DERIVED LANGLANDS II: SHEFFIELD LECTURES

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Contents

1. Lecture One: G finite or $G/Z(G)$ finite.

Arbitrary k algebraically closed field $\underline{\phi} : Z(G) \longrightarrow k^*$
$\hat{G} = \mathrm{Hom}(G, k^*)$ continuous homomorphisms

<u>hyperHecke algebra $\mathcal{H}_{cmc}(G)$</u>
$\mathcal{H}$ k-vector space on triples $[(K,\psi), g, (H,\phi)]$ such that $Z(G) \subseteq H, K$, ϕ, ψ restrict to give $\underline{\phi}$ on $Z(G)$

$$(K,\psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

which means that $K \leq g^{-1}Hg$ and that $\psi(k) = \phi(h)$ where $k = g^{-1}hg$ for $h \in H, k \in K$.

product

$$[(H,\phi), g_1, (J,\mu)] \cdot [(K,\psi), g_2, (H,\phi)] = [(K,\psi), g_1g_2, (J,\mu)]$$

and zero otherwise.

$\mathcal{H}_{cmc}(G)$ is algebra given by $\mathcal{H}$ modulo relations

$$[(K,\psi), gk, (H,\phi)] = \psi(k^{-1})[(K,\psi), g, (H,\phi)]$$

and

$$[(K,\psi), hg, (H,\phi)] = \phi(h^{-1})[(K,\psi), g, (H,\phi)].$$

<u>The usual Hecke algebra $\mathcal{H}_G$</u> is the subalgebra of $\mathcal{H}_{cmc}(G)$ where all the ϕ's and ψ's are trivial.

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Induced representations and Comparison of inductions

In the case of finite groups this Appendix compares the "tensor product of modules" model of an induced representation with the "function space" model[1].

Suppose that $H \subseteq G$ are finite groups and that W is a vector space over an algebraically closed field k together with a left H-action given by a homomorphism

$$\phi : H \longrightarrow \mathrm{Aut}_k(W).$$

In this case the functional model for the induced representation is given by the k-vector space of functions $X_{(H,\phi)}$ consisting of functions of the form $f : G \longrightarrow W$ such that $f(hg) = \phi(h)(f(g))$. The left G-action on these functions is given by $(g \cdot f)(x) = f(xg)$.

For $w \in W$ we have a function f_w, supported in H and satisfying $(h \cdot f_w) = f_{\phi(h)(w)}$ for $h \in H$ so that $f_w(1) = w$. We have a left $k[H]$-module map

$$f : W \longrightarrow X_{(H,\phi)}$$

defined by $w \mapsto f_w$.

The map f induces a left $k[G]$-module map, which is an isomorphism,

$$\hat{f} : Ind_H^G(W) = k[G] \otimes_{k[H]} W \stackrel{\cong}{\longrightarrow} X_{(H,\phi)}$$

given by $\hat{f}(g \otimes_{k[H]} w) = g \cdot f_w$.

Henceforth, in this Appendix, I shall consider only the case when $\dim_k(W) = 1$. In this case $W = k_\phi$ will denote the H-representation given by the action $h \cdot v = \phi(h)v$ for $h \in H, v \in k$.

As in Definition §2, write $[(K,\psi), g, (H,\phi)]$ for any triple consisting of $g \in G$, characters ϕ, ψ on subgroups $H, K \leq G$, respectively such that

$$(K,\psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

which means that $K \leq g^{-1}Hg$ and that $\psi(k) = \phi(h)$ where $k = g^{-1}hg$ for $h \in H, k \in K$.

We have a well-defined left $k[G]$-module homomorphism

$$[(K,\psi), g, (H,\phi)] : k[G] \otimes_{k[K]} k_\psi \longrightarrow k[G] \otimes_{k[H]} k_\phi$$

given by the formula $[(K,\psi), g, (H,\phi)](g' \otimes_{k[K]} v) = g'g^{-1} \otimes_{k[H]} v$.

[1] In ([19] Chapter Two, Definition 1.1) my unreliable typography resulted in a superfluous suffix "-1" which gives the right action. This this essay I have been more careful to give the correct formula for the left action, since left actions are my usual preference.

In order to define a left $k[G]$-homomorphism

$$[(K,\psi),g,(H,\phi)] : X_{(K,\psi)} \longrightarrow X_{(H,\phi)}$$

satisfying the relation

$$\hat{f}\cdot[(K,\psi),g,(H,\phi)] = [(K,\psi),g,(H,\phi)]\cdot\hat{f} : k[G]\otimes_{k[K]} k_{\psi} \longrightarrow X_{(H,\phi)}$$

we set

$$[(K,\psi),g,(H,\phi)](g_1\cdot f_v) = (g_1 g^{-1})\cdot f_v.$$

It is easy to see that transporting the map $[(K,\psi),g,(H,\phi)]$ from the tensor product model of the induced representation to the function space model gives the left $k[G]$-homomorphism whose well-definedness we have just verified.

Among the left $k[G]$-maps

$$k[G]\otimes_{k[K]} k_{\psi} \longrightarrow k[G]\otimes_{k[H]} k_{\phi}$$

we have the relations, $h\in H, k\in K$

$$[(K,\psi),gk,(H,\phi)] = [(K,\psi),g,(H,\phi)]\cdot(1\otimes_{k[K]}\psi(k^{-1}))$$

and

$$[(K,\psi),hg,(H,\phi)] = (1\otimes_{k[H]}\phi(h^{-1}))\cdot[(K,\psi),g,(H,\phi)].$$

Theorem Let M be the k-vector space which is given by the direct sum of copies of the $X_{(H,\phi)}$'s. Then M is a left module over the hyperHecke algebra $\mathcal{H}_{cmc}(G)$.

We shall be interested in the case when M contains at let one copy of $X_{(H,\phi)}$ for each (H,ϕ).

Roughly: ${}_{k[G]}$mon, the monomial category of G has objects given by the these M's and morphisms given by the hyperHecke algebra

The Double Coset Formula ([18] Theorem 1.2.40) is a functorial isomorphism describing the restriction of an induced representation. It is a consequence of the J-orbit structure of the left action of a subgroup $J\subseteq G$ on G/H. This is a left $k[J]$-isomorphism of the form

$$\mathrm{Res}^G_J\mathrm{Ind}^G_H(k_{\phi}) \stackrel{\alpha}{\longrightarrow} \oplus_{z\in J\backslash G/H}\ \mathrm{Ind}^J_{J\bigcap zHz^{-1}}((z^{-1})^*(k_{\phi}))$$

given by $\alpha(g\otimes_H v) = j\otimes_{J\bigcap zHz^{-1}}\phi(h)(v)$ for $g = jzh, j\in J, h\in H$. The inverse of α is given by $\alpha^{-1}(j\otimes_{J\bigcap zHz^{-1}} v) = jz\otimes_H (v)$.

Remark: (i) For finite groups we can forget about the conditions on (H,ϕ) relating to the centre and $\underline{\phi}$. This is only needed when $Z(G)$ is infinite.

(ii) The objective is to define what we mean by an resolution of a left $k[G]$-representation by an exact complex in ${}_{k[G]}$**mon**.

(iii) A natural construct as in (ii) would be of interest when G is finite and k has positive characteristic, even though the resolution would have infinite length in that case, but be of finite type. For a finite group and k of characteristic zero the resolution will be finite.

(iv) The irreducible (admissible) modular representations of a p-adic GL_n were classified in [11]. As we shall see, such representations also have monomial resolutions (presumably of infinite length in general) whose behaviour would be interesting.

2. Lecture Two: The bar-monomial resolution: I. finite modulo the centre case

The poset of $\mathcal{M}_{\underline{\phi}}(G)$ of pairs (H, ϕ) admits a left G-action by conjugation for which the G-orbit of (H, ϕ) will be denoted by $(H, \phi)^G$.

Definition

A finite $(G, \underline{\phi})$-lineable left $k[G]$-module M[2] is a left $k[G]$-module together with a fixed finite direct sum decomposition

$$M = M_1 \oplus \cdots \oplus M_m$$

where each of the M_i is a free k-module of rank one on which $Z(G)$ acts via $\underline{\phi}$ and the G-action permutes the M_i. The M_i's are called the lines of M. For $1 \leq i \leq m$ let H_i denote the subgroup of G with stabilises the line M_i. Then there exists a unique $\phi_i \in \hat{H}_{i\underline{\phi}}$ such that $h \cdot v = \phi_i(h)v$ for all $v \in M_i, h \in H_i$. The pair $(H_i, \phi_i) \in \mathcal{M}_{\underline{\phi}}(G)$ is called the stabilising pair of M_i.

The k-submodule of M given by

$$M^{((H,\phi))} = \oplus_{1 \leq i \leq m, \ (H,\phi) \leq (H_i, \phi_i)} \ M_i$$

is called the (H, ϕ)-fixed points of M.

A morphism between $(G, \underline{\phi})$-lineable modules from M to $N = N_1 \oplus \cdots \oplus N_n$ is defined to be a $k[G]$-module homomorphism $f : M \longrightarrow N$ such that

$$f(M^{((H,\phi))}) \subseteq N^{((H,\phi))}$$

for all $(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)$.

The (left) finite $(G, \underline{\phi})$-lineable modules and their morphisms define an additive category denoted by ${}_{k[G],\underline{\phi}}\mathbf{mon}$.

By definition each $(G, \underline{\phi})$-lineable module is a k-free $k[G]$-module so there is a forgetful functor

$$\mathcal{V} : \ {}_{k[G],\underline{\phi}}\mathbf{mon} \longrightarrow \ {}_{k[G],\underline{\phi}}\mathbf{mod}.$$

[2] Here I have taken my own terminological advice given in the footnote to ([19] Chapter One, Definition 1.2).

The usual natural operations and constructions for modules have analogues in ${}_{k[G],\underline{\phi}}\mathbf{mon}$.

The M_i's are isomorphic to $X_{(H,\phi)}$'s and the morphisms are given by the equivalence classes of the triples $[(K,\psi),g,(H,\phi)]$ in the hyperHecke algebra. In fact they are the ${}_{k[G],\underline{\phi}}\mathbf{mon}$-indecomposables.

Proposition
(i) The set of $(G,\underline{\phi})$-lineeable modules given by

$$\{X_{(H,\phi)} = \underline{\mathrm{Ind}}_H^G(k_\phi) \mid (H,\phi) \in G\backslash \mathcal{M}_{\underline{\phi}}(G)\}$$

is a full set of pairwise non-isomorphic representatives for the isomorphism classes of indecomposable objects in ${}_{k[G],\underline{\phi}}\mathbf{mon}$. Moreover any object in ${}_{k[G],\underline{\phi}}\mathbf{mon}$ is canonically isomorphic to the direct sum of objects in this set.

(ii) Let $[(K,\psi),g,(H,\phi)]$ be one of the basic generators of the hyperHecke algebra $\mathcal{H}_{cmc})(G)$ of §2 then we have a morphism

$$[(K,\psi),g,(H,\phi)] \in \mathrm{Hom}_{{}_{k[G],\underline{\phi}}\mathbf{mon}}(\underline{\mathrm{Ind}}_K^G(k_\psi), \underline{\mathrm{Ind}}_H^G(k_\phi))$$

defined by the same formula as in the case of induced modules (see, Appendix: Comparison of Inductions). In addition the composition of morphisms in ${}_{k[G],\underline{\phi}}\mathbf{mon}$ coincides with the product in the hyperHecke algebra.

(iii) Let $(K,\psi) \in \mathcal{M}_{\underline{\phi}}(G)$ and let N be an object of ${}_{k[G],\underline{\phi}}\mathbf{mon}$. Then there is a k-linear isomorphism

$$\mathrm{Hom}_{{}_{k[G],\underline{\phi}}\mathbf{mon}}(\underline{\mathrm{Ind}}_K^G(k_\psi), N) \stackrel{\cong}{\longrightarrow} N^{((K,\psi))}$$

given by $f \mapsto f(1 \otimes_K 1)$. The inverse isomorphism is given by

$$n \mapsto ((g \otimes_K v \mapsto vg \cdot n)).$$

Lemma Projectivity in ${}_{k[G],\underline{\phi}}\mathbf{mon}$
Consider the diagram

$$M \stackrel{h}{\longrightarrow} N \stackrel{f}{\longleftarrow} P$$

in which $M, P \in {}_{k[G],\underline{\phi}}$ **mon** and $N \in {}_{k[G],\underline{\phi}}$ **mod** with h, f being morphisms in ${}_{k[G],\underline{\phi}}\mathbf{mod}$. Assume, for all $(H,\phi) \in \mathcal{M}_{\underline{\phi}}(G)$, that

$$f(P^{((H,\phi))}) \subseteq h(M^{((H,\phi))}).$$

Then there exists $j \in \mathrm{Hom}_{{}_{k[G],\underline{\phi}}\mathbf{mon}}(P, M)$ such that $h \cdot j = f$.

In particular we include the situation where $N' \in_{k[G],\underline{\phi}}$ **mon** with h, f being morphisms to N' in ${}_{k[G],\underline{\phi}}\mathbf{mon}$ and the diagram above being the result of applying the forgetful functor $\mathcal{V}$ with $N = \mathcal{V}(N')$.

For $V \in_{k[G],\underline{\phi}}$ **mod** and $(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)$ define the (H, ϕ)-fixed points of V by

$$V^{(H,\phi)} = \{v \in V \mid h \cdot v = \phi(h)v \text{ for all } h \in H\}.$$

Definition ([19] Chapter One §2)

Let $V \in_{k[G],\underline{\phi}}$ **mod**. A ${}_{k[G],\underline{\phi}}\mathbf{mon}$-resolution of V is a chain complex

$$M_* : \qquad \ldots \xrightarrow{\partial_{i+1}} M_{i+1} \xrightarrow{\partial_i} M_i \xrightarrow{\partial_{i-1}} \ldots \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0$$

with $M_i \in_{k[G],\underline{\phi}}$ **mon** and $\partial_i \in \mathrm{Hom}_{{}_{k[G],\underline{\phi}}\mathbf{mon}}(M_{i+1}, M_i)$ for all $i \geq 0$ together with $\epsilon \in \mathrm{Hom}_{{}_{k[G],\underline{\phi}}\mathbf{mod}}(\mathcal{V}(M_0), V)$ such that

$$\ldots \xrightarrow{\partial_i} M_i^{((H,\phi))} \xrightarrow{\partial_{i-1}} \ldots \xrightarrow{\partial_1} M_1^{((H,\phi))} \xrightarrow{\partial_0} M_0^{((H,\phi))} \xrightarrow{\epsilon} V^{(H,\phi)} \longrightarrow 0$$

is an exact sequence of k-modules for each $(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)$. In particular, when $(H, \phi) = (Z(G), \underline{\phi})$ we see that

$$\ldots \xrightarrow{\partial_i} M_i \xrightarrow{\partial_{i-1}} \ldots \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_0} M_0 \xrightarrow{\epsilon} V \longrightarrow 0$$

is an exact sequence in ${}_{k[G],\underline{\phi}}\mathbf{mod}$.

Proposition

Let $V \in_{k[G],\underline{\phi}}$ **mod** and let

$$\ldots \longrightarrow M_n \xrightarrow{\partial_{n-1}} M_{n-1} \xrightarrow{\partial_{n-2}} \ldots \xrightarrow{\partial_0} M_0 \xrightarrow{\epsilon} V \longrightarrow 0$$

be a ${}_{k[G],\underline{\phi}}\mathbf{mon}$-resolution of V. Suppose that

$$\ldots \longrightarrow C_n \xrightarrow{\partial'_{n-1}} C_{n-1} \xrightarrow{\partial'_{n-2}} \ldots \xrightarrow{\partial'_0} C_0 \xrightarrow{\epsilon'} V \longrightarrow 0$$

a chain complex where each ∂'_i and C_i belong to ${}_{k[G],\underline{\phi}}\mathbf{mon}$ and ϵ' is a ${}_{k[G],\underline{\phi}}\mathbf{mod}$ homomorphism such that $\epsilon'(C_0^{((H,\phi))}) \subseteq V^{(H,\phi)}$ for each $(H, \phi) \in \mathcal{M}_{\underline{\phi}}(G)$.

Then there exists a chain map of ${}_{k[G],\underline{\phi}}\mathbf{mon}$-morphisms $\{f_i : C_i \longrightarrow M_i, i \geq 0\}$ such that

$$\epsilon \cdot f_0 = \epsilon', \; f_{i-1} \cdot \partial'_i = \partial_i \cdot f_i \text{ for all } i \geq 1.$$

In addition, if $\{f'_i : C_i \longrightarrow M_i, i \geq 0\}$ is another chain map of ${}_{k[G],\underline{\phi}}\mathbf{mon}$-morphisms such that $\epsilon \cdot f_0 = \epsilon \cdot f'_0$ then there exists a ${}_{k[G],\underline{\phi}}\mathbf{mon}$-chain homotopy $\{s_i : C_i \longrightarrow M_{i+1}, \text{ for all } i \geq 0\}$ such that $\partial_i \cdot s_i + s_{i-1} \cdot \partial'_i = f_i - f'_i$ for all $i \geq 1$ and $f_0 - f'_0 = \partial_0 \cdot s_0$.

Remark

(i) Needless to say, the proposition has an analogue to the effect that every ${}_{k[G],\underline{\phi}}\mathbf{mod}$-homomorphism $V \longrightarrow V'$ extends to a ${}_{k[G],\underline{\phi}}\mathbf{mon}$-morphism between the monomial resolutions of V and V', if they exist, and the extension is unique up to ${}_{k[G],\underline{\phi}}\mathbf{mon}$-chain homotopy.

(ii) The category ${}_{k[G],\underline{\phi}}\mathbf{mon}$ is additive but not abelian. Homological algebra (e.g. a projective resolution) is more conveniently accomplished in an abelian category. To overcome this difficulty we shall embed ${}_{k[G],\underline{\phi}}\mathbf{mon}$ into more convenient abelian categories. This is reminiscent of the Freyd-Mitchell Theorem which embeds every abelian category into a category of modules.

A complex of functors

Let $M \in_{k[G],\underline{\phi}} \mathbf{mon}, V \in_{k[G],\underline{\phi}} \mathbf{mod}$ and let $\mathcal{A}_M = \mathrm{Hom}_{{}_{k[G],\underline{\phi}}\mathbf{mon}}(M, M)$, the ring of endomorphisms on M under composition. For $i \geq 0$ define $\tilde{M}_{M,i} \in {}_k\mathbf{mod}$ by (i copies of $\mathcal{A}_M$)

$$\tilde{M}_{M,i} = \mathrm{Hom}_{{}_{k[G],\underline{\phi}}\mathbf{mod}}(\mathcal{V}(M), V) \otimes_k \mathcal{A}_M \otimes_k \ldots \otimes_k \mathcal{A}_M$$

and set

$$\underline{M}_{M,i} = \tilde{M}_{M,i} \otimes_k \mathrm{Hom}_{{}_{k[G],\underline{\phi}}\mathbf{mon}}(-, M).$$

Hence $\underline{M}_{M,i} \in funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon},{}_k \mathbf{mod})$ and in fact the values of this functor are not merely objects in ${}_k\mathbf{mod}$ because they have a natural right $\mathcal{A}_M$-module structure, defined as in §**??**.

If $i \geq 1$ we defined natural transformations $d_{M,0}, d_{M,1}, \ldots, d_{M,i}$ in the following way. Define

$$d_{M,0} : \underline{M}_{M,i} \longrightarrow \underline{M}_{M,i-1}$$

by

$$d_{M,0}(f \otimes \alpha_1 \otimes \ldots \otimes \alpha_i \otimes u) = f(- \cdot \alpha_1) \otimes \alpha_2 \ldots \otimes \alpha_i \otimes u.$$

The map $f(- \cdot \alpha_1) : \mathcal{V}(M) \longrightarrow V$ is a ${}_{k[G],\underline{\phi}}\mathbf{mod}$- homomorphism since α_i acts on the right of M.

For $1 \leq j \leq i-1$ we define

$$d_{M,j} : \underline{M}_{M,i} \longrightarrow \underline{M}_{M,i-1}$$

by

$$d_{M,j}(f \otimes \alpha_1 \otimes \ldots \otimes \alpha_i \otimes u) = f \otimes \alpha_1 \ldots \otimes \alpha_j\alpha_{j+1} \otimes \ldots \otimes \alpha_i \otimes u.$$

Finally

$$d_{M,i} : \underline{M}_{M,i} \longrightarrow \underline{M}_{M,i-1}$$

is given by

$$d_i(M)(f \otimes \alpha_1 \otimes \ldots \otimes \alpha_i \otimes u) = f \otimes \alpha_1 \otimes \ldots \otimes \alpha_{i-1} \otimes \alpha_i \cdot u.$$

Since u is a ${}_{k[G],\underline{\phi}}\mathbf{mon}$-morphism so is $\alpha_i \cdot u$ because

$$(\alpha_i \cdot u)(\alpha m) = \alpha_i(u(\alpha m)) = \alpha_i(\alpha u(m)) = \alpha\alpha_i(u(m)) = \alpha(\alpha_i \cdot u)(m)$$

since α_i is a ${}_{k[G],\underline{\phi}}\mathbf{mon}$ endomorphism of M.

Next we define a natural transformation

$$\epsilon_M : \underline{M}_{M,0} \longrightarrow \mathcal{I}(V) = \mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mod}}(\mathcal{V}(-), V)$$

by sending $f \otimes u \in \underline{M}_{M,0}$ to $f \cdot \mathcal{V}(u) \in \mathcal{I}(V)$.

Finally we define

$$d_M = \sum_{j=0}^{i} (-1)^j d_{M,j} : \underline{M}_{M,i} \longrightarrow \underline{M}_{M,i-1}.$$

Theorem (Relation with the bar resolution)

The sequence

$$\ldots \xrightarrow{d_M} \underline{M}_{M,i}(M) \xrightarrow{d_M} \underline{M}_{M,i-1}(M) \ldots \xrightarrow{d_M} \underline{M}_{M,0}(M) \xrightarrow{\epsilon_M} \mathcal{I}(V)(M) \longrightarrow 0$$

is the right $\mathcal{A}_M$-module bar resolution of $\mathcal{I}(V)(M)$.

Proposition (The abelian category)

Let $\mathcal{I}$ denote the functor of introduced above and define a functor

$$\mathcal{J} :_{k[G],\underline{\phi}} \mathbf{mon} \longrightarrow funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon},_k \mathbf{mod})$$

by $\mathcal{J}(M) = \mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mon}}(-, M)$.

Then the category $funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon},_k \mathbf{mod})$ is abelian. Furthermore both $\mathcal{I}$ and $\mathcal{J}$ are full embeddings (i.e. bijective on morphisms and hence injective on isomorphism classes of objects).

Proposition (Projectivity)

For $M \in_{k[G],\underline{\phi}} \mathbf{mon}$ the functor $\mathcal{J}(M)$ in $funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon},_k \mathbf{mod})$ is projective.

Definition $\mathcal{K}_{M,V}$

Let $M \in_{k[G],\underline{\phi}} \mathbf{mon}, V \in_{k[G],\underline{\phi}} \mathbf{mod}$. Define a k-linear isomorphism $\mathcal{K}_{M,V}$ of the form

$$\mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mod}}(\mathcal{V}(M), V) \xrightarrow{\mathcal{K}_{M,V}} \mathrm{Hom}_{funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon},_k\mathbf{mod})}(\mathcal{J}(M), \mathcal{I}(V))$$

by sending $f : \mathcal{V}(M) \longrightarrow V$ to the natural transformation

$$\mathcal{K}_{M,V}(N) : \mathcal{J}(M)(N) \longrightarrow \mathcal{I}(V)(N)$$

given by $h \mapsto f \cdot \mathcal{V}(h)$ for all $N \in_{k[G],\underline{\phi}} \mathbf{mon}$

$$\mathcal{J}(M)(N) = \mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mon}}(N, M) \longrightarrow \mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mod}}(\mathcal{V}(N), V) = \mathcal{I}(V)(N).$$

The inverse isomorphism is given by $\mathcal{K}_{M,V}^{-1}(\phi) = \phi(M)(1_M)$ where 1_M denotes the identity morphism on M.

In fact $\mathcal{K}$ is a functorial equivalence of the form

$$\mathcal{K} : \mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mod}}(\mathcal{V}(-), -) \xrightarrow{\cong} \mathrm{Hom}_{funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon},_k\mathbf{mod})}(\mathcal{J}(-), \mathcal{I}(-))$$

Recognising a monomial resolution

Theorem
Let

$$\ldots \stackrel{\partial_i}{\longrightarrow} M_i \stackrel{\partial_{i-1}}{\longrightarrow} \ldots \stackrel{\partial_1}{\longrightarrow} M_1 \stackrel{\partial_0}{\longrightarrow} M_0 \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

be a chain complex with $M_i \in_{k[G],\underline{\phi}}$ **mon** for $i \geq 0$, $V \in_{k[G],\underline{\phi}}$ **mod**, $\partial_i \in \mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mon}}(M_{i+1}, M_i)$ and $\epsilon \in \mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mod}}(\mathcal{V}(M_0), V)$. Then the following are equivalent:

(i) $M_* \longrightarrow V$ is a ${}_{k[G],\underline{\phi}}$**mon**-resolution of V.

(ii) The sequence

$$\ldots \stackrel{\mathcal{J}(\partial_i)}{\longrightarrow} \mathcal{J}(M_i) \stackrel{\mathcal{J}(\partial_{i-1})}{\longrightarrow} \ldots \stackrel{\mathcal{J}(\partial_1)}{\longrightarrow} \mathcal{J}(M_1) \stackrel{\mathcal{J}(\partial_0)}{\longrightarrow} \mathcal{J}(M_0) \stackrel{\mathcal{K}_{M_0,V}(\epsilon)}{\longrightarrow} \mathcal{I}(V) \longrightarrow 0$$

is exact in $funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon}, {}_k\mathbf{mod})$.

The functor Φ_M

Let $M \in_{k[G],\underline{\phi}}$ **mon** and let $\mathcal{A}_M = \mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mon}}(M, M)$, the ring of endomorphisms on M under composition. In the present context$\mathcal{A}_M$ is a finitely generated k-algebra.

I shall show that there is an equivalence of categories between $funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon}, {}_k\mathbf{mod})$ and the category of right modules $\mathbf{mod}_{\mathcal{A}_M}$ for a suitable choice of M.

We have a functor

$$\Phi_M : funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon}, {}_k\mathbf{mod}) \longrightarrow \mathbf{mod}_{\mathcal{A}_M}$$

given by $\Phi(\mathcal{F}) = \mathcal{F}(M)$. Right multiplication by $z \in \mathcal{A}_M$ on $v \in \mathcal{F}(M)$ is given by

$$v\#z = \mathcal{F}(z)(v)$$

where $\mathcal{F}(z) : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$ is the left k-module morphism obtained by applying $\mathcal{F}$ to the endomorphism z. This is a right-$\mathcal{A}_M$ action since

$$v\#(zz_1) = \mathcal{F}(zz_1)(v) = (\mathcal{F}(z_1) \cdot \mathcal{F}(z))(v) = \mathcal{F}(z_1)(\mathcal{F}(z)(v)) = (v\#z)\#z_1.$$

In the other direction define a functor

$$\Psi_M : \mathbf{mod}_{\mathcal{A}_M} \longrightarrow funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon}, {}_k\mathbf{mod}),$$

for $P \in \mathbf{mod}_{\mathcal{A}_M}$, by

$$\Psi_M(P) = \mathrm{Hom}_{\mathcal{A}_M}(\mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mon}}(M, -), P).$$

Here, for $N \in_{k[G],\underline{\phi}}$ **mon**, $\mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mon}}(M, -)$ is a right $\mathcal{A}_M$-module via pre-composition by endomorphisms of M. For a homomorphism of $\mathcal{A}_M$-modules $f : P \longrightarrow Q$ the map $\Psi_M(f)$ is given by composition with f.

Next we consider the composite functor

$$\Phi_M \cdot \Psi_M : \mathbf{mod}_{\mathcal{A}_M} \longrightarrow \mathbf{mod}_{\mathcal{A}_M}.$$

This is given by $P \mapsto \mathrm{Hom}_{\mathcal{A}_M}(\mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mon}}(M,M),P) = \mathrm{Hom}_{\mathcal{A}_M}(\mathcal{A}_M, P)$ so that there is an obvious natural transformation $\eta : 1 \stackrel{\cong}{\longrightarrow} \Phi_M \cdot \Psi_M$ such that $\eta(P)$ is an isomorphism for each module P.

Now consider the composite functor

$$\Psi_M \cdot \Phi_M : funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon}, {}_k\,\mathbf{mod}) \longrightarrow funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon}, {}_k\,\mathbf{mod}).$$

For a functor $\mathcal{F}$ we shall define a natural transformation

$$\epsilon_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathrm{Hom}_{\mathcal{A}_M}(\mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mon}}(M,-),\mathcal{F}(M)) = \Psi_M \cdot \Phi_M(\mathcal{F}).$$

For $N \in_{k[G],\underline{\phi}}$ **mon** we define

$$\epsilon_{\mathcal{F}}(N) : \mathcal{F}(N) \longrightarrow \mathrm{Hom}_{\mathcal{A}_M}(\mathrm{Hom}_{k[G],\underline{\phi}\mathbf{mon}}(M,N),\mathcal{F}(M))$$

by the formula $v \mapsto (f \mapsto \mathcal{F}(f)(v))$.

Theorem (Functors to modules and back)

Let $S \in_{k[G],\underline{\phi}}$ **mon** be the finite $(G,\underline{\phi})$-lineable k-module given by

$$S = \oplus_{(H,\phi)\in\mathcal{M}_{\underline{\phi}}(G)} \ \underline{\mathrm{Ind}}_H^G(k_\phi).$$

Then

$$\Phi_S : funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon}, {}_k\,\mathbf{mod}) \longrightarrow \mathbf{mod}_{\mathcal{A}_S}$$

and

$$\Psi_S : \mathbf{mod}_{\mathcal{A}_S} \longrightarrow funct_k^o({}_{k[G],\underline{\phi}}\mathbf{mon}, {}_k\,\mathbf{mod})$$

are inverse equivalences of categories. In fact, the natural transformations η and ϵ are isomorphisms of functors when $M = S$.

Remark

The theorem remains true when S is replaced by any M which is the direct sum of $\underline{\mathrm{Ind}}_H^G(k_\phi)$'s containing at least one pair (H,ϕ) from each G-orbit of $\mathcal{M}_{\underline{\phi}}(G)$. That is, for any $(G,\underline{\phi})$-lineable k-module containing

$$\oplus_{(H,\phi)\in G\backslash\mathcal{M}_{\underline{\phi}}(G)} \ \underline{\mathrm{Ind}}_H^G(k_\phi)$$

as a summand. This remark is established by Morita theory.

Let V be a finite rank left $k[G]$-module. Let $M \in {}_{k[G],\underline{\phi}}\mathbf{mon}$ and $W \in {}_k\mathbf{lat}$. Define another object $W \otimes_k M \in {}_{k[G],\underline{\phi}}\mathbf{mon}$ by letting G act only on the M-factor, $g(w \otimes m) = w \otimes gm$, and defining the Lines of $W \otimes_k M$ to consist of the one-dimensional subspaces $\langle w \otimes L\rangle$ where $w \in W$, runs through a k-basis of W, and L is a Line of M.

Theorem (Existence of the bar-monomial resolution)

Let k be a field. The chain complex, which we met earlier in connection with the "chain complex of functors" paragraph,

$$\ldots \stackrel{d}{\longrightarrow} \tilde{M}_{S,i} \otimes_k S \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} \tilde{M}_{S,1} \otimes_k S \stackrel{d}{\longrightarrow} \tilde{M}_{S,0} \otimes_k S \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is a ${}_{k[G],\underline{\phi}}\mathbf{mon}$-resolution of V.

Remark

(i) Since the theorem "from functors to modules and back" remains true when S is replaced by any $M \in {}_{k[G],\underline{\phi}}\mathbf{mon}$ which contains S as a summand one may replace S by such an M in the above theorem to maintain another ${}_{k[G],\underline{\phi}}\mathbf{mon}$-resolution of V.

(ii) The bar-monomial resolution of bar-monomial resolution possesses a number of the usual naturality properties, as an object in the derived category of ${}_{k[G],\underline{\phi}}\mathbf{mon}$.

(iii) As mentioned earlier for finite groups we may forget about the central character $\underline{\phi}$.

3. Lecture Three: $GL_n\mathbb{F}_q$ analogues of the Langlands Programme

PSH-algebras over the integers

3.1. A PSH-algebra is a connected, positive self-adjoint Hopf algebra over $\mathbb{Z}$. The notion was introduced in [20]. Let $R = \oplus_{n \geq 0} R_n$ be an augmented graded ring over $\mathbb{Z}$ with multiplication

$$m : R \otimes R \longrightarrow R.$$

Suppose also that R is connected, which means that there is an augmentation ring homomorphism of the form

$$\epsilon : \mathbb{Z} \stackrel{\cong}{\longrightarrow} R_0 \subset R.$$

These maps satisfy associativity and unit conditions.

Associativity: $m(m \otimes 1) = m(1 \otimes m) : R \otimes R \otimes R \longrightarrow R.$

Unit: $m(1 \otimes \epsilon) = 1 = m(\epsilon \otimes 1); R \otimes \mathbb{Z} \cong R \cong \mathbb{Z} \otimes R \longrightarrow R \otimes R \longrightarrow R.$

R is a Hopf algebra if, in addition, there exist comultiplication and counit homomorphisms $m^* : R \longrightarrow R \otimes R$ and $\epsilon^* : R \longrightarrow \mathbb{Z}$ such that

Hopf m^* is a ring homomorphism with respect to the product $(x \otimes y)(x' \otimes y') = xx' \otimes yy'$ on $R \otimes R$ and ϵ^* is a ring homomorphism restricting to an isomorphism on R_0. The homomorphism m is a coalgebra homomorphism with respect to m^*.

The m^* and ϵ^* also satisfy

Coassociativity: $(m^* \otimes 1)m^* = (1 \otimes m^*)m^* : R \longrightarrow R \otimes R \otimes R \longrightarrow R \otimes R \otimes R$

Counit: $m(1 \otimes \epsilon) = 1 = m(\epsilon \otimes 1); R \otimes \mathbb{Z} \cong R \cong \mathbb{Z} \otimes R \longrightarrow R \otimes R \longrightarrow R.$

R is a cocomutative if

Cocommutative: $m^* = T \cdot m^* : R \longrightarrow R \otimes R$ where $T(x \otimes y) = y \otimes x$ on $R \otimes R$.

Suppose now that each R_n (and hence R by direct-sum of bases) is a free abelian group with a distinguished $\mathbb{Z}$-basis denoted by $\Omega(R_n)$. Hence $\Omega(R)$ is the disjoint union of the $\Omega(R_n)$'s. With respect to the choice of basis the positive elements R^+ of R are defined by

$$R^+ = \{r \in R \mid r = \sum m_\omega \omega,\ m_\omega \geq 0, \omega \in \Omega(R)\}.$$

Motivated by the representation theoretic examples the elements of $\Omega(R)$ are called the irreducible elements of R and if $r = \sum m_\omega \omega \in R^+$ the elements $\omega \in \Omega(R)$ with $m_\omega > 0$ are called the irreducible constituents of r.

Using the tensor products of basis elements as a basis for $R \otimes R$ we can similarly define $(R \otimes R)^+$ and irreducible constituents etc.

Positivity:

R is a positive Hopf algebra if $m((R \otimes R)^+) \subset R^+, m^*(R^+) \subset (R \otimes R)^+, \epsilon(\mathbb{Z}^+) \subset R^+, \epsilon^*(R^+) \subset \mathbb{Z}^+$.

Define inner products $\langle -, - \rangle$ on R, $R \otimes R$ and $\mathbb{Z}$ by requiring the chosen basis ($\Omega(\mathbb{Z}) = \{1\}$) to be an orthonormal basis.

A positive Hopf $\mathbb{Z}$-algebra is self-adjoint if

Self-adjoint: m and m^* are adjoint to each other and so are ϵ and ϵ^*.

The subgroup of **primitive elements** $P \subset R$ is given by

$$P = \{r \in R \mid m^*(r) = r \otimes 1 + 1 \otimes r\}$$

Let $\{R_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of PSH algebras. Define the tensor product PSH algebra

$$R = \otimes_{\alpha \in \mathcal{A}} R_\alpha$$

to be the inductive limit of the finite tensor products $\otimes_{\alpha \in S} R_\alpha$ with $S \subset \mathcal{A}$ a finite subset. Define $\Omega(R)$ to be the disjoint union over finite subsets S of $\prod_{\alpha \in S} \Omega(R_\alpha)$.

The following result of the PSH analogue of a structure theorem for Hopf algebras over the rationals due to Milnor-Moore.

Theorem (The Decomposition Theorem)

Any PSH algebra R decomposes into the tensor product of PSH algebras with only one irreducible primitive element. Precisely, let $\mathcal{C} = \Omega \bigcap P$ denote the set of irreducible primitive elements in R. For any $\rho \in \mathcal{C}$ set

$$\Omega(\rho) = \{\omega \in \Omega \mid \langle \omega, \rho^n \rangle \neq 0 \text{ for some } n \geq 0\}$$

and

$$R(\rho) = \oplus_{\omega \in \Omega(\rho)} \ \mathbb{Z} \cdot \omega.$$

Then $R(\rho)$ is a PSH algebra with set of irreducible elements $\Omega(\rho)$, whose unique irreducible primitive is ρ and

$$R = \otimes_{\rho \in \mathcal{C}} \ R(\rho).$$

<u>The PSH algebra $R = \oplus_n \ R(GL_n\mathbb{F}_q)$</u>

Let $R(G)$ denote the complex representation ring of a finite group G. Set $R = \oplus_{m \geq 0} \ R(GL_m\mathbb{F}_q)$ with the interpretation that $R_0 \cong \mathbb{Z}$, an isomorphism which gives both a choice of unit and counit for R.

Let $U_{k,m-k} \subset GL_m\mathbb{F}_q$ denote the subgroup of matrices of the form

$$X = \begin{pmatrix} I_k & W \\ & \\ 0 & I_{m-k} \end{pmatrix}$$

where W is an $k \times (m-k)$ matrix. Let $P_{k,m-k}$ denote the parabolic subgroup of $GL_m\mathbb{F}_q$ given by matrices obtained by replacing the identity matrices I_k and I_{m-k} in the condition for membership of $U_{k,m-k}$ by matrices from $GL_k\mathbb{F}_q$ and $GL_{m-k}\mathbb{F}_q$ respectively. Hence there is a group extension of the form

$$U_{k,m-k} \longrightarrow P_{k,m-k} \longrightarrow GL_k\mathbb{F}_q \times GL_{m-k}\mathbb{F}_q.$$

If V is a complex representation of $GL_m\mathbb{F}_q$ then the fixed points $V^{U_{k,m-k}}$ is a representation of $GL_k\mathbb{F}_q \times GL_{m-k}\mathbb{F}_q$ which gives the $(k, m-k)$ component of

$$m^* : R \longrightarrow R \otimes R.$$

Given a representation W of $GL_k\mathbb{F}_q \times GL_{m-k}\mathbb{F}_q$ so that $W \in R_k \otimes R_{m-k}$ we may form

$$\mathrm{Ind}_{P_{k,m-k}}^{GL_m\mathbb{F}_q}(\mathrm{Inf}_{GL_k\mathbb{F}_q \times GL_{m-k}\mathbb{F}_q}^{P_{k,m-k}}(W))$$

which gives the $(k, m-k)$ component of

$$m : R \otimes R \longrightarrow R.$$

We choose a basis for R_m to be the irreducible representations of $GL_m\mathbb{F}_q$ so that R^+ consists of the classes of representations (rather than virtual ones). Therefore it is clear that $m, m^*, \epsilon, \epsilon^*$ satisfy positivity. The inner product on R is given by the Schur inner product so that for two representations V, W of $GL_m\mathbb{F}_q$ we have

$$\langle V, W \rangle = \dim_{\mathbb{C}}(\mathrm{Hom}_{GL_m\mathbb{F}_q}(V, W))$$

and for $m \neq n$ R_n is orthogonal to R_m. As is well-known, with these choice of inner product, the basis of irreducible representations for R is an orthonormal basis.

The irreducible primitive elements are represented by irreducible complex representations of $GL_m\mathbb{F}_q$ which have no non-zero fixed vector for any of the subgroups $U_{k,m-k}$. These representations are usually called cuspidal.

The decomposition theorem shows how all representations are derived from cuspidal ones. This fact has an analogue ([3] and [4]) for GL_n of a local field.

Shintani base change/Shintani coorespondence ([19] Chapter Nine §6)

Let $\mathrm{Irr}(G)$ denote the set of irreducible complex representations of G.

Theorem ([16] Theorem 1)
There is a bijection

$$Sh : \mathrm{Irr}(GL_n\mathbb{F}_{q^m})^{\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)} \stackrel{\cong}{\longrightarrow} \mathrm{Irr}(GL_n\mathbb{F}_q).$$

This fact also has an analogue, called "base change" [1], for GL_n of a local field.

Theorem ([19] Chapter Nine §6.4)
The $\mathbb{Z}$-linear extension of the inverse Shintani correspondence yields an injective algebra homomorphism

$$Sh^{-1} : R' = \oplus_n\ R(GL_n\mathbb{F}_q) \longrightarrow R = \oplus_n\ R(GL_n\mathbb{F}_{q^m})$$

between the PSH-Hopf algebras introduced above.

NOT A HOMOMORPHISM OF HOPF ALGEBRAS!!

Remark: In ([19] Chapter Eight §3.12) it is shown that the existence of the Shintani correspondence is equivalent to an integrality property of certain numbers derived from monomial-resolutions.

Kondo-Gauss sums for $GL_n\mathbb{F}_q$

Definition

Let $\rho : H \longrightarrow GL_n\mathbb{C}$ denote a representation of a subgroup H of $GL_n\mathbb{F}_q$. If q is a power of the prime p we have the (additive) trace map

$$\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \longrightarrow \mathbb{F}_p.$$

In addition we have the matrix trace map

$$\mathrm{Trace} : GL_n\mathbb{F}_q \longrightarrow \mathbb{F}_q.$$

Define a measure map Ψ on matrices $X \in GL_n\mathbb{F}_q$ by

$$\Psi(X) = e^{\frac{2\pi\sqrt{-1}\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Trace}(X))}{p}}$$

which is denoted by $e_1[X]$ in [12]. Let χ_ρ denote the character function of ρ which assigns to X the trace of the complex matrix $\rho(X)$.

Define a complex number $W_H(\rho)$ by the formula

$$W_H(\rho) = \frac{1}{\dim_{\mathbb{C}}(\rho)} \sum_{X \in H} \chi_\rho(X)\Psi(X).$$

When $H = GL_n\mathbb{F}_q$ and ρ is irreducible $W_{GL_n\mathbb{F}_q}(\rho) = w(\rho)$, the Kondo-Gauss sum which is introduced and computed in [12].

Theorem 3.2.

Let σ be a finite-dimensional representation of $H \subseteq GL_n\mathbb{F}_q$. Then for any subgroup J such that $H \subseteq J \subseteq GL_n\mathbb{F}_q$

$$W_H(\sigma) = W_J(\mathrm{Ind}_H^J(\sigma)).$$

Remark:

(i) The Kondo-Gauss sum has an analogue, called the epsilon factor, in the case of admissible representations of p-adic GL_n.

(ii) In the case of the field of one element (i.e. GL_n is the symmetric group Σ_n) the associated PSH algebra is particularly simple [20]. Furthermore there is a very nice formula, which I learned from Francesco Mezzadri, for the Kondo-Gauss sum of an irreducible representation in terms of the partition representing it ([19] Appendix III §1.7).

The Bernstein centre

Let $\mathcal{A}$ be an abelian category then its centre $Z(\mathcal{A})$ is the ring of endomorphisms of the identity functor of A. Explicitly, for each object A of $\mathcal{A}$ there is given an endomorphism $z_A \in \mathrm{Hom}_{\mathcal{A}}(A, A)$ such that for any $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ one has $z_B f = f z_A$.

If the category $\mathcal{A}$ is the product of abelian categories $(\mathcal{A}_i)_{i \in \mathcal{I}}$ then one has $Z(\mathcal{A}) = \prod_{i \in \mathcal{I}} Z(\mathcal{A}_i)$.

Suppose the category $\mathcal{A}$ admits direct sums indexed by $\mathcal{I}$ such that any morphism $f : X \longrightarrow \oplus_{i\in\mathcal{I}} Y_i$ is zero if and only if all the projections

$$X \stackrel{f}{\longrightarrow} \oplus_{i\in\mathcal{I}} Y_i \stackrel{pr_i}{\longrightarrow} Y_i$$

are zero.

This property holds for the category of algebraic (i.e. smooth) representations of a reductive group over a non-Archimedean local field ([9] p.5).

Under the above condition $\mathcal{A}$ is the product of full subcategories $\mathcal{A}_i$ for $i \in \mathcal{I}$ such that

(i) if $X \in A_i$ and $Y \in A_j$ then $\mathrm{Hom}_{\mathcal{A}}(X,Y) = 0$ if $i \neq j$ and

(ii) for all objects X we have $X = \oplus_{i\in\mathcal{I}} X_i$ with X_i in $\mathcal{A}_i$.

Resolutions and the centre of $\mathcal{A}$

Suppose that $\mathcal{A}$ is an abelian category and that $\mathcal{B}$ is an additive category together with a forgetful functor $\nu : \mathcal{B} \longrightarrow \mathcal{A}$ and suppose that for each object $V \in \mathrm{Ob}(\mathcal{A})$ we have a $\mathcal{B}$-resolution of V. This means a chain complex in $\mathcal{B}$

$$\stackrel{d}{\longrightarrow} M_i \stackrel{d}{\longrightarrow} M_{i-1} \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} M_0 \longrightarrow 0$$

such that

$$\longrightarrow \nu(M_i) \longrightarrow \nu(M_{i-1}) \longrightarrow \ldots \longrightarrow \nu(M_0) \longrightarrow V \longrightarrow 0$$

is exact in $\mathcal{A}$. In addition suppose that the association $V \mapsto M_*$ is functorial into the derived category of $\mathcal{B}$.

Thus any two choices of $\mathcal{B}$-resolution for V are chain homotopy equivalent in $\mathcal{B}$ and any morphism $f : V \longrightarrow V'$ in $\mathcal{A}$ induces a $\mathcal{B}$-chain map, f_* unique up to chain homotopy, between the resolutions.

Now consider a family giving an element in the centre of $\mathcal{A}$ which yields $z_V : V \longrightarrow V$ and $z_{V'} : V' \longrightarrow V'$ satisfying $fz_V = z_{V'}f$ for all f. Fix resolutions for V and V'. Then z_V induces a chain map $(z_V)_*$ on M_* and another $(z_{V'})_*$ on M'_*. The morphism f induces a chain map $f_* : M_* \longrightarrow M'_*$ and because $f_*(z_V)_*$ is chain homotopic to $(z_{V'})_*f_*$ the pair of $\mathcal{A}$-morphisms $\nu(f_i)\nu(z_V)_i$ and $\nu(z_{V'})_i\nu(f_i)$ for $i = 0, 1$ induce $fz_V = z_{V'}f$ and so $\nu(z_V)_i$ and $\nu(z_{V'})_i$ for $i = 0, 1$ induce the elements $z_V, z_{V'}$ of the central family.

Conversely the degree 0 and 1, for any choice of resolution of V determine a central morphism z_V. When $\mathcal{A}$ is the category of (smooth) representations of G the morphisms $(z_V)_i$ for $i = 0, 1$ are described in terms of elements of the hyperHecke algebra satisfying certain commutativity conditions (I call them the monocentric conditions), -which I shall now describe.

The monocentre of a group

As (K,ψ) varies over $\mathcal{M}_{cmc,\underline{\phi}}(G)$ suppose that we have a family of elements of G, $\{x_{(K,\psi)} \in \mathrm{stab}_G(K,\psi)\}$ indexed by pairs (K,ψ) where $\mathrm{stab}_G(K,\psi)$ denotes the stabiliser of (K,ψ)

$$\mathrm{stab}_G(K,\psi) = \{z \in G \mid zKz^{-1} = K, \psi(zkz^{-1}) = \psi(k) \text{ for all } k \in K\}.$$

This is equivalent to $K \leq x^{-1}_{(K,\psi)} K x_{(K,\psi)}$ and, for all $k \in K$,

$$\psi(x^{-1}_{(K,\psi)} k x_{(K,\psi)}) = \psi(k) = x^{*}_{(K,\psi)}(\psi)(x^{-1}_{(K,\psi)} k x_{(K,\psi)})$$

so that $[(K,\psi), x_{(K,\psi)}, (K,\psi)]$ is one of the basis vectors for $\mathcal{H}$ of §2.

Next suppose that $(H,\phi) \in \mathcal{M}_{cmc,\underline{\phi}}(G)$ and $x_{(H,\phi)}$ are similar data for another pair and that $[(K,\psi), g, (H,\phi)]$ is another basis element of $\mathcal{H}$.

The **monocentre condition** relating these elements is defined by

(i) $g x_{(K,\psi)} g^{-1} \in \mathrm{stab}_G(H,\phi)$

and

(ii) $g x_{(K,\psi)} g^{-1} = x_{(H,\phi)} \in \mathrm{stab}_G(H,\phi)/\mathrm{Ker}(\phi)$.

Observe that $\mathrm{Ker}(\phi)$ is a normal subgroup of $\mathrm{stab}_G(H,\phi)$. Therefore if $[(K,\psi), g, (H,\phi)]$, $x_{(K,\psi)}$ and $x_{(H,\phi)}$ satisfy the monocentre condition then so do $[(K,\psi), g, (H,\phi)]$, $x^{-1}_{(K,\psi)}$ and $x^{-1}_{(H,\phi)}$.

Furthermore, if $[(K,\psi), g, (H,\phi)]$, $x_{(K,\psi)}$ and $x_{(H,\phi)}$ satisfy the monocentre condition and $w \in \mathrm{Ker}(\psi) \leq K$ then $[(K,\psi), g, (H,\phi)]$, $x_{(K,\psi)}w$ and $x_{(H,\phi)} g w g^{-1}$ also satisfy the condition and $gwg^{-1} \in \mathrm{Ker}(\phi) \leq H$.

Proposition

The monocentre condition implies that the two compositions

$$[(K,\psi), g, (H,\phi)] \cdot [(K,\psi), x_{(K,\psi)}, (K,\psi)]$$

and

$$[(H,\phi), x_{(H,\phi)}, (H,\phi)] \cdot [(K,\psi), g, (H,\phi)]$$

are equal in the algebra $\mathcal{H}_{cmc}(G)$.

Definition (The monocentre group of G)

The monocentre of G, denoted by $Z_{\mathcal{M}}(G)$, is the set of families $\{x_{(K,\psi)} \in \mathrm{stab}_G(K,\psi)/\mathrm{Ker}(\psi)\}$ such that for every $x_{(K,\psi)}$, $x_{(H,\phi)}$ and g such that $(K,\psi) \leq (g^{-1}Hg, (g)^*(\phi))$ the monocentre condition holds, as introduced above.

Multiplication in G induces a group structure on $Z_{\mathcal{M}}(G)$.

As we shall see in more detail, because the monocentre condition includes a central character which is common to the pairs (K,ψ) and (H,ϕ), $Z_{\mathcal{M}}(G)$ is the product of subgroups $Z_{\mathcal{M}_{cmc,\underline{\phi}}}(G)$ indexed by the set of central characters, $\underline{\phi}$.

Theorem

The monocentre group, $Z_{\mathcal{M}}(G)$, is the product of the subgroups $Z_{\mathcal{M}_{cmc,\underline{\phi}}}(G)$ as $\underline{\phi}$ varies over the central characters. Also the set of elements in a family $\{x_{(K,\psi)} \in \mathrm{stab}_G(K,\psi)/\mathrm{Ker}(\psi)\}$ representing an element of $Z_{\mathcal{M}_{cmc,\underline{\phi}}}(G)$ are determined by the

$$x_{(Z(G),\underline{\phi})} \in G/\mathrm{Ker}(\underline{\phi})$$

such that the image of $x_{(Z(G),\underline{\phi})}$ represents an element $x_{(K,\psi)} \in \mathrm{stab}_G(K,\psi)/\mathrm{Ker}(\psi)$ for every $(K,\phi) \in \mathcal{M}_{cmc,\underline{\phi}}$.

Example

The dihedral group of order eight is given by

$$D_8 = \langle x, y \mid x^4 = 1 = y^2,\ yxy = x^3\rangle.$$

Therefore we obtain

$$Z_{\mathcal{M}}(D_8) = Z_{\mathcal{M}_{cmc,1}}(D_8) \times Z_{\mathcal{M}_{cmc,\chi}}(D_8) \cong D_8/\langle x^2\rangle \times \langle x^2\rangle.$$

Remark

(i) The monocentre group is an entertaining construction, but it will turn out to be too restrictive for our purposes. Although it might be less trivial - even useful! - in the case of modular representations.

(ii) More important is the situation "**resolutions and the centre of $\mathcal{A}$**". Fix a central character $\underline{\phi}$ as usual.

In terms of monocentric conditions this situation is equivalent to the following:

Suppose, for $i = 1, 2$, that we are given

$$[(K_i,\psi_i), g_i, (H_i,\phi_i)] \text{ and}$$

$$\{x_{(K_i,\psi_i)} \in \mathrm{stab}_G(K_i,\psi_i)/\mathrm{Ker}(\psi_i)\} \text{ and}$$

$$\{x_{(H_i,\phi_i)} \in \mathrm{stab}_G(H_i,\phi_i)/\mathrm{Ker}(\phi_i)\}$$

which satisfy both

$$[(H_1,\phi_1), x_{(H_1,\phi_1)}, (H_1,\phi_1)] \cdot [(K_1,\psi_1), g_1, (H_1,\phi_1)]$$

$$= [(K_1,\psi_1), g_1, (H_1,\phi_1)] \cdot [(K_1,\psi_1), x_{(K_1,\psi_1)}, (K_1,\psi_1)]$$

and

$$[(H_2,\phi_2), x_{(H_2,\phi_2)}, (H_2,\phi_2)] \cdot [(K_2,\psi_2), g_2, (H_2,\phi_2)]$$

$$= [(K_2,\psi_2), g_2, (H_2,\phi_2)] \cdot [(K_2,\psi_2), x_{(K_2,\psi_2)}, (K_2,\psi_2)].$$

Under these conditions we require that for all

$$[(H_1,\phi_1), g_3, (H_2,\phi_2)] \text{ and } [(K_1,\psi_1), g_4, (K_2,\psi_2)]$$

such that

$$[(H_1,\phi_1),g_3,(H_2,\phi_2)]\cdot[(K_1,\psi_1),g_1,(H_1,\phi_1)]$$

$$=[(K_2,\psi_2),g_2,(H_2,\phi_2)]\cdot[(K_1,\psi_1),g_4,(K_2,\psi_2)]$$

the $\{x_{(K_i,\psi_i)},x_{(H_i,\phi_i)}\}$ satisfy

$$[(H_2,\phi_2),x_{(H_2,\phi_2)},(H_2,\phi_2)]\cdot[(H_1,\phi_1),g_3,(H_2,\phi_2)]$$

$$=[(H_1,\phi_1),g_3,(H_2,\phi_2)]\cdot[(H_1,\phi_1),x_{(H_1,\phi_1)},(H_1,\phi_1)]$$

and also that

$$[(K_2,\psi_2),x_{(K_2,\psi_2)},(K_2,\psi_2)]\cdot[(K_1,\psi_1),g_4,(K_2,\psi_2)]$$

$$=[(K_1,\psi_1),g_4,(K_2,\psi_2)]\cdot[(K_1,\psi_1),x_{(K_1,\psi_1)},(K_1,\psi_1)].$$

4. Lecture Four: Smooth representations of locally p-dic groups

Extending the definition of admissibility

If G is a locally profinite group and k is an algebraically closed field then a k-representation of G is a vector space V with a left, k-linear G-action. Let $\underline{\phi}:Z(G)\longrightarrow k^*$ be a continuous character on the centre of G. Let $\mathcal{M}_{cmc,\underline{\phi}}(G)$, as in §2, denote the poset of pairs (H,ϕ) where H is a subgroup of G, such that $Z(G)\subseteq H$, which is compact open modulo the centre and $\phi:H\longrightarrow k^*$ is a continuous character which extends $\underline{\phi}$.

Suppose that V is acted upon by $g\in Z(G)$ via multiplication by $\underline{\phi}(g)$. The representation V is called smooth if

$$V=\bigcup_{K\subset G,\ K\ \text{compact,open}} V^K.$$

V is called admissible if $\dim_k(V^K)<\infty$ for all compact open subgroups K. Define a subspace of V, denoted by $V^{(H,\phi)}$, for $(H,\phi)\in\mathcal{M}_{cmc,\underline{\phi}}(G)$ by

$$V^{(H,\phi)}=\{v\in V\mid g\cdot v=\phi(g)v \text{ for all } g\in H\}.$$

Hence $V^K=V^{(Z(G)\cdot K,\phi)}$ if ϕ is a continuous character which is trivial on K.

We shall say that V is $\mathcal{M}_{cmc,\underline{\phi}}(G)$-smooth if

$$V=\bigcup_{(H,\phi)\in\mathcal{M}_{cmc,\underline{\phi}}(G)} V^{(H,\phi)}.$$

In addition we shall say that V is $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissible if $\dim_k V^{(H,\phi)}<\infty$ for all $(H,\phi)\in\mathcal{M}_{cmc,\underline{\phi}}(G)$.

Proposition 4.1.

Let G be a locally profinite group and let k be an algebraically closed field. Let V be a k-representation of G with central character $\underline{\phi}$. Suppose that every continuous, k-valued character of a compact open subgroup of G has finite image. Then V is $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissible if and only if it is admissible.

Proof:

If K is compact open then $K\bigcap Z(G)$ is also compact open. It is certainly compact, being a closed subset of a compact subspace. For $G = GL_nF$ with F a p-adic local field the assumption it true. More generally, it holds if the quotient of $Z(G)$ by its maximal compact subgroup is discrete[3].

Suppose that V is admissible. If H is a subgroup of G which is compact open modulo the centre then $H = Z(G)\cdot K$ for some compact open subgroup. In this case supose that ϕ is a character of H extending the central character. Then $V^{(H,\phi)} = V^{(K,\mu)}$ where $\mu = \mathrm{Res}^H_K(\phi)$. Since the image of μ is finite the kernel of μ is compact open and $V^{(K,\mu)} \subseteq V^{\mathrm{Ker}(\mu)}$, which is finite-dimensional.

Next suppose that $0 \neq v \in V$. There exists a compact open subgroup K such that $v \in V^K$. Set $H = Z(G)\cdot K$, which is compact open modulo $Z(G) \subset H$. If $g \in Z(G)\bigcap K$ then $v = g\cdot v = \underline{\phi}(g)\cdot v$ so that the central character is trivial on $Z(G)\bigcap K$. Hence the central character induces a character λ on H which factors through $K/Z(G)\bigcap K \cong Z(G)\cdot K/K$ and so $v \in V^{(H,\lambda)}$, which completes the proof of $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissibility.

Assume that V is $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissible. If $0 \neq v \in V$ belongs to $V^{(H,\phi)}$ where H is compact open modulo the centre then $H = Z(G)\cdot K$ where K is compact open. Hence $v \in V^J$ where J is the compact open subgroup given by $J = \mathrm{Ker}(\mathrm{Res}^H_K(\phi))$.

Next suppose that K is a compact open subgroup. If V^K is non-trivial then $V^K \subseteq V^{(Z(G)\cdot K,\lambda)}$ where $\lambda : H = Z(G)\cdot K \longrightarrow k^*$ is the character which was constructed in the first half of the proof. Since $V^{(Z(G)\cdot K,\lambda)}$ is assumed to be finite-dimensional this concludes the proof of admissibility. □

Question 4.2. *Di-p-adic Langlands*

In the last 20 years I believe that several authors have studied the "p-adic Langlands programme". This is the situation where, for example, one studies "admissible" representations of a locally p-adic Lie group on vector spaces over the algebraic closure of a p-adic local field (or its residue field).

I intend to called this the di-p-adic situation since it is no more complicated to say and indicates the involvement of p-adic fields twice. In addition to [2] there are lots of papers on this subject[4] and a useful source for these (brought to my attention by Rob Kurinczuk) is the bibliography of [8].

The question arises: Are the sort of representations considered by the di-p-adic professionals $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissible?

[3]If this condition is not true in general it is true in the main cases of interest. Therefore let us treat it as an unimportant assumption for the time being!

[4]Regrettably I have not got round to reading any of them!

Smooth representations and Hecke modules

In this Appendix, for my convenience, representations are complex representations.

Now let Γ be a compact totally disconnected group. Denote by $\hat{\Gamma}$ the set of equivalence classes of finite-dimensional irreducible representations of Γ whose kernel is open - and hence of finite index in Γ.

Suppose now that Γ is finite and (π, V) is a representation of Γ on a possible infinite dimensional vector space V. If $\rho \in \hat{\Gamma}$ let $V(\rho)$ be the sum of all invariant subspaces of V that are isomorphic as Γ-modules to V_ρ. $V(\rho)$ is the ρ-isotypic subspace of V. We have

$$V \cong \oplus_{\rho\in\hat{\Gamma}} V_\rho.$$

Now we generalise this to smooth representations of a totally disconnected locally compact group. Choose a compact open subgroup K of G. The compact open normal subgroups of K form a basis of neighbourhoods of the identity in K. Let $\rho \in \hat{K}$ then the kernel of ρ is K_ρ a compact open normal subgroup of finite index.

Proposition 4.3. *(*[7] *Proposition 4.2.2)*

Let (π, V) be a smooth representation of G. Then

$$V \cong \oplus_{\rho\in\hat{K}} V_\rho.$$

The representation π is admissible if and only if each $V(\rho)$ is finite-dimensional.

Let (π, V) be a smooth representation of G. If $\hat{v} : V \longrightarrow \mathbb{C}$ is a linear functional we write $\langle v, \hat{v}\rangle = \hat{v}(v)$ for $v \in V$. We say $\hat{v}$ is smooth if there exists an open neighbourhood U of $1 \in G$ such that for all $g \in U$

$$\langle \pi(g)(v), \hat{v}\rangle = \hat{v}(v).$$

Let $\hat{V}$ denote the space of smooth linear functionals on V.

Define the contragredient representation $(\hat{\pi}, \hat{V})$ is defined by

$$\langle v, \hat{\pi}(g)(\hat{v})\rangle = \langle \pi(g^{-1})(v), \hat{v}\rangle.$$

The contragredient representation of a smooth representation is a smooth representation. Also

$$\hat{V} \cong \oplus_{\rho\in\hat{K}} V_\rho^*$$

where V_ρ^* is the dual space of V_ρ.

Since the dual of a finite-dimensional V_ρ is again finite-dimensional the contragredient of an admissible representation is also admissible. Also $\hat{\hat{\pi}} = \pi$.

If X is a totally disconnected space a complex valued function f on X is smooth if it is locally constant. Let $\mathcal{H}_G$ be, as before, the space of smooth compactly supported complex-valued functions on $X = G$. Assuming G is unimodular $\mathcal{H}_G$ is an algebra without unit under the convolution product

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(gh^{-1})\phi_2)(h)dh.$$

This is the Hecke algebra - an idempotented algebra (see §6).

If $\phi \in \mathcal{H}$ define $\pi(\phi) \in \mathrm{End}(V)$ with V as above

$$\pi(\phi)(v) = \int_G \phi(g)\pi(g)(v)dg.$$

Then

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1) \cdot \pi(\phi_2)$$

so that V is an $\mathcal{H}$-representation.

The integral defining ϕ may be replaced by a finite sum as follows. Choose an open subgroup K_0 fixing v. Choosing K_0 small enough we may assume that the support of ϕ is contained in a finite union of left cosets $\{g_i K_0 \mid 1 \leq i \leq t\}$. Then

$$\pi(\phi)(v) = \frac{1}{\mathrm{vol}(K_0)} \sum_{i=1}^{t} \phi(g_i)\pi(g_i)(v).$$

Finite group example:

Let (π, V) be a finite-dimensional representation of a finite group G. Write $\mathcal{H}$ for the space of functions from G to $\mathbb{C}$. If $\phi_1, \phi_2 \in \mathcal{H}$ define $\phi_1 * \phi_2 \in \mathcal{H}$ by

$$(\phi_1 * \phi_2)(g) = \sum_{h \in G} \phi_1(gh^{-1})\phi_2(h).$$

For $\phi \in \mathcal{H}$ define $\pi(\phi) \in \mathrm{End}_{\mathbb{C}}(V)$ by

$$\pi(\phi)(v) = \sum_{g \in G} \phi(g)\pi(g)(v).$$

Hence

$$\pi(\phi_1(\pi(\phi_2)(v))$$

$$= \pi(\phi_1)(\textstyle\sum_{g \in G} \phi_2(g)\pi(g)(v))$$

$$= \textstyle\sum_{g \in G} \phi_2(g)\pi(\phi_1(\pi(g)(v))$$

$$= \textstyle\sum_{g \in G} \phi_2(g) \sum_{\tilde{g} \in G} \phi_1(\tilde{g})(\pi(\tilde{g}(\pi(g)(v))$$

$$= \textstyle\sum_{g, \tilde{g} \in G} \phi_2(g)\phi_1(\tilde{g})(\pi(\tilde{g}g)(v)).$$

Now

$$\pi(\phi_1 * \phi_2)(v)$$

$$= \textstyle\sum_{g_1 \in G} (\phi_1 * \phi_2)(g_1)\pi(g_1)(v)$$

$$= \textstyle\sum_{g_1, h \in G} \phi_1(h_1 h^{-1})\phi_2(h)\pi(g_1)(v).$$

Setting $g = h$, $\tilde{g}g = g_1$ shows that

$$\pi(\phi_1 * \phi_2) = \pi(\phi_1 \cdot \pi(\phi_2).$$

Also $\mathcal{H} \cong \mathbb{C}[G]$ because if $f_g(x) = 0$ if $g \neq x$ and $f_g(g) = 1$ then

$$f_g * f_{g'} = f_{gg'}.$$

Proposition 4.4. *([7] Proposition 4.2.3)*
Let (π, V) be a smooth non-zero representation of G. Then equivalent are:
(i) π is irreducible.
(ii) V is a simple $\mathcal{H}$-module.
(iii) V^{K_0} is either zero or simple as an $\mathcal{H}_{K_0}$-module for all open subgroups K_0. Here $\mathcal{H}_{K_0} = e_{K_0} * \mathcal{H} * e_{K_0}$.

Schur's Lemma holds ([7] §4.2.4)for (π, V) an irreducible admissible representation of a totally disconnected group G.

Proposition 4.5. *([7] Proposition 4.2.5)*
Let (π, V) be an admissible representation of the totally disconnected locally compact group G with contragredient $(\hat{\pi}, \hat{V})$. Let $K_0 \subseteq G$ be a compact open subgroup. Then the canonical pairing between V and $\hat{V}$ induces a non-degenerate pairing betweem V^{K_0} and $\hat{V}^{K_0}$.

The trace

As with representations of finite groups the character of an admissible representation of a totally disconnected locally compact group G is an important invariant. It is a distribution. It is a theorem of Harish-Chandra that if G is a reductive p-adic group then the character is in fact a locally integrable function defined on a dense subset of G.

We shall define the character as a distribution on $\mathcal{H}_G = C_c^\infty(G)$. Suppose that U is a finite-dimensional vector space and let $f : U \longrightarrow U$ be a linear map. Suppose $\mathrm{Im}(f) \subseteq U_0 \subseteq U$. Then we have

$$\mathrm{Trace}(f : U_0 \longrightarrow U_0) = \mathrm{Trace}(f : U \longrightarrow U).$$

Therefore we may define the trace of any endomorphism f of V which has finite rank by choosing any finite-dimensional U_0 such that $\mathrm{Im}(f) \subseteq U_0 \subseteq V$ and by defining

$$\mathrm{Trace}(f) = \mathrm{Trace}(f : U_0 \longrightarrow U_0).$$

Now let (π, V) be an admissible representation of G. Let $\phi \in \mathcal{H}_G$. Since ϕ is compactly supported and locally constant there exists a compact open K_0 such that $\phi \in \mathcal{H}_{K_0}$. The endomorphism $\pi(\phi)$ has image in V^{K_0} which is finite-dimensional - by admissibility - so we define the trace distribution

$$\chi_V : \mathcal{H} \longrightarrow \mathbb{C}$$

by

$$\chi_V(\phi) = \mathrm{Trace}(\pi(\phi)).$$

Proposition 4.6. *([7] Proposition 4.2.6)*
Let R be an algebra over a field k. Let E_1 and E_2 be simple R-modules that are finite-dimensional over k. For each $\phi \in R$ if

$$\mathrm{Trace}((\phi \cdot -) : E_1 \longrightarrow E_1) = \mathrm{Trace}((\phi \cdot -) : E_2 \longrightarrow E_2)$$

then the E_i are isomorphic R-modules.

Proposition 4.7. *([7] Proposition 4.2.7)*
Let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of G (as above) such that, for each compact open K_1, $V_1^{K_1} \cong V_2^{K_1}$ as $\mathcal{H}_{K_1}$-modules then $(\pi_1, V_1) \cong (\pi_2, V_2)$.

Theorem 4.8. *([7] Theorem 4.2.1)*
Let (π_1, V_1) and (π_2, V_2) be irreducible admissible representations of G (as above) such that $\chi_{V_1} = \chi_{V_2}$ then $(\pi_1, V_1) \cong (\pi_2, V_2)$.

From this one sees that the contragredient of an admissible irreducible (π, V) of $GL_n K$ (K a p-adic local field) is given by $\pi_1(g) = \pi((g^{-1})^{tr})$ on the same vector space V.

Induced representations and locally profinite groups

Let G be a locally profinite group. In this section we are going to study admissible representations of G and its subgroups in relation to induction. These representations will be given by left-actions of the groups on vector spaces over k, which is an algebraically closed field of arbitrary characteristic.

Let us begin by recalling, from ([19] Chapter Two §1), induced and compactly induced smooth representations.

Definition 4.9.
Let G be a locally profinite group and $H \subseteq G$ a closed subgroup. Thus H is also locally profinite. Let

$$\sigma : H \longrightarrow \mathrm{Aut}_k(W)$$

be a smooth representation of H. Set X equal to the space of functions $f : G \longrightarrow W$ such that (writing simply $h \cdot w$ for $\sigma(h)(w)$ if $h \in H, w \in W$)
(i) $f(hg) = h \cdot f(g)$ for all $h \in H, g \in G$,
(ii) there is a compact open subgroup $K_f \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K_f$.
The (left) action of G on X is given by $(g \cdot f)(x) = f(xg)$ and

$$\Sigma : G \longrightarrow \mathrm{Aut}_k(X)$$

gives a smooth representation of G.

The representation Σ is called the representation of G smoothly induced from σ and is usually denoted by $\Sigma = \mathrm{Ind}_H^G(\sigma)$.

4.10.

$$(g \cdot f)(hg_1) = f(hg_1g) = hf(g_1g) = h(g \cdot f)(g_1)$$

so that $(g \cdot f)$ satisfies condition (i) of Definition 4.9.

Also, by the same discussion as in the finite group case, the formula will give a left G-representation, providing that $g \cdot f \in X$ when $f \in X$. However, condition (ii) asserts that there exists a compact open subgroup K_f such that $k \cdot f = f$ for all $k \in K_f$. The subgroup gK_fg^{-1} is also a compact open subgroup and, if $k \in K_f$, we have

$$(gkg^{-1}) \cdot (g \cdot f) = (gkg^{-1}g) \cdot f = (gk) \cdot f = (g \cdot (k \cdot f)) = (g \cdot f)$$

so that $g \cdot f \in X$, as required.

The smooth representations of G form an abelian category $\mathrm{Rep}(G)$.

Proposition 4.11.
The functor

$$\mathrm{Ind}_H^G : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G)$$

is additive and exact.

Proposition 4.12. *(Frobenius Reciprocity)*
There is an isomorphism

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\sigma)) \xrightarrow{\cong} \mathrm{Hom}_H(\pi, \sigma)$$

given by $\phi \mapsto \alpha \cdot \phi$ where α is the H-map

$$\mathrm{Ind}_H^G(\sigma) \longrightarrow \sigma$$

given by $\alpha(f) = f(1)$.

4.13. In general, if $H \subseteq Q$ are two closed subgroups there is a Q-map

$$\mathrm{Ind}_H^G(\sigma) \longrightarrow \mathrm{Ind}_H^Q(\sigma)$$

given by restriction of functions. Note that α in Proposition 4.12 is the special case where $H = Q$.

4.14. *The c-Ind variation*
Inside X let X_c denote the set of functions which are compactly supported modulo H. This means that the image of the support

$$\mathrm{supp}(f) = \{g \in G \mid f(g) \neq 0\}$$

has compact image in $H \backslash G$. Alternatively there is a compact subset $C \subseteq G$ such that $\mathrm{supp}(f) \subseteq H \cdot C$.

The Σ-action on X preserves X_c, since $\mathrm{supp}(g \cdot f) = \mathrm{supp}(f)g^{-1} \subseteq HCg^{-1}$, and we obtain $X_c = c - \mathrm{Ind}_H^G(W)$, the compact induction of W from H to G.

This construction is of particular interest when H is open. There is a canonical left H-map (see the Appendix in induction in the case of finite groups)

$$f : W \longrightarrow c - \mathrm{Ind}_H^G(W)$$

given by $w \mapsto f_w$ where f_w is supported in H and $f_w(h) = h \cdot w$ (so $f_w(g) = 0$ if $g \notin H$).

For $g \in G$ we have

$$(g \cdot f_w)(x) = f_w(xg) = \begin{cases} 0 & \text{if } xg \notin H, \\ (xg^{-1}) \cdot w & \text{if } xg \in H, \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \notin Hg^{-1}, \\ (xg^{-1}) \cdot w & \text{if } x \in Hg^{-1}. \end{cases}$$

We shall be particularly interested in the case when $\dim_k(W) = 1$. In this case we write $W = k_\phi$ where $\phi : H \longrightarrow k^*$ is a continuous/smooth character and, as a vector space with a left H-action $W = k$ on which $h \in H$ acts by multiplication by $\phi(h)$. In this case α_c is an injective left $k[H]$-module homomorphism of the form

$$f : k_\phi \longrightarrow c - \mathrm{Ind}_H^G(k_\phi).$$

Lemma 4.15.

Let H be an open subgroup of G. Then

(i) $f : w \mapsto f_w$ is an H-isomorphism onto the space of functions $f \in c - \mathrm{Ind}_H^G(W)$ such that $\mathrm{supp}(f) \subseteq H$.

(ii) If $w \in W$ and $h \in H$ then $h \cdot f_w = f_{h \cdot w}$.

(iii) If $\mathcal{W}$ is a k-basis of W and $\mathcal{G}$ is a set of coset representatives for $H \backslash G$ then

$$\{g \cdot f_w \mid w \in \mathcal{W},\ g \in \mathcal{G}\}$$

is a k-basis of $c - \mathrm{Ind}_H^G(W)$.

Proof

If $\mathrm{supp}(f)$ is compact modulo H there exists a compact subset C such that

$$\mathrm{supp}(f) \subseteq HC = \bigcup_{c \in C} Hc.$$

Each Hc is open so the open covering of C by the Hc's refines to a finite covering and so

$$C = Hc_1 \bigcup \ldots \bigcup Hc_n$$

and so

$$\mathrm{supp}(f) \subseteq HC = Hc_1 \bigcup \ldots \bigcup Hc_n.$$

For part (i), the map f is an H-homomorphism to the space of functions supported in H with inverse map $f \mapsto f(1)$.

For part (ii), from §**??** we have

$$(h \cdot f_w)(x) = f_w(xh) = \begin{cases} 0 & \text{if } x \notin H, \\ xh \cdot w & \text{if } x \in H. \end{cases}$$

so that, for all $x \in G$, $(h \cdot f_w)(x) = f_{h \cdot w}(x)$, as required.

For part (iii), the support of any $f \in c-\mathrm{Ind}_H^G(W)$ is a finite union of cosets Hg where the g's are chosen from the set of coset representatives $\mathcal{G}$ of $H\backslash G$. The restriction of f to any one of these Hg's also lies in $c-\mathrm{Ind}_H^G(W)$. If $\mathrm{supp}(f) \subseteq Hg$ then $(g \cdot f)(z) \neq 0$ implies that $zg \in Hg$ so that $g \cdot f$ has support contained in H. Hence $g \cdot f$ on H is a finite linear combination of the functions f_w with $w \in \mathcal{W}$. Therefore f is a finite linear combination of $g \cdot f_w$'s where $w \in \mathcal{W}, g \in \mathcal{G}$. Clearly the set of functions $g \cdot f_w$ with $g \in \mathcal{G}$ and $w \in \mathcal{W}$ is linearly independent. □

Example 4.16. Let K be a p-adic local field with valuation ring $\mathcal{O}_K$ and π_K a generator of the maximal ideal of $\mathcal{O}_K$. Suppose that $G = GL_nK$ and that H is a subgroup containing the centre of G (that is, the scalar matrices K^*). If H is compact, open modulo K^* then there is a subgroup H' of finite index in H such that $H' = K^*H_1$ with H_1 compact, open in SL_nK. This can be established by studying the simplicial action of GL_nK on a suitable barycentric subdivision of the Bruhat-Tits building of SL_nK (see [19] Chapter Four §1).

To show that H is both open and closed it suffices to verify this for H'. Firstly H' is open, since it is $H' = \bigcup_{z \in K^*} zH_1 = \bigcup_{s \in \mathbb{Z}} \pi_K^s H_1$.

Also $H' = K^*H_1$ is closed. Suppose that $X' \notin K^*H_1$. K^*H_1 is closed under mutiplication by the multiplicative group generated by π_K so that $\pi_K^m X' \notin K^*H_1$ for all m. By conjugation we may assume that H_1 is a subgroup of $SL_n\mathcal{O}_K$, which is the maximal compact open subgroup of SL_nK, unique up to conjugacy. Choose the smallest non-negative integer m such that every entry of $X = \pi_K^m X'$ lies in $\mathcal{O}_K$. Therefore we may write $0 \neq \det(X) = \pi_K^s u$ where $u \in \mathcal{O}_K^*$ and $1 \leq s$. Now suppose that V is an $n \times n$ matrix with entries in $\mathcal{O}_K$ such that $X + \pi_K^t V \in K^*H_1$. Then

$$\det(X + \pi_K^t V) \equiv \pi_K^s u \text{ (modulo } \pi_K^t).$$

So that if $t > s$ then s must have the form $s = nw$ for some integer w and $\pi_K^{-w}(X + \pi_K^t V) \in GL_n\mathcal{O}_K \bigcap K^*H_1 = H_1$. Therefore all the entries in $\pi_K^{-w}X$ lie in $\mathcal{O}_K$ and $\pi_K^{-w}X \in GL_n\mathcal{O}_K$. Enlarging t, if necessary, we can ensure that $\pi_K^{-w}X \in H_1$, since H_1 is closed (being compact), and therefore $X' \in K^*H_1$, which is a contradiction.

Since H is both closed and open in GL_nK we may form the admissible representation $c - \mathrm{Ind}_H^{GL_nK}(k_\phi)$ for any continuous character $\phi : H \longrightarrow k^*$ and apply Lemma **??**.

If $g \in GL_nK, h \in H$ then $(g \cdot f_1)(x) = \phi(xg)$ if $xg \in H$ and zero otherwise. On the other hand, $(gh \cdot f_1)(x) = \phi(xgh) = \phi(h)\phi(xg)$ if $xg \in H$ and zero otherwise. Therefore as a left GL_nK-representation $c - \mathrm{Ind}_H^{GL_nK}(k_\phi)$ is isomorphic to

$$k[GL_nK]/(\phi(h)g - gh \mid g \in GL_nK,\ h \in H)$$

with left action induced by $g_1 \cdot g = g_1 g$.

This vector space is isomorphic to the k-vector space whose basis is given by k-bilinear tensors over H of the form $g \otimes_{k[H]} 1$ as in the case of finite groups. The basis vector $g \cdot f_1$ corresponds to $g \otimes_H 1$ and $GL_n K$ acts on the tensors by left multiplication, as usual (see Appendix §4 in the finite group case).

Proposition 4.17.
The functor

$$c - \mathrm{Ind}_H^G : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G)$$

is additive and exact.

Proposition 4.18.
Let $H \subseteq G$ be an open subgroup and (σ, W) smooth. Then there is a functorial isomorphism

$$\mathrm{Hom}_G(c - \mathrm{Ind}_H^G(W), \pi) \xrightarrow{\cong} \mathrm{Hom}_H(W, \pi)$$

given by $F \mapsto F \cdot f$, the composition with the H-map f of Lemma 4.15.

Example 4.19. $c - \underline{\mathrm{Ind}}_H^G(\phi)$
Suppose that $\phi : H \longrightarrow k^*$ is a continuous character (i.e. a one-dimensional smooth representation of H).

Suppose that we are in a situation analogous to that of Example 4.16. Namely suppose that H is open and closed, contains $Z(G)$,the centre of G, and is compact open modulo $Z(G)$. A basis for k is given by $1 \in k^*$ and we have the function $f_1 \in X_c$ given by $f_1(h) = \phi(h)$ if $h \in H$ and $f_1(g) = 0$ if $g \notin H$.

If, following Lemma 4.15, $\mathcal{G}$ is a set of coset representatives for $H \backslash G$ then a k-basis for $c - \underline{\mathrm{Ind}}_H^G(\phi)$ is given by

$$\{g \cdot f_1 \mid g \in \mathcal{G}\}.$$

For $g \in G$ we have

$$(g \cdot f_1)(x) = f_1(xg) = \begin{cases} 0 & \text{if } xg \notin H, \\ \phi(xg) & \text{if } xg \in H, \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \notin Hg^{-1}, \\ \phi(xg) & \text{if } x \in Hg^{-1}. \end{cases}$$

Before going further let us introduce the presence of (H, ϕ) into the notation.

Definition 4.20. Let H be a closed subgroup of G containing the centre, $Z(G)$, which is compact open modulo $Z(G)$. Let $\phi : H \longrightarrow k^*$ be a continuous character of H. Denote by $X_c(H, \phi)$ the k-vector space of functions $f : G \longrightarrow k$ such that

(i) $f(hg) = \phi(h)f(g)$ for all $h \in H, g \in G$,

(ii) there is a compact open subgroup $K_f \subseteq G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K_f$,

(ii) f is compactly supported modulo H.

As in §4.14, the left action of G on $X_c(H, \phi)$ is given by $(g \cdot f)(x) = f(xg)$ and therefore

$$\Sigma : G \longrightarrow \mathrm{Aut}_k(X_c(H, \phi))$$

gives a smooth representation of G - denoted by $\Sigma = c - \mathrm{Ind}_H^G(\phi)$.

Henceforth we shall denote the map written as f_1 in Example 4.19 by $f_{(H,\phi)} \in X_c(H, \phi)$.

Therefore, for $g \in G$, we have

$$\begin{aligned}(g \cdot f_{(H,\phi)})(x) = f_{(H,\phi)}(xg) &= \begin{cases} 0 & \text{if } xg \notin H, \\ \phi(xg) & \text{if } xg \in H, \end{cases} \\ &= \begin{cases} 0 & \text{if } x \notin Hg^{-1}, \\ \phi(xg) & \text{if } x \in Hg^{-1}. \end{cases}\end{aligned}$$

Definition 4.21. For (H, ϕ) and (K, ψ) as in Definition 4.20, write $[(K, \psi), g, (H, \phi)]$ for any triple consisting of $g \in G$, characters ϕ, ψ on subgroups $H, K \leq G$, respectively such that

$$(K, \psi) \leq (g^{-1}Hg, (g)^*(\phi))$$

which means that $K \leq g^{-1}Hg$ and that $\psi(k) = \phi(h)$ where $k = g^{-1}hg$ for $h \in H, k \in K$.

Let $\mathcal{H}$ denote the k-vector space with basis given by these triples. Define a product on these triples by the formula

$$[(H, \phi), g_1, (J, \mu)] \cdot [(K, \psi), g_2, (H, \phi)] = [(K, \psi), g_1g_2, (J, \mu)]$$

and zero otherwise. This product makes sense because

(i) if $K \leq g_2^{-1}Hg_2$ and $H \leq g_1^{-1}Jg_1$ then $K \leq g_2^{-1}Hg_2 \leq g_2^{-1}g_1^{-1}Jg_1g_2$ and

(ii) if $\psi(k) = \phi(h) = \mu(j)$, where $k = g_2^{-1}hg_2, h = g_1^{-1}jg_1$ then $k = g_2^{-1}g_1^{-1}jg_1g_2$.

This product is clearly associative and we define an algebra $\mathcal{H}_{cmc}(G)$ to be $\mathcal{H}$ modulo the relations

$$[(K, \psi), gk, (H, \phi)] = \psi(k^{-1})[(K, \psi), g, (H, \phi)]$$

and

$$[(K, \psi), hg, (H, \phi)] = \phi(h^{-1})[(K, \psi), g, (H, \phi)].$$

We observe that

$$[(K, \psi), g, (H, \phi)] = [(g^{-1}Hg, g^*\phi), g, (H, \phi)] \cdot [(K, \psi), 1, (g^{-1}Hg, g^*\phi)]$$

We shall refer to this algebra as the compactly supported modulo the centre (CSMC-algebra) of G.

Lemma 4.22.
Let $[(K,\psi),g,(H,\phi)]$ be a triple as in Definition 4.21. Associated to this triple define a left $k[G]$-homomorphism

$$[(K,\psi),g,(H,\phi)] : X_c(K,\psi) \longrightarrow X_c(H,\phi)$$

by the formula $g_1 \cdot f_{(K,\psi)} \mapsto (g_1 g^{-1}) \cdot f_{(H,\phi)}$.

For a proof, which is the same as in the case when G is finite, can be found in (the Appendix on induction in the case of finite groups).

Theorem 4.23.
Let $\mathcal{M}_c(G)$ denote the partially order set of pairs (H,ϕ) as in Definitions 4.20 and 4.21 (so that $X_c(H,\phi) = c - \mathrm{Ind}_H^G(\phi))$. Then, when each $n_\alpha = 1$,

$$M_c(\underline{n},G) = \oplus_{\alpha\in\mathcal{A},(H,\phi)\in\mathcal{M}_c(G)} underline n_\alpha X_c(H,\phi)$$

is a left $k[G] \times \mathcal{H}_{cmc}(G)$-module. For a general distribution of multiplicities $\{n_\alpha\}$ it is Morita equivalent to a left $k[G] \times \mathcal{H}_{cmc}(G)$-module.

Proof
We have only to verify associativity of the module multiplication, which is obvious. □

Definition 4.24. ${}_{k[G]}$**mon**, *the monomial category of* G
The monomial category of G is the additive category (non-abelian) whose objects are the k-vector spaces given by direct sums of $X_c(H,\phi)$'s of §4.23 and whose morphisms are elements of the hyperHecke algebra $\mathcal{H}_{cmc}(G)$. In other words the subcategory of the category of $k[G] \times \mathcal{H}_{cmc}(G)$-modules of which one example is $M_c(\underline{n},G)$ in §4.23.

The bar-monomial resolution: II. The compact, open modulo the centre case

Let G be a locally profinite group and let k be an algebraically closed field. Let V be a k-representation of G with central character $\underline{\phi}$ and that V is a $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissible representation as in Proposition 4.9.
GOT TO HERE
Let $\mathcal{H}_{cmc}(G)$ be the hyperHecke algebra, introduced earlier. Let

$$M_c(\underline{n},G) = \oplus_{\alpha\in\mathcal{A},(H,\phi)\in\mathcal{M}_c(G)} \underline{n}_\alpha X_c(H,\phi)$$

be the left $k[G] \times \mathcal{H}_{cmc}(G)$-module of Theorem 4.23 form some family of strictly positive integers, $\{\underline{n}_\alpha\}$.

Theorem 4.25. Replacing the previous S by $M_c(\underline{n},G)$ and replacing the ring $\mathcal{A}_M$ (when $M = S$) by $\mathcal{H}_{cmc}(G)$ we may imitate the previous construction to make a ${}_{k[G],\underline{\phi}}\mathbf{mon}$-resolution of V

$$\ldots \stackrel{d}{\longrightarrow} \tilde{M}_{M_c(\underline{n},G),i} \otimes_k M_c(\underline{n},G) \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} \tilde{M}_{M_c(\underline{n},G),1} \otimes_k M_c(\underline{n},G)$$

$$\stackrel{d}{\longrightarrow} \tilde{M}_{M_c(\underline{n},G),0} \otimes_k M_c(\underline{n},G) \stackrel{\epsilon}{\longrightarrow} V \longrightarrow 0$$

This result is proved using the analogues of the earlier ones.

Remark 4.26. In [19] this result was proved[5] by reduction to the finite modulo the centre case. Also an explicit bare hands homological construction was given in the case of GL_2 of a local field. I think that the use of the hyperHecke algebra simplifies the construction both in the compact, open modulo the centre case of this section and the general case of the next.

The monomial resolution in the general case

Once again let G be a locally profinite group and let k be an algebraically closed field. Let V be a k-representation of G with central character $\underline{\phi}$ and that V is a $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissible representation as in Proposition 4.9.

First I shall recall the properties of Tammo tom Dieck's space $\underline{E}(G,\mathcal{C})$ ([19] Appendix IV) which is defined for a group G and a family of subgroups $\mathcal{C}$ which is closed under conjugation and passage to subgroups. This space is a simplicial complex on which G acts simplicially in such a way that for any subgroup $H \in \mathcal{C}$ the fixed-point set $\underline{E}(G,\mathcal{C})^H$ is non-empty and contractible. In our case $\mathcal{C}$ will be the family of compact, open modulo the centre subgroups.

$\underline{E}(G,\mathcal{C})$ is unique up to G-equivariant homotopy equivalence. In the case of GL_n of a local field, for example, the Bruhat-Tits building gives a finite-dimensional model for the tom Dieck space.

If the set of conjugacy classes maximal compact, open modulo the centre subgroups of G is finite, as in the case of GL_nK for example, one can find a local system which assigns to each compact, open modulo the centre J a ${}_{k[J],\underline{\phi}}\mathbf{mon}$-resolution of $\mathrm{Res}^G_J V$

$$\ldots \stackrel{d}{\longrightarrow} \tilde{M}_{M_c(\underline{n},J),i} \otimes_k M_c(\underline{n},J) \stackrel{d}{\longrightarrow} \ldots \stackrel{d}{\longrightarrow} \tilde{M}_{M_c(\underline{n},J),1} \otimes_k M_c(\underline{n},J)$$

$$\stackrel{d}{\longrightarrow} \tilde{M}_{M_c(\underline{n},J),0} \otimes_k M_c(\underline{n},J) \stackrel{\epsilon}{\longrightarrow} \mathrm{Res}^G_J V \longrightarrow 0.$$

Next one forms the double complex ([19] Chapter Four Theorem 3.2) given by the simplicial chain complex of the tom Dieck space in one grading and the compact, open modulo the centre ${}_{k[J],\underline{\phi}}\mathbf{mon}$-resolutions in the other grading. The contribution of the resolutions corresponding to the orbit of one J-fixed simplex gives the compactly supported induction of that resolution.

[5] I believe!

Theorem 4.27. *([19] Chapter Four Theorem 3.2)*

Let V be a $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissible representation as in Proposition 4.9. Then the total complex of the above double complex is ${}_{k[G],\underline{\phi}}$**mon**-resolution of V.

Idempotented algebras ([7] p.309)

Definition 4.28. Let k be a field and H a k-algebra. Let $\mathcal{E}$ denote a set of idempotents of H. Assume that if $e_1, e_2 \in \mathcal{E}$ then there exists $e_0 \in \mathcal{E}$ such that $e_0e_1 = e_1e_0 = e_1$ and $e_0e_2 = e_2e_0 = e_2$. In addition assume for every $\phi \in H$ that there exists $e \in \mathcal{E}$ such that $e\phi = \phi e = \phi$.

With these assumptions H is called an idempotented k-algebra.

Write $f \leq e$ if $ef = fe = f$. This gives $\mathcal{E}$ the structure of a partially ordered set (i.e. a poset).

If R is a ring and e an idempotent denote eRe by $R[e]$. If M is a left R-module write $M[e]$ for the $R[e]$-module eM. If H is an idempotented algebra then $H[e]$ is a k-algebra with unit e and $M[e]$ is an $H[e]$-module.

M is smooth if $M = \bigcup_{e \in \mathcal{E}} M[e]$ and is admissible if it is smooth and for each $e \in \mathcal{E}$ we have $\dim_k(M[e]) < \infty$.

If $(H_i, \mathcal{E}_i)$ are idempotented algebras for $i = 1, 2$ then so is $H_1 \otimes H_2$ with idempotents $e_1 \otimes e_2$ for $e_i \in \mathcal{E}_i$.

4.29. *The idempotented algebra $\mathcal{H}_{cmc}(G)$*

Let $\mathcal{E}$ be the collection of finite additive combinations in $\mathcal{H}_{cmc}(G)$, the algebra of Definition 4.21, of the form

$$e = \sum_{i=1}^{n} [(H_i, \phi_i), 1, (H_i, \phi_i)]$$

in which $(H_i, \phi_i) = (H_j, \phi_j)$ if and only if $i = j$. Then $e \cdot e = e$ and all the idempotents in $\mathcal{H}_{cmc}(G)$ have this form.

We shall write $e_{(H,\phi)}$ for the idempotent $[(H, \phi), 1, (H, \phi)]$.

Define the homomorphism

$$[(K', \psi'), g, (H', \phi')] : X_c(K, \psi) \longrightarrow X_c(H, \phi)$$

to be zero unless $K', \psi') = (K, \psi)$ and $(H, \phi) = (H', \phi')$. The following result is clear.

Theorem 4.30.

(i) In §4.29 $(\mathcal{H}_{cmc}(G), \mathcal{E})$ is an idempotented algebra and $M_c(G)$ is an $\mathcal{H}_{cmc}(G)$-module in the category of smooth $k[G]$-modules.

(ii) In this idempotented algebra $e = \sum_{i=1}^{n} [(H_i, \phi_i), 1, (H_i, \phi_i)]$ and f satisfy $ef = fe = f$ in and only if the idempotent f is a subsum of e, which fits very nicely with the $f \leq e$ notation.

(iii) If $M_c(\underline{n}, G)$ is the module of Theorem 4.23 then $M_c(\underline{n}, G)[e]$ is the direct sum of the $\underline{n}_\alpha X_c(H, \phi)$'s for which $e_{(H,\phi)}$ appears in the sum for e.

4.31. *Hecke algebras*

The Hecke algebra of a locally compact, totally disconnected group is a related idempotented algebra.

Let G be a locally compact, totally disconnected group. Assume that G is unimodular - that is, the left invariant Haar measure equals the right-invariant Haar measure of G ([7] p.137).

The Hecke algebra of G, denoted by $\mathcal{H}_G$ is the space $C_c^\infty(G)$ of locally constant,compactly supported k-valued functions on G with the convolution product ([7] p.140 and p.255)

$$(\phi_1 * \phi_2)(g) = \int_G \phi_1(gh)\phi_2(h^{-1})dh = \int_G \phi_1(h)\phi_2(h^{-1}g)dh.$$

This integral requires only one of ϕ_1, ϕ_2 to be compactly supported in order to land in $\mathcal{H}_G$.

Suppose that $K_0 \subseteq G$ is a compact, open subgroup. Define an idempotent

$$e_{K_0} = \frac{1}{\mathrm{vol}(K_0)} \cdot \chi_{K_0}$$

where χ_{K_0} is the characteristic function of K_0. If $K_0 \subseteq K_1$ then $e_{K_0} * e_{K_1} = e_{K_1}$.

This is seen using left invariance of the Haar measure

$$\int_G \frac{\chi_K(zh)}{\mathrm{vol}(K)} \frac{\chi_H(h^{-1})}{\mathrm{vol}(H)} = \int_G \frac{\chi_K(h)}{\mathrm{vol}(K)} \frac{\chi_H(h^{-1}z)}{\mathrm{vol}(H)}.$$

The integrand is zero unless $h \in K$ and then it is zero unless $z \in H$. When $z \in H$ we are integrating

$$\int_G \frac{\chi_K(h)}{\mathrm{vol}(K)} \frac{1}{\mathrm{vol}(H)} = \frac{\chi_H(z)}{\mathrm{vol}(H)},$$

as required.

$\mathcal{H}_G$ is an idempotented algebra because G has a base of neighbourhoods consisting of compact open subgroups.

A function $f \in \mathcal{H}_G$ is called K-finite if the subspace spanned by all its (left) translates by K is finite-dimensional ([7] p.299).

Monomial morphisms as convolution products

It is my belief and eventual intention that the material of this section will remain true for the general G as in §2 provided that all continuous k-valued characters on compact, open subgroups have finite image.

However, throughout this section I shall assume that G is a locally profinite group whose centre $Z(G)$ is compact. Let H be a subgroup which is compact, open modulo the centre. Let k be an algebraically closed field for which all continuous characters $\phi : H \longrightarrow k^*$ have finite image when H is compact, open.

The following two results give some examples of G for which $Z(G)$ is compact.

Lemma 4.32.
Let K be a p-adic local field. Then $Z(SL_nK)$ is finite. In particular it is compact.

Proof
Consider the relation

$$\begin{pmatrix} x_1 & 0 & \cdot & \cdots & 0 \\ 0 & x_2 & \cdot & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & x_n \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & \cdot & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

$$= \begin{pmatrix} a_{1,1} & a_{1,2} & \cdot & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & \cdots & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 & 0 & \cdot & \cdots & 0 \\ 0 & x_2 & \cdot & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & x_n \end{pmatrix}$$

In the (i,j) entry we find $x_i a_{i,j} = a_{i,j} x_j$ and since we may suppose $a_{i,j} \neq 0$ we see that $x_1 = x_2 = \ldots = x_n$ and $x_1^n = 1$. Therefore $Z(SL_nK) = \mu_n(K)$, the group of n-th roots of unity in K. □

Lemma 4.33.
Let K be a p-adic local field with ring of integers $\mathcal{O}_K$ and prime π_K. Then $Z(GL_nK/\langle\pi_K\rangle) \cong \mathcal{O}_K^*$. In particular it is compact. Here $\langle\pi_K\rangle$ denotes the centre subgroup generated by π_K times the identity matrix.

Proof
The relation used in the proof of §4.32 implies that for each (i,j) we have $\pi_K^\alpha x_i a_{i,j} = a_{i,j} x_j \pi_K^\beta$ for some pair α, β. Therefore we may suppose that $x_1 \in \mathcal{O}_K^*$ and that $x_j = x_1 \pi_K^{e_j}$. Now taking a matrix with $a_{1,j} a_{j,1} \neq 0$ for $j = 2, 3, \ldots, n$ we find that $\pi_K^\alpha x_1 = x_j \pi_K^\beta = x_1 \pi_K^{\beta + e_j}$ for $j = 2, 3, \ldots, n$. This implies that $x_1 \pi_K^e = x_2 = x_3 = \ldots = x_n$ which implies that $e = 0$. □

The next two results ensure that we are free to use convolution products in our context.

Lemma 4.34.
Let G be a locally profinite group whose centre $Z(G)$ is compact. If H is a subgroup of G, containing $Z(G)$, which is compact, open modulo the centre then H is compact, open.

Proof
The is a compact open subset C of G such that $H = Z(G) \cdot C$. Multiplication is a continuous map from the compact space $Z(G) \times C$ onto H so that

H is compact. Furthermore any point of H may be written as $h = z \cdot c$ with $z \in Z(G)$ and $c \in C$. Therefore $z \cdot N \subseteq H$ for any open neighbourhood of c in C is an open neighbourhood of h in H, which is therefore open. $\square$

Lemma 4.35.
Let G be a locally profinite group whose centre $Z(G)$ is compact and let H be a subgroup which is compact, open modulo the centre. Let k be an algebraically closed field for which all continuous characters $\phi : H \longrightarrow k^*$ have finite image when H is compact, open. Then the vector space, X_c, of §4.14 on which $c - \mathrm{Ind}_H^G(k_\phi)$ is defined is a subspace of the Hecke algebra of G, $\mathcal{H}_G$, the space of locally constant,compactly supported k-valued functions on G.

Proof:
By §4.15 it suffices to verify that the function f_w of §4.14 is locally constant, compactly supported for $w = 1 \in k^*$. This function is given by the formula

$$f_1(x) = \begin{cases} 0 & \text{if } x \notin H, \\ \\ \phi(x) & \text{if } x \in H, \end{cases}$$

By §4.34 H, the support of f_1, is compact. Since the image of ϕ is finite the function f_1 is locally constant. $\square$

Recall from §§4.21-4.22 that we have defined

$$[(K, \psi), g, (H, \phi)] : X_c(K, \psi) \longrightarrow X_c(H, \phi)$$

by the formula $g_1 \cdot f_{(K,\psi)} \mapsto (g_1 g^{-1}) \cdot f_{(H,\phi)}$.

If χ_W is the characteristic function of $W \subseteq G$ we may define $g_1 \cdot f_{(K,\psi)}$ using characteristic functions in the following manner. By definition

$$g_1 \cdot f_{(K,\psi)}(x) = \begin{cases} \psi(x g_1^{-1}) & \text{if } x g_1^{-1} \in K, \\ 0 & \text{if } x g_1^{-1} \notin K. \end{cases}$$

Suppose that $v_1, \ldots, v_t$ are coset representatives for $K/\mathrm{Ker}(\psi)$. Then, if $x g_1^{-1} \in K$ we must have $x g_1^{-1} \in \mathrm{Ker}(\psi) v_{j(x g_1^{-1})}$ for some $1 \leq j(x g_1^{-1}) \leq t$ and therefore $\psi(x g_1^{-1}) = \psi(v_{j(x g_1^{-1})})$. Hence we have the fomula

$$g_1 \cdot f_{(K,\psi)} = \sum_{j=1}^{t} \psi(v_j) \chi_{\mathrm{Ker}(\psi) v_j g_1}$$

because $\bigcup \mathrm{Ker}(\psi) v_j g_1 = K g_1$ so that the right hand side is zero unless $x g_1^{-1} \in K$ and is $\psi(v_{j_0})$ precisely when $j_0 = j(x g_1^{-1})$.

Next, from Definition 4.21

$$(K, \psi) \leq (g^{-1} H g, (g)^*(\phi))$$

implies that $\psi(k) = \phi(h)$ where $k = g^{-1} h g$ for $h \in H, k \in K$. Therefore if $k \in \mathrm{Ker}(\psi)$ then $h \in \mathrm{Ker}(\phi)$ and so $\mathrm{Ker}(\psi) \leq g^{-1} \mathrm{Ker}(\phi) g$.

Consider the convolution product

$$\chi_{g_1\mathrm{Ker}(\psi)} * \chi_{g^{-1}\mathrm{Ker}(\phi)}(z) = \int_G \chi_{g_1\mathrm{Ker}(\psi)}(h)\chi_{g^{-1}\mathrm{Ker}(\phi)}(h^{-1}z)dh.$$

The integrand is zero unless $h \in g_1\mathrm{Ker}(\psi)$ in addition to the condition $z \in hg^{-1}\mathrm{Ker}(\phi) \subseteq g_1\mathrm{Ker}(\psi)g^{-1}\mathrm{Ker}(\phi) = g_1g^{-1}g\mathrm{Ker}(\psi)g^{-1}\mathrm{Ker}(\phi) \subseteq g_1g^{-1}\mathrm{Ker}(\phi)$ and conversely. Therefore

$$\chi_{g_1\mathrm{Ker}(\psi)} * \chi_{g^{-1}\mathrm{Ker}(\phi)} = \mathrm{vol}(g_1\mathrm{Ker}(\psi))\chi_{g_1g^{-1}\mathrm{Ker}(\phi)}.$$

Similarly, if $v \in K$ and $u \in H$, we have a convolution product

$$\chi_{g_1\mathrm{Ker}(\psi)v} * \chi_{g^{-1}\mathrm{Ker}(\phi)u}(z) = \int_G \chi_{g_1\mathrm{Ker}(\psi)v}(h)\chi_{g^{-1}\mathrm{Ker}(\phi)u}(h^{-1}z)dh.$$

The integrand is zero unless $h \in g_1\mathrm{Ker}(\psi)v$ in addition to the condition

$$z \in hg^{-1}\mathrm{Ker}(\phi)u \subseteq g_1\mathrm{Ker}(\psi)vg^{-1}\mathrm{Ker}(\phi)u \subseteq g_1g^{-1}\mathrm{Ker}(\phi)(gvg^{-1}) \cdot u$$

and conversely. Therefore

$$\chi_{g_1\mathrm{Ker}(\psi)v} * \chi_{g^{-1}\mathrm{Ker}(\phi)u} = \mathrm{vol}(g_1\mathrm{Ker}(\psi)v)\chi_{g_1g^{-1}\mathrm{Ker}(\phi)gvg^{-1}u}.$$

Lemma 4.36.

Suppose that $v_1, \ldots, v_t \in K$ is a set of coset representatives for $K/\mathrm{Ker}(\psi)$. Then

$$g_1 \cdot f_{(K,\psi)} = \sum_{j=1}^{t} \psi(v_j) \cdot \chi_{\mathrm{Ker}(\psi)v_jg_1^{-1}}.$$

Proof:

Consider the functions in the equation applied to $x \in G$. The left hand side is zero if $xg_1 \notin K$ which is equivalent to there being no j such that $xg_1 \in \mathrm{Ker}(\psi)v_j$ or $x \in \mathrm{Ker}(\psi)v_jg_1^{-1}$. Under these conditions every characteristic function on the right hand side also vanishes on x. On the other hand if $xg_1 \in K$ there exists a unique j_0 such that $x \in \mathrm{Ker}(\psi)v_{j_0}g_1^{-1}$ and so, evaluated at xg_1, there is one and only one term on the right hand side which contributes. It yields $\psi(v_{j_0})$ which is the value of $g_1 \cdot f_{(K,\psi)}$ at x, as required. □

4.37. The image $\phi(H)$ is a finite cyclic group, being a finite subgroup of k^*, which contains $\phi(gKg^{-1}) = \psi(K)$. Therefore there exist $v_1, \ldots, v_t$ which are coset representatives for $K/\mathrm{Ker}(\psi)$ and $u_1, \ldots, u_s$ which give distinct cosets in $H/\mathrm{Ker}(\phi)$ such that the set $\{(gv_ig^{-1})u_j \mid 1 \leq i \leq t,\ 1 \leq j \leq s\}$ is a set of coset representatives for $H/\mathrm{Ker}(\phi)$.

Definition 4.38. Define an involution $T : C_c^\infty(G) \longrightarrow C_c^\infty(G)$ by $T(F)(x) = F(x^{-1})$. For example $T(\chi_{\mathrm{Ker}(\psi)v_jg_1^{-1}}) = \chi_{g_1\mathrm{Ker}(\psi)v_j^{-1}}$.

In the notation of §4.37 set

$$\Phi_{[(K,\psi),g,(H,\phi)]} = \sum_{j=1}^{s} \phi(u_j) \cdot \chi_{g^{-1}\mathrm{Ker}(\phi)u_j}.$$

Theorem 4.39.

In the notation of Definition 4.38

$$[(K,\psi),g,(H,\phi)](g_1\cdot f_{(K,\psi)}) = g_1g^{-1}\cdot f_{(H,\phi)}$$

$$= \frac{1}{\mathrm{vol}(Ker(\psi))}T(T(g_1\cdot f_{(K,\psi)})*\Phi_{[(K,\psi),g,(H,\phi)]}).$$

Proof:

We observe that $\psi(v_i)(\chi_{\mathrm{Ker}(\psi)v_ig_1^{-1}})(x)=\psi(v_i)=\psi(xg_1)$ if $x\in\mathrm{Ker}(\psi)v_ig_1^{-1}=v_i\mathrm{Ker}(\psi)g_1^{-1}$ and zero otherwise. Therefore

$$T(\psi(v_i)(\chi_{\mathrm{Ker}(\psi)v_ig_1^{-1}}))(x)=\psi(v_i)(\chi_{\mathrm{Ker}(\psi)v_ig_1^{-1}})(x^{-1})=\psi(v_i)$$

if $x^{-1}\in\mathrm{Ker}(\psi)v_ig_1^{-1}$ and zero otherwise. In the non-zero case $x\in g_1\mathrm{Ker}(\psi)v_i^{-1}$ and $\psi(v_i)=\psi(g_1^{-1}x)^{-1}$ so that

$$T(\psi(v_i)(\chi_{\mathrm{Ker}(\psi)v_ig_1^{-1}}))=\psi(v_i)^{-1}\chi_{g_1\mathrm{Ker}(\psi)v_i^{-1}}.$$

From Lemma 4.32 we have

$$T(g_1\cdot f_{(K,\psi)})=\sum_{i=1}^t\ \psi(v_i)^{-1}\cdot\chi_{g_1\mathrm{Ker}(\psi)v_i^{-1}}.$$

Therefore

$$T(g_1\cdot f_{(K,\psi)})*\Phi_{[(K,\psi),g,(H,\phi)]}$$

$$=\sum_{i=1}^t\ \sum_{j=1}^s\ \psi(v_i)^{-1}\phi(u_j)(\chi_{g_1\mathrm{Ker}(\psi)v_i^{-1}}*\chi_{g^{-1}\mathrm{Ker}(\phi)u_j})$$

$$=\sum_{i=1}^t\ \sum_{j=1}^s\ \psi(v_i)^{-1}\phi(u_j)\mathrm{vol}(\mathrm{Ker}(\psi))\chi_{g_1g^{-1}\mathrm{Ker}(\phi)(gv_i^{-1}g^{-1})u_j}.$$

Hence

$$T(T(g_1\cdot f_{(K,\psi)})*\Phi_{[(K,\psi),g,(H,\phi)]})$$

$$=T(\sum_{i=1}^t\ \sum_{j=1}^s\ \psi(v_i)^{-1}\phi(u_j)\mathrm{vol}(\mathrm{Ker}(\psi))\chi_{g_1g^{-1}\mathrm{Ker}(\phi)(gv_i^{-1}g^{-1})u_j})$$

$$=T(\sum_{i=1}^t\ \sum_{j=1}^s\ \phi(gv_ig^{-1})^{-1}\phi(u_j)\mathrm{vol}(\mathrm{Ker}(\psi))\chi_{g_1g^{-1}\mathrm{Ker}(\phi)(gv_i^{-1}g^{-1})u_j})$$

$$=\mathrm{vol}(\mathrm{Ker}(\psi))\sum_{i=1}^t\ \sum_{j=1}^s\ T(\phi((gv_i^{-1}g^{-1})u_j)\chi_{g_1g^{-1}\mathrm{Ker}(\phi)(gv_i^{-1}g^{-1})u_j})$$

$$=\mathrm{vol}(\mathrm{Ker}(\psi))\sum_{i=1}^t\ \sum_{j=1}^s\ \phi((gv_i^{-1}g^{-1})u_j)^{-1}\chi_{u_j^{-1}(gv_ig^{-1})\mathrm{Ker}(\phi)gg_1^{-1}}$$

$$=\mathrm{vol}(Ker(\psi))g_1g^{-1}\cdot f_{(H,\phi)},$$

by Lemma 4.32. □

Remark 4.40. (i) Theorem 4.39 has shown that, under the special conditions which were stated at the start of this section, the morphisms of the monomial category ${}_{k[G]}\mathbf{mon}$ of Definition 4.24 are given in terms of the convolution product of §4.31 of the Hecke algebra $\mathcal{H}_G$.

(ii) My belief is that Theorem 4.39 remains true in general, in some sense, providing that all continuous characters $\phi : H \longrightarrow k^*$ have finite image when H is compact, open. This belief is based on the following: [19] claims to construct for each admissible representation V of G a monomial resolution in the derived category ${}_{k[G]}\mathbf{mon}$[6] and (see §9; also [9] pp.2-3) such V are intimately related to Hecke modules. Therefore one should expect a connection between the morphisms in that resolution and convolutions products.

The difficulty, in the case of a general locally profinite group G, with the treatment of this section is that $X_c(H, \phi)$'s are spaces of locally constant functions which are compactly supported modulo H, rather than actually being compactly supported.

It might be that I can get away with using the Schwartz space of locally constant, compactly supported functions of $G/Z(G)$, but I have not yet had time to examine this generalisation[7].

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[6] In a later section I shall give a self-contained construction of these resolutions based on the hyperHecke algebra and which applies to any V is $\mathcal{M}_{cmc,\underline{\phi}}(G)$-admissible V.

[7] To that end, as a novice, I should re-read §9, several sections of [7] on Hecke modules and the material of ([17] §1.11 p.63).

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