{aligned} m_2 &= j-1 \\ m_1 &= k-j \\ v_1 &= |\{i_p = 1 \,|\, 1 \leq p \leq j-1\}| \\ v_2 &= |\{i_p = 2 \,|\, 1 \leq p \leq j-1\}| \\ u_1 &= |\{i_p = 1 \,|\, j+1 \leq p \leq k\}| \\ u_2 &= |\{i_p = 2 \,|\, j+1 \leq p \leq k\}| \end{aligned}$$

Define $n_1 = u_1 + v_1$ and $n_2 = u_2 + v_2$, that is,

$$\begin{aligned} n_1 &= |\{i_p = 1 \,|\, 1 \leq p \leq k\}| \\ n_2 &= |\{i_p = 2 \,|\, 1 \leq p \leq k\}| - 1 . \end{aligned}$$

Let us denote by χ_2 the characteristic function on 2 ($\chi_2(2) = 1$, $\chi_2(i) = 0$ if $i \neq 2$). Using (19) and the previous calculations, we can rewrite (18) as

$$\sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=0}^{2} \kappa_{i_1,\ldots,i_k} \operatorname{Tr}(U_1 U_{i_1} \ldots U_{i_k})$$

where

$$\begin{aligned} \kappa_{i_1,\ldots,i_k} &= \sum_{j=1}^{k} \chi_2(i_j)(-1)^k \frac{|k-2j+1|}{k+2} \frac{(u_1+v_1)!(u_2+v_2)!}{(u_1+v_1+u_2+v_2+2)!} \\ &= \frac{(-1)^k}{k+2} \frac{(n_1)!(n_2)!}{(n_1+n_2+2)!} \sum_{j=1}^{k} \chi_2(i_j)\, |k-2j+1| . \end{aligned}$$

Note that we have use that, for positive definite hermitian matrices X_i of size n and the spectral norm, $\operatorname{Tr}\left(\sum_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \operatorname{Tr}(X_n)$, since $\|X\| \leq \operatorname{Tr}(X)$ and $\operatorname{Tr}(X) \leq n\|X\|$.

Proposition 14 (Infinite series for $n = 3$) *If $\|A\| < 1$ then*

$$\operatorname{Tr} \int_M \left((1-tA)^{-1}(dtA + tdA)\right)^3 =$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k+2} \sum_{i_1,\ldots,i_k=0}^{2} \frac{(n_1)!(n_2)!}{(n_1+n_2+2)!}\, c \operatorname{Tr}(U_1 U_{i_1} \ldots U_{i_k}) , \tag{21}$$

where n_1, n_2 and c depend on each tuple $(i_1, \ldots, i_{k+1})$ as

$$\begin{aligned} n_1 &= |\{i_p = 1 \,|\, 1 \leq p \leq k\}|, \\ n_2 &= |\{i_p = 2 \,|\, 1 \leq p \leq k\}| - 1, \\ c &= \sum_{j=1}^{k} \chi_2(i_j)\, |k-2j+1| . \end{aligned}$$

7 Computacional aspects and algorithm

We discuss the computational aspects of our algorithm. We write $\| \ \|$ for the spectral norm.

7.1 Initial steps

The program is given matrices $Y_0, Y_1, Y_2, Y_3 \in GL_N(\mathbb{C})$ (normally $N = 7$). It first computes the positive definite hermitian matrices X_0, X_1, X_2, X_3 such that $X_i^* X_i = Y_i^* Y_i$ for all $0 \leq i \leq 3$ (see Remark 2). This is done via the Schur decomposition of $X_i^* X_i$. Then the following values are stored

$$\begin{aligned} d_{ij} &= \|X_i - X_j\| = \sqrt{\lambda_{max}\left((X_i - X_j)^*(X_i - X_j)\right)}, \\ \mu_i &= \|X_i\| = \lambda_{max}(X_i), \\ \nu_i &= \|X_i^{-1}\| = 1/\lambda_{min}(X_i), \end{aligned}$$

for $i, j \in \{0, 1, 2, 3\}$.

7.2 Barycentric subdivision

This is a recursive step. Given a 4-simplex $\Delta[\widetilde{X}_0, \dots, \widetilde{X}_3]$ of the k^{th}-barycentric subdivision of $\Delta = \Delta[X_0, \dots, X_3]$ (just the initial simplex Δ when $k = 0$), we define the corresponding matrix $\widetilde{A}$ as in Proposition 12 and try to prove that $\|\widetilde{A}\| < 1$ using

$$\|\widetilde{A}\| \leq \|\widetilde{X}_n^{-1}\|^2 \cdot \max_{1 \leq i \leq n} \|\widetilde{X}_i - \widetilde{X}_0\| \cdot \left(\|\widetilde{X}_n\| + \max_{1 \leq j \leq n-1} \|\widetilde{X}_j\| + 2\|\widetilde{X}_0\| \right) .$$

If we succeed, we move to the next step. If we fail, we calculate the barycentric subdivision of $\Delta[\widetilde{X}_0, \dots, \widetilde{X}_3]$ and apply this step recursively to each of the resulting 24 simplices.

Remark 15 *Proposition 12 guarantees that the recursion finishes: the maximal depth required in the barycentric subdivision is bounded above by* $\left| \frac{\log(4dMm^2)}{\log(n+1) - \log(n)} \right|$.

7.3 Infinite Series

We are given $(X_0, \dots, X_n)$ with the associated matrix A satisfying $\|A\| < 1$. Our aim is to approximate the value of the infinite series in Proposition 14. We keep the same notation as there. Also, let us write $f_{i_1 \dots i_k}$ for the factorial coefficient, that is,

$$f_{i_1 \dots i_k} = \frac{(n_1)!(n_2)!}{(n_1 + n_2 + 2)!}.$$

The $(k+1)$th term in the series (21) can be obtained from the kth term as follows. For each $i_1, \dots, i_k \in \{0, 1, 2\}$, we have three new matrices

$$(U_1 U_{i_1} \dots U_{i_k})\, U_j \qquad j = 0, 1, 2$$

and three new factorial coefficients

$$f_{i_1 \dots i_k j} = \begin{cases} f_{i_1 \dots i_k} & j = 0 \\ f_{i_1 \dots i_k} \frac{n_j}{n_1 + n_2 + 2} & j = 1, 2\,, \end{cases} \tag{22}$$

where n_1 and n_2 are the ones defined for the tuple $(i_1, \dots, i_k)$. Hence for each such tuple $(i_1, \dots, i_k)$, we should store the values of $U_1 U_{i_1} \dots U_{i_k}$ and $f_{i_1 \dots i_k}$. The other terms in the series can be inexpensively calculated in each iteration, see (21).

7.4 Error estimate

It remains the question of bounding the error after each iteration. We now verify the absolute convergence of the series (21) with respect to the spectral norm by finding a majorising series, which serves as means of estimating the error after each iteration.

If X is a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ then

$$|\mathrm{Tr}(X)| = \left| \sum_{i=1}^{n} \lambda_i \right| \leq n|\lambda_{max}|\,.$$

So if X is hermitian, $|\mathrm{Tr}(X)| \leq n\|X\|$. Write $A = U_0 + s_1U_1 + s_2U_2$ as in Section 4. Suppose that $\|U_i\| \leq m$, $i = 0, 1, 2$ (we will need $m < 1/2$). Then

$$|\mathrm{Tr}\,(U_1U_{i_1} \ldots U_{i_k})| \leq 4\,\|U_1U_{i_1} \ldots U_{i_k}\| \leq 4\,m^{k+1}\,.$$

On the other hand

$$\sum_{j=1}^{k} |k - 2j + 1| = \begin{cases} k^2/2 & \text{if } k \text{ even,} \\ (k^2 - 1)/2 & \text{if } k \text{ odd,} \end{cases}$$

so we have,

$$c = \sum_{j=1}^{k} \chi_2(i_j)\,|k - 2j + 1| \leq \frac{k^2}{2}\,.$$

Additionally, one can prove that

$$\sum_{i_1,\ldots,i_k=0}^{2} \frac{(n_1)!(n_2)!}{(n_1 + n_2 + 2)!} \leq 2^{k-1}$$

by induction on k: for $k = 1$ we get $5/6 \leq 1$ and the induction step follows from (22),

$$\sum_{i_{k+1}=0}^{2} f_{i_1 \ldots i_{k+1}} = f_{i_1 \ldots i_k}\left(1 + \frac{n_1 + n_2 + 2}{n_1 + n_2 + 3}\right) \leq 2f_{i_1 \ldots i_k}$$

which implies

$$\sum_{i_1,\ldots,i_{k+1}=0}^{2} f_{i_1 \ldots i_{k+1}} \leq \sum_{i_1,\ldots,i_k=0}^{2} 2f_{i_1 \ldots i_k} \leq 2^k\,.$$

All in all, if we call a_k the kth coefficient of the series (21), we have

$$\begin{aligned} |a_k| &= \frac{1}{k+2} \sum_{i_1,\ldots,i_k=0}^{2} f_{i_1 \ldots i_k} \frac{k^2}{2} |\mathrm{Tr}(U_1U_{i_1} \ldots U_{i_k})| \\ &\leq \frac{k^2}{2(k+2)} \left(\sum_{i_1,\ldots,i_k=0}^{2} f_{i_1 \ldots i_k} \right) 4m^{k+1} \leq \frac{k^2}{2(k+2)} 2^{k-1} 4m^{k+1} = \frac{k^2}{2(k+2)} (2m)^{k+1} \\ &\leq \frac{k}{2} (2m)^{k+1}\,. \end{aligned}$$

Therefore, the series (21) is majorised by the convergent series

$$\sum_{k=1}^{\infty} \frac{k}{2} (2m)^{k+1} = \frac{2m^2}{(1 - 2m)^2}\,, \tag{23}$$

if $m < 1/2$. Hence we need each $\|U_i\| \leq m < 1/2$, as expected, since this immediately implies $\|A\| < 1$.

Let $S_n = \sum_{k=1}^{n} a_k$ be the value of the series (21) after n iterations and $S = \sum_{k=1}^{\infty} a_k$ the exact value. Since (23) is a majorising series, the error after n iterations $|S - S_n|$ is bounded above by

$$E_n = \sum_{k=1}^{\infty} \frac{k}{2} (2m)^{k+1} - \sum_{k=1}^{n} \frac{k}{2} (2m)^{k+1} = \frac{2m^2}{(1 - 2m)^2} - \sum_{k=1}^{n} \frac{k}{2} (2m)^{k+1}.$$

8 Test cases

We have implemented an algorithm in Matlab [11] to compute the series in Proposition 14. We have performed some test cases where we compare the program's output with the expected value of Hamida's function φ.

8.1 Test case 1

Let d be a prime, $u \in \mathbb{Q}(\sqrt{-d})^*$ and $n, m \in \mathbb{Z}$. Set

$$g_1 = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \text{ and } g_3 = \begin{pmatrix} 1 & n\sqrt{-d} \\ 0 & 1 \end{pmatrix}.$$

Let

$$\begin{aligned} z = &(1, g_1, g_1g_2, g_1g_2g_3) - (1, g_1, g_1g_3, g_1g_2g_3) + (1, g_3, g_1g_3, g_1g_2g_3) \\ &-(1, g_2, g_1g_2, g_1g_2g_3) + (1, g_2, g_2g_3, g_1g_2g_3) - (1, g_3, g_2g_3, g_1g_2g_3). \end{aligned}$$

Then z is a 3-cycle and $\varphi(z) = 0$ [4]. Write the cycle as $z = c_1 - c_2 + c_3 - c_4 + c_5 - c_6$. We obtain answers close to zero for Hamida's function already at each c_i, that is, $\varphi(c_i) \sim 0$ for each $1 \leq i \leq 6$. The reason is that Hamida's function is unitarily normalized. If

$$u_2 = \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \text{ and } u_3 = \begin{pmatrix} 1 & -n\sqrt{-d} \\ 0 & 1 \end{pmatrix},$$

then u_2g_2 and u_3g_3 are the identity, and every c_i above can me make into a cycle c_i' with two consecutive repeated elements. In this case $\varphi(c_i) = \varphi(c_i') = 0$, since Hamida's integral vanishes for any chain with two repeated values (an n-form in $n-1$ variables is zero).

The algorithm fails to detect the vanishing directly but gives answers close to zero. We keep track of the positive respectively negative partial sums (the sum of the positive respectively negative terms of the series in Proposition 14) so that the answer is close to zero up to an error relative to the order of the positive and negative sums.

Additionally, we had to limit the maximum number of iterations to obtain an answer in a sensible time. Sometimes we modified the bound test (see Section 7.2) to force a deeper barycentric subdivision in order to compensate for the lack of iterations.

Case $d = 3$, $u = 1$, $m = 1$ and $n = 1$

In this case there are two repeated entries in each chain and we obtain answers close to the machine epsilon in our computer.

	POS	NEG	POS+NEG
c_1	79.212543798236482	−79.212543798232005	$4.476 \cdot 10^{-12}$
c_2	35.255902765497538	−35.255902765496728	$0.810 \cdot 10^{-12}$
c_3	25.761457824702696	−25.761457824702926	$-0.231 \cdot 10^{-12}$
c_4	22.698322652159643	−22.698322652159863	$-0.220 \cdot 10^{-12}$
c_5	35.423517445833234	−35.423517445834165	$-0.931 \cdot 10^{-12}$
c_6	38.702788638164705	−38.702788638165714	$-1.009 \cdot 10^{-12}$

Case $d = 3$, $u = 2$, $m = 1$ and $n = 1$

This time the algorithm was not simplify by repeated entries. We had to forced an increased barycenter subdivision and limit the iterations to 4 to finish in a reasonable time. The output is summarized in the following table.

	POS	NEG	POS+NEG
c_1	402.6517675	−402.6512473	0.0005202
c_2	778.9142690	−778.9130614	0.0012076
c_3	933.8326019	−933.8463496	−0.0013748
c_4	489.5582685	−489.5751380	−0.0168695
c_5	494.6188740	−494.6213859	−0.0025119
c_6	449.8738903	−449.8708292	0.0030611

The solutions are nevertheless close to zero relative to the positive and negative values.

Case $d = 3$, $u = \sqrt{-3}$, $m = -1$ and $n = 2$

The same remarks apply to this case. The output is give again in table format.

	POS	NEG	POS+NEG
c_1	1390.3704696	−1390.3769837	−0.0065141
c_2	4185.0669067	−4185.0619767	0.0049300
c_3	5879.8887455	−5879.9673981	−0.0786525
c_4	1701.0187054	−1701.0253901	−0.0066847
c_5	2774.6657137	−2774.6719721	−0.0062584
c_6	2601.1767976	−2601.1739894	0.0028082

As before, the answers are close to zero with small error relative to the positive and negative values.

8.2 Test case 2

Let F be a field and $u \in F$ with $u^n = 1$. Set

$$a = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u^{-1} \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & v \\ 0 & -v^{-1} & 0 \end{pmatrix}.$$

Note that $ab = ba$, $a^n = e$, $waw^{-1} = b$ and $wbw^{-1} = a$. Define (in bar resolution notation)

$$\begin{aligned} z_1 &= [w|a|b] - [a|b|w] - [w|b|a] + [b|a|w] + [a|w|a] - [b|w|b], \\ z_2 &= [e|e|b] + [b|e|e] + \sum_{r=1}^{n-1} [a^r|a|b] - [a^r|b|a] + [b|a^r|a]. \end{aligned}$$

Then $d(z_1) = 2\{a, b\}$ and $d(z_2) = n\{a, b\}$ so $[nz_1 - 2z_2] \in H_3(SL_3(F); \mathbb{Z})$ [4]. If F is a number field, this class should be in the kernel of the Borel regulator.

We have done several test cases for $F = \mathbb{Q}(u)$, $u = e^{2\pi i/n}$ and $v = 1$. In all cases, no barycentric subdivision is needed (the matrices are already close enough) and the algorithm takes just a few seconds to finish (hence no need in general to limit the number of iterations). Moreover, the values of φ at each chain forming z_1 and z_2 are deeply related, as we show.

To facilitate reference we write $z_1 = c_1 - c_2 - c_3 + c_4 + c_5 - c_6$ and $z_2 = d_1 + d_2 + d_3 - d_4 + d_5 + \ldots + d_{3(n-1)+2}$.

Case $n = 3$

The results are summarized in the following two tables. For once, we show the positive and negative answers, altough this is not needed anymore.

	POS	NEG	POS+NEG
c_1	0	0	0
c_2	$0.54738221262 \cdot 10^{-47}$	$-0.18246073754 \cdot 10^{-47}$	$0.36492147508 \cdot 10^{-47}$
c_3	0	0	0
c_4	$0.54738221263 \cdot 10^{-47}$	$-0.18246073754 \cdot 10^{-47}$	$0.36492147508 \cdot 10^{-47}$
c_5	$0.54738221263 \cdot 10^{-47}$	$-0.18246073754 \cdot 10^{-47}$	$0.36492147508 \cdot 10^{-47}$
c_6	$0.54738221263 \cdot 10^{-47}$	$-0.18246073754 \cdot 10^{-47}$	$0.36492147508 \cdot 10^{-47}$
z_1	0	0	0

	POS	NEG	POS+NEG
d_1	0	0	0
d_2	$0.54738221262 \cdot 10^{-47}$	$-0.18246073754 \cdot 10^{-47}$	$0.36492147508 \cdot 10^{-47}$
d_3	$0.54738221262 \cdot 10^{-47}$	$-0.18246073754 \cdot 10^{-47}$	$0.36492147508 \cdot 10^{-47}$
d_4	$0.54738221262 \cdot 10^{-47}$	$-0.18246073754 \cdot 10^{-47}$	$0.36492147508 \cdot 10^{-47}$
d_5	$0.54738221262 \cdot 10^{-47}$	$-0.18246073754 \cdot 10^{-47}$	$0.36492147508 \cdot 10^{-47}$
d_6	$0.32842932758 \cdot 10^{-46}$	$-0.10947644252 \cdot 10^{-46}$	$0.21895288505 \cdot 10^{-46}$
d_7	$0.27369110631 \cdot 10^{-46}$	$-0.07298429502 \cdot 10^{-46}$	$0.20070681130 \cdot 10^{-46}$
d_8	$0.21895288505 \cdot 10^{-46}$	$-0.05473822126 \cdot 10^{-46}$	$0.16421466379 \cdot 10^{-46}$
z_2	$0.38316754884 \cdot 10^{-46}$	$-0.12772251628 \cdot 10^{-46}$	$0.25544503256 \cdot 10^{-46}$

Note that, if $x = 0.364921475084588 \cdot 10^{-47}$, then $d_6 = 6x$, $d_7 = \frac{11}{2}x$ and $d_8 = \frac{9}{2}x$. Write $c = (\varphi(c_1), \ldots, \varphi(c_6))$ and $d = (\varphi(d_1), \ldots, \varphi(d_8))$. Then we have showed that for $n = 3$, $c = (0, x, 0, x, x, x)$ and $d = (0, x, x, x, x, 6x, 11/2x, 9/2x)$.

Case $n = 4$

We obtain $c_1 = \ldots = c_6 = d_1 = \ldots = d_{11} = 0$ all zeroes, including the positive and negative subtotals.

Case $n = 5$

The output for $n = 5$ is

c_1	0	d_5	$0.36492147508 \cdot 10^{-47}$
c_2	$0.36492147508 \cdot 10^{-47}$	d_6	$-0.91230368771 \cdot 10^{-47}$
c_3	0	d_7	$-0.91230368771 \cdot 10^{-47}$
c_4	$0.36492147508 \cdot 10^{-47}$	d_8	$-0.18246073754 \cdot 10^{-47}$
c_5	$0.36492147508 \cdot 10^{-47}$	d_9	$-4.01413622593 \cdot 10^{-47}$
c_6	$0.36492147508 \cdot 10^{-47}$	d_{10}	$-0.72984295017 \cdot 10^{-47}$
d_1	0	d_{11}	0
d_2	$0.36492147508 \cdot 10^{-47}$	d_{12}	$-5.47382212627 \cdot 10^{-47}$
d_3	$0.36492147508 \cdot 10^{-47}$	d_{13}	$-5.47382212627 \cdot 10^{-47}$
d_4	$0.36492147508 \cdot 10^{-47}$	d_{14}	$0.36492147508 \cdot 10^{-47}$

That is, for $n = 5$ we obtain

$$\begin{aligned} c &= (0, x, 0, x, x, x) \quad \text{and} \\ d &= (0, x, x, x, x, -5/2x, -5/2x, -1/2x, -11x, -2x, 0, -15x, -15x, x). \end{aligned}$$

Cases $n \leq 11$

We always obtain results which are always rational multiples of x. We summarize the information in the following tables.

n	c	$\varphi(z_1)$
3	$(0, x, 0, x, x, x)$	0
4	$(0, 0, 0, 0, 0, 0)$	0
5	$(0, x, 0, x, x, x)$	0
6	$(0, x, 0, x, 0, 0)$	0
7	$(0, 0, 0, 0, 0, 0)$	0
8	$(0, 0, 0, 0, 0, 0)$	0
9	$(0, -1/2x, 0, -1/2x, x, x)$	0
10	$(0, x, 0, x, 0, 0)$	0
11	$(0, x, 0, x, 0, 0)$	0

n	d	$\varphi(z_2)$
3	$(0, x, x, x, x, 6x, \frac{11}{2}x, \frac{9}{2}x)$	$7x$
4	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	0
5	$(0, x, x, x, x, -\frac{5}{2}x, -\frac{5}{2}x, -12x, -11x, -2x, 0, -15x, -15x, x)$	$-\frac{13}{2}x$
6	$(0, x, \frac{3}{2}x, x, x, 2x, 2x, x, \frac{21}{2}x, 8x, x, 15x, \frac{27}{2}x, x, \frac{27}{2}x, \frac{27}{2}x, x)$	$\frac{21}{2}x$
7	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -4x, -4x, 0, -10x, -4x, 0, -24x, -20x, 0)$	$-10x$
8	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	0
9	$(0, -x, \frac{5}{2}x, \frac{5}{2}x, \frac{5}{2}x, -\frac{3}{2}x, \frac{5}{2}x, 4x, -\frac{75}{2}x, -\frac{53}{2}x, 4x, -\frac{219}{2}x, -\frac{195}{2}x, 4x, -207x,$ $-191x, 11x, -205x, -354x, 13x, -370x, \frac{695}{2}x, \frac{29}{2}x, -622x, -598x, 13x)$	$\frac{251}{2}x$
10	$(0, x, x, x, x, x, x, x, \frac{3}{2}x, x, x, 0, 0, x, \frac{9}{2}x, 0, x, \frac{23}{2}x,$ $\frac{23}{2}x, x, 3x, \frac{23}{2}x, x, 29x, 24x, x, 34x, 34x, x)$	$\frac{31}{2}x$
11	$(0, x, \frac{3}{2}x, x, x, \frac{21}{2}x, 8x, x, 15x, \frac{29}{2}x, x, 16x, 16x, 1, 48x, 40x, x, 54x,$ $108x, x, 231x, 108x, x, 126x, 245x, x, 234x, 224x, x, 243x, 243x, x)$	$-\frac{21}{2}x$

[INTERPRETATION]

Remark 16 *I guess there's something wrong with the signs, since we should get* $n\varphi(z_1) - 2\varphi(z_2) = 0$*, but instead we get* $\varphi(z_1)$ *always zero and* $\varphi(z_2) \neq 0$*.*

References

[1] R. Bott and L.W. Tu: *Differential Forms in Algebraic Topology*; Grad. Texts in Math. #82 (1982) Springer Verlag.

[2] K. S. Brown, *Cohomology of groups*, Springer-Verlag, 1982.

[3] J.I. Burgos Gil: *The Regulators of Beilinson and Borel*; Centre de Récherches Montréal Monographs #15, Amer. Math. Soc. (2002).

[4] Z. Choo, W. Mannan, R. J. Sánchez-García and V. P. Snaith, *Computing Borel's Regulator II*, in preparation.

[5] N. Hamida, Description explicite du regulateur de Borel, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 3, 169172.

[6] Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge University Press, 1985.

[7] K.P. Knudson: *Homology of Linear Groups*; Prospects in Math. #193 Birkhäuser (2001).

[8] M. Kolster, T. Nguyen Quang Do and V. Fleckinger: Twisted S-units, *p*-adic class number formulas and the Lichtenbaum conjectures; Duke J. Math. 84 (1996) 679-717 (erratum Duke J. Math. 90 (1997) 641-643 plus further corrigenda in [**?**] and [**?**]).

[9] S. Lichtenbaum: Values of zeta functions, étale cohomology and algebraic K-theory; *Algebraic K-theory II* Lecture Notes in Maths. #342, Springer Verlag (1973) 489-501.

[10] J. Martinet: Character theory and Artin L-functions; London Math. Soc. 1975 Durham Symposium *Algebraic Number Fields* (ed. A. Fröhlich) Academic Press (1977) 1-87.

[11] The Mathworks Inc, Matlab, Version R2008b, 2008. (http://www.mathworks.com/products/matlab/)

[12] I. Rada: The Lichtenbaum conjecture at the prime 2; PhD thesis McMaster University (2002).

[13] Equivariant motivic phenomena; NATO Sci. Series II #131, *Axiomatic, Enriched and Motivic Homotopy* (ed. J.P.C. Greenlees) Kluwer (2004) 335-383.

[14] Relative K_0, annihilators, Fitting ideals and the Stickelberger phenomena; Proc. London Math. Soc. (3) 90 (2005) 545-590.

[15] Stark's conjecture and new Stickelberger phenomena; Can. J. Math. 58 (2) 419-448 (2006).

[16] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1989.

[17] J.T. Tate: *Les Conjectures de Stark sur les Fonctions L d'Artin en* $s = 0$; Progress in Mathematics # 47 (1984) Birkhauser.