THE SECOND CHERN CLASS IN THE COHOMOLOGY OF GROUPS

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Contents

1. Introduction

This article has a string of Appendices. These are for the reader's convenience - giving formulae, group actions etc etc. However many are very elementary and I leave it to the reader's discretion to ignore them as appropriate!

Let G be a finite group and ρ a finite-dimensional complex representation of G. Then the first Chern class $c_1(\rho) \in H^2(G;\mathbb{Z})$ is equal to the coboundary of the determinant homomorphism $\det(\rho) : G \longrightarrow \mathbb{C}^*$ in the long exact cohomology sequence of

$$\mathbb{Z} \cong \mathbb{Z} \cdot (2\pi\sqrt{-1}) \longrightarrow \mathbb{C} \stackrel{exp}{\longrightarrow} \mathbb{C}^*.$$

This article recalls how explicitly to calculate the second Chern class $c_2(\rho) \in H^4(G;\mathbb{Z})$ in the case when $\rho = \mathrm{Ind}_H^G(\phi)$, the representation of G induced from a one-dimensional complex representation ϕ of H in terms of $c_1(\phi) = \lambda$. In general the Explicit Brauer Induction formula of ([?], [?], [?], [?]) extends the material of this article to give an explicit formula for $c_2(\rho)$ for an arbitrary ρ.

If G is a finite group denote by $H^{**}_{ev}(G;\mathbb{Z})$ the set of elements of $\prod_{r \geq 0} H^{2r}(G;\mathbb{Z})$ of the form $\alpha = 1 + \alpha_1 +_2 + \ldots$ with $\alpha_r \in H^{2r}(G;\mathbb{Z})$. Multiplication in $H^*(G;\mathbb{Z})$ gives a group structure on $H^{**}_{ev}(G;\mathbb{Z})$ (see Appendix Seven). The norm map[1] $\mathcal{N}_H^G$ is a natural construction going from $H^{**}_{ev}(H;\mathbb{Z})$ to $H^{**}_{ev}(G;\mathbb{Z})$ which respects the above group multiplication.

In Appendix Eight an isomorphism

$$\mathcal{A}_1 : \mathrm{Hom}_{\Sigma_n}(W, \mathrm{Hom}_H(X,A)^{\otimes n}) \longrightarrow \mathrm{Hom}_{\Sigma_n \int H}(W \otimes X^{\otimes n}, A^{\otimes n})$$

is described. We shall use the particular case in which, if $\underline{B}_*G$ denotes the inhomogeneous bar resolution (see Appendix Seven) of G, $W = \underline{B}_*\Sigma_n$, $X = \underline{B}_*H$ and $A = \mathbb{Z}$ with the trivial group action.

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[1]Often referred to as the Evens transfer.

Set $\alpha = 1 + \lambda \in H^{**}_{ev}(H;\mathbb{Z})$ where $c_1(\phi) = \lambda$ was introduced earlier. Using the map A_1 an element $1 \int_{\Sigma_n}(1+\lambda) \in H^{**}_{ev}(\Sigma_n \int H;\mathbb{Z})$ is constructed in Appendix Seven. In Appendix Four, if $n = [G : H]$, a group homomorphism

$$\Phi : G \longrightarrow \Sigma_n \int H$$

is constructed, using the definitions and group actions of Appendix One, Appendix Two and Appendix Three.

The Evens transfer of $1 + \lambda$ is

$$\mathcal{N}_H^G(1+\lambda) = \Phi^*(1 \int_{\Sigma_n} (1+\lambda)) \in H^{**}_{ev}(G;\mathbb{Z})$$

as described in Appendix Seven.

To be more specific, the 4-dimensional component of $1 \int_{\Sigma_n}(1+\lambda)$ is obtained by choosing a $\Sigma_n \int H$-chain homotopy equivalence

$$\tilde{F} : \underline{B}_*\Sigma_n \int H \xrightarrow{\simeq} \underline{B}_*\Sigma_n \otimes (\underline{B}_*H)^{\otimes n}$$

where the $\Sigma_n \int H$-action on the target is described in Appendix Four and Appendix Five. Form $F = (\epsilon \otimes 1)\tilde{F}$ where ϵ is the augmentation in $\underline{B}_*\Sigma_n$.

The 4-dimensional component of $1 \int_{\Sigma_n}(1+\lambda)$ is represented by the map which sends a 4-chain z to $F(z) \in (\underline{B}_*H)^{\otimes n}$ and then composes $A_1^{-1}(F(z))$ with the 4-dimensional component of

$$(1+\lambda)^{\otimes n} \in \mathrm{Hom}(\underline{B}_*H, \mathbb{Z})^{\otimes n}.$$

In Appendix Nine F_* is computed in dimensions less than or equal to four when $n = 2$. The result is to show in this case that $1 \int_{\Sigma_n}(1+\lambda)$ is represented by the 4-cocycle of Example **??**.

In general we have the following result[2]

Theorem (See Appendix Ten: Example **??**)

With the notation introduced above the cohomology class $1 \int_{\Sigma_n} (1+\lambda) \in H^4(\Sigma_n \int H; \mathbb{Z})$ is represented by the 4-cocycle

$$\tilde{c}_4 \in \mathrm{Hom}_{\mathbb{Z}[\Sigma_n \int H]}(\underline{B}_4 \Sigma_n \int H, \mathbb{Z})$$

sending the 4-chain ($\hat{\sigma}, \sigma, \sigma', \sigma'', \sigma''' \in \Sigma_n$ and $\hat{h}_j, h_j, h'_j.h''_j, h'''_j \in H$)

$$z = (\hat{\sigma}, \hat{h}_1, \ldots)[(\sigma, h_1, \ldots)|(\sigma', h'_1, \ldots)|(\sigma'', h''_1, \ldots)|(\sigma''', h'''_1, \ldots)]$$

to

$$\tilde{c}_4(z) = \sum_{1 \leq i \neq j \leq n} \lambda[h_{\sigma^{-1}(i)} | h'_{(\sigma\sigma')^{-1}(i)}] \cdot \lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(j)} | h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(j)}].$$

If $\rho = \mathrm{Ind}_H^G(\phi)$ is the complex representation of the finite group G which was introduced earlier we denote by $c_n(\rho) \in H^{2n}(G; \mathbb{Z})$ the n-th Chern class of ρ.

Let $\mathcal{N}_2(\lambda)$, the component of $\mathcal{N}_H^G(1+\lambda)$[3] in dimension four. The above theorem states that $\mathcal{N}_2(\lambda) = \Phi^*[\tilde{c}_4]$.

From ([**?**] Theorem 4 p.190) we have the following formulae:

Let $\phi : H \longrightarrow \mathbb{C}^*$ be a character of a finite group H which is a subgroup of the finite group G of index n. As above let $\lambda = c_1(\phi) \in H^2(H; \mathbb{Z})$. Let $\pi = \mathrm{Ind}_H^G(1)$ then

$$c_2(\rho) = c_2(\mathrm{Ind}_H^G(\phi)) = c_{2,0}(\lambda) + c_{1,1}(\lambda) + c_{0,2}(\lambda)$$

where

(a) $c_{0,2}(\lambda) = \mathcal{N}_2(\lambda) = \Phi^*[\tilde{c}_4]$,
(b) $c_{1,1}(\lambda) = c_1(\pi) \cdot \mathrm{Tr}_H^G(\lambda)$.
(c) $c_{2,0}(\lambda) = c_2(\pi)$.

The above formulae simplify considerably if we map to $H^4(G; \mathbb{Z}[1/6])$. This follows from two facts about the (co)homology of the infinite symmetric group, which is defined to be the union $\Sigma_\infty = \bigcup_{n \geq 0} \Sigma_n$ where Σ_n embeds into Σ_{n+k} by extending each permutation to be the identity on $\{n+1, \ldots, n+k\}$. The first fact is that for all n the homology map $H_*(\Sigma_n; R) \longrightarrow H_*(\Sigma_{n+1}; R)$ is split injective for any coefficent group R. This was first proved in [**?**] but it also follows immediately (see [**?**]) from the Snaith splitting in stable homotopy theory [**?**], which states inter alia that the canonical map from the classifying space of Σ_n to that of Σ_{n+1} is split injective in the stable homotopy category. The second fact is that the (co)homology of Σ_∞ has only 2- and 3-primary

[2] I shall not give a proof of this result in this article. An easier approach is to verify that this explicit 4-cocycle gives rise to a cohomology class $\tilde{c}_2(\rho)$ which satisfies the axiomatic characterisation of the second Chern class.

[3] $\mathcal{N}_{H \to G}(1+\lambda)$ in ([**?**] p.59)

torsion in dimensions up to and including four [**?**]. By naturality the classes $c_{2,0}(\lambda) = c_2(\pi)$ and $c_1(\pi)$ are pulled back from the cohomology of a symmetric group in dimension less than or equal to four and are therefore annihilated by inverting six.

Theorem

$$c_2(\rho) = c_2(\mathrm{Ind}_H^G(\phi)) = \Phi^*[\tilde{c}_4] \in H^4(G; \mathbb{Z}[1/6]).$$

2. Appendix One: Symmetric Groups

This very elementary section is just a reminder of the left action of Σ_n on the set $\{1, 2, \ldots, n\}$.

The simplest way to depict the symmetric group acting by a left action on a set of n objects is to take the $n \times n$ permutation matrices acting by left matrix multiplication on the column vectors

$$\underline{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

where the 1 is in the i-th entry of $\underline{e}_i$. Then, to a permutation σ, we assign the matrix X_σ where $(X_\sigma)_{u,v} = 1$ if $u = \sigma(v)$ and zero otherwise. Hence the matrix product $X_\sigma \cdot X_{\sigma'}$ satisfies

$$(X_\sigma \cdot X_{\sigma'})_{u,v} = \sum_{a=1}^{n} (X_\sigma)_{u,a}(X_{\sigma'})_{a,v} = \sum_{u=\sigma(a), a=\sigma'(v)} (X_\sigma)_{u,a}(X_{\sigma'})_{a,v}$$

which is equal to 1 if $u = \sigma(\sigma'(v))$ and zero otherwise. Hence

$$X_\sigma \cdot X_{\sigma'} = X_{\sigma \cdot \sigma'}$$

where $(\sigma \cdot \sigma')(v) = \sigma(\sigma'(v))$ that is, first apply σ' then apply σ.

We have the matrix product

$$(X_\sigma \underline{e}_i)_j = \sum_{a=1}^{n} (X_\sigma)_{j,a}(\underline{e}_i)_a = (X_\sigma)_{j,i}$$

which 1 if $j = \sigma(i)$ and zero otherwise. Hence

$$X_\sigma \underline{e}_i = \underline{e}_{\sigma(i)}.$$

The symmetric group Σ_n acts as left bijective transformations on the set of ordered n-tuples chosen from the set $\{1, 2, \ldots, n\}$ so that $(\sigma\sigma')(i) = \sigma(\sigma'(i))$.

Hence, under the bijection

$$\{1, \ldots, n\} \leftrightarrow \{\underline{e}_1, \ldots, \underline{e}_n\}$$

sending i to $\underline{e}_i$, the left matrix multiplication action on the right-hand set corresponds to the left action of Σ_n on the left-hand set by $\sigma \mapsto X_\sigma$.

3. Appendix Two: Wreath Products of Symmetric Groups

In [?] the wreath product $\Sigma_{n_1} \int \Sigma_{n_2}$ may be considered as the group of permutations of $\{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\}$ given by

$$\theta(i, j) = (\sigma(i), \tau_i(j)).$$

On this set we shall take the lexicographical order

$$\begin{array}{ll}
j = 1 & 1, 2, \ldots, \ldots, n_1, \\
j = 2 & n_1 + 1, n_1 + 2, \ldots, \ldots, 2n_1, \\
j = 3 & 2n_1 + 1, 2n_1 + 2, \ldots, \ldots, 3n_1, \\
 & \vdots \quad \vdots \quad \vdots \\
j & (j-1)n_1 + 1, (j-1)n_2 + 2, \ldots, \ldots, jn_1, \\
 & \vdots \quad \vdots \quad \vdots \\
j = n_2 - 1 & (n_2 - 2)n_1 + 1, (n_2 - 2)n_1 + 2, \ldots, \ldots, n_1(n_2 - 1), \\
j = n_2 & (n_2 - 1)n_1 + 1, (n_2 - 1)n_1 + 2, \ldots, \ldots, n_1 n_2,
\end{array}$$

where the order in this array increases left to right in each row and everything in the j-th row is greater than anything in the $(j-1)$-th row. We consider (i, j) as being the i-th entry in the j-th row so that $(i, j) \equiv (j-1)n_1 + i$. Therefore if

$$\theta = (\sigma, \tau_1, \tau_2, \ldots, \tau_{n_1}) \in \Sigma_{n_1} \times \Sigma_{n_2} \times \ldots \Sigma_{n_2}$$

then $\theta(i, j) = (\sigma(i), \tau_i(j)) \equiv (\tau_i(j) - 1)n_1 + \sigma(i)$.

Hence the composition

$$\theta\theta' = (\sigma, \tau_1, \tau_2, \ldots, \tau_{n_1})(\sigma', \tau_1', \tau_2', \ldots, \tau_{n_1}')$$

is given by

$$(i, j) \equiv (j-1)n_1 + i$$

$$\theta' \quad \downarrow$$

$$(\sigma'(i), \tau_i'(j)) \equiv (\tau_i'(j) - 1)n_1 + \sigma'(i)$$

$$\theta \quad \downarrow$$

$$(\sigma(\sigma'(i)), \tau_{\sigma'(i)}(\tau_i'(j)) \equiv (\tau_{\sigma'(i)}(\tau_i'(j)) - 1)n_1 + \sigma(\sigma'(i)).$$

In other words

$$(\sigma, \tau_1, \tau_2, \ldots)(\sigma', \tau_1', \tau_2', \ldots) = (\sigma\sigma', \tau_{\sigma'(1)}\tau_1', \tau_{\sigma'(2)}\tau_2', \ldots).$$

Alternatively we can think of the above lexicographically ordered set as the standard basis for $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ given by $\underline{e}_i \otimes \underline{d}_j$ for $1 \leq i \leq n_1$, $1 \leq j \leq n_2$.

Match these sets by the display

$$
\begin{array}{ll}
j=1 & \underline{e}_1\otimes\underline{d}_1, \underline{e}_2\otimes\underline{d}_1, \ldots, \underline{e}_{n_1}\otimes\underline{d}_1 \\
j=2 & \underline{e}_1\otimes\underline{d}_2, \underline{e}_2\otimes\underline{d}_2, \ldots, \underline{e}_{n_1}\otimes\underline{d}_2 \\
j=3 & \underline{e}_1\otimes\underline{d}_3, \underline{e}_2\otimes\underline{d}_3, \ldots, \underline{e}_{n_1}\otimes\underline{d}_3 \\
 & \vdots \quad \vdots \quad \vdots \\
j & \underline{e}_1\otimes\underline{d}_j, \underline{e}_2\otimes\underline{d}_j, \ldots, \underline{e}_{n_1}\otimes\underline{d}_j \\
 & \vdots \quad \vdots \quad \vdots \\
j=n_2-1 & \underline{e}_1\otimes\underline{d}_{n_2-1}, \underline{e}_2\otimes\underline{d}_{n_2-1}, \ldots, \underline{e}_{n_1}\otimes\underline{d}_{n_2-1} \\
j=n_2 & \underline{e}_1\otimes\underline{d}_{n_2}, \underline{e}_2\otimes\underline{d}_{n_2}, \ldots, \underline{e}_{n_1}\otimes\underline{d}_{n_2}
\end{array}
$$

Consider the left permutation action on the above set of vectors where $(\sigma, \tau_1, \ldots, \tau_{n_2})$ sends $\underline{e}_i \otimes \underline{d}_j$ to $(X_\sigma \cdot \underline{e}_i) \otimes (Y_{\tau_i} \cdot \underline{d}_j)$ where Y_{τ_i} is the $n_2 \times n_2$ permutation matrix for which $(Y_{\tau_i})_{u,v} = 1$ if $u = \tau_i(v)$ and zero otherwise.

Therefore we obtain

$$(X_\sigma \cdot \underline{e}_i) \otimes (Y_{\tau_i} \cdot \underline{d}_j) = \underline{e}_{\sigma(i)} \otimes \underline{d}_{\tau_i(j)}$$

which corresponds to $(i, j) \mapsto (\sigma(i), \tau_i(j))$.

Now consider the matrix multiplication composition

$$(\sigma, \tau_1, \tau_2, \ldots)(\sigma', \tau_1', \tau_2', \ldots)$$

which first applies

$$(X_{\sigma'} \cdot \underline{e}_i) \otimes (Y_{\tau_i'} \cdot \underline{d}_j) = \underline{e}_{\sigma'(i)} \otimes \underline{d}_{\tau_i'(j)}$$

and then applies

$$(X_\sigma \cdot \underline{e}_{\sigma'(i)}) \otimes (Y_{\tau_{\sigma'(i)}} \cdot \underline{d}_{\tau_i'(j)}) = \underline{e}_{\sigma(\sigma'(i))} \otimes \underline{d}_{\tau_{\sigma'(i)}(\tau_i'(j))}$$

Therefore the product is given by

$$(\sigma, \tau_1, \tau_2, \ldots)(\sigma', \tau_1', \tau_2', \ldots) = (\sigma\sigma', \tau_{\sigma'(1)}\tau_1', \tau_{\sigma'(2)}\tau_2', \ldots),$$

which coincides with the product from the previous point of view.

This product is associative because left composition by bijective transformations (or left matrix multiplication) is always associative. The associativity formula is given by

$$\theta(\theta'(\theta''(i, j))) = (\sigma(\sigma'(\sigma''(i)), \tau_{\sigma'(\sigma''(i))}(\tau'_{\sigma''(i)}(\tau_i''(j)))).$$

4. Appendix Three: Semi-Direct Products in General

Similarly we may form the semi-direct product $\Sigma_n \int H$ given by the set

$$\Sigma_n \times H \times H \times \ldots \times H \qquad (n \text{ copies of } H)$$

with multiplication given by the following convention

$$(\sigma, h_1, \ldots, h_n) \cdot (\sigma', h_1', \ldots, h_n') = (\sigma\sigma', h_{\sigma'(1)}h_1', h_{\sigma'(2)}h_2', \ldots).$$

Let us pause to verify that this definition works for general H[4]. We have

$$((\sigma, h_1, \ldots, h_n) \cdot (\sigma', h'_1, \ldots, h'_n)) \cdot (\sigma'', h''_1, \ldots, h''_n)$$

$$= (\sigma\sigma', h_{\sigma'(1)}h'_1, h_{\sigma'(2)}h'_2, \ldots) \cdot (\sigma'', h''_1, \ldots, h''_n)$$

$$= (\sigma\sigma'\sigma'', h_{\sigma'(\sigma''(1))}h'_{\sigma''(1)}h''_1, \ldots, h_{\sigma'(\sigma''(n))}h'_{\sigma''(n)}h''_n)$$

while

$$(\sigma, h_1, \ldots, h_n) \cdot ((\sigma', h'_1, \ldots, h'_n)) \cdot (\sigma'', h''_1, \ldots, h''_n))$$

$$= (\sigma, h_1, \ldots, h_n) \cdot (\sigma'\sigma'', h'_{\sigma''(1)}h''_1, \ldots, h'_{\sigma''(n)}h''_n)$$

$$= (\sigma\sigma'\sigma'', h_{(\sigma'\sigma'')(1)}h'_{\sigma''(1)}h''_1, \ldots, h_{(\sigma'\sigma'')(n)}h'_{\sigma''(n)}h''_n).$$

Therefore this multiplication is associative.

Note that

$$((\sigma^{-1}, 1, \ldots, 1) \cdot (1, h'_1, \ldots, h'_n)) \cdot (\sigma, 1, \ldots, 1)$$

$$= (1, h'_{\sigma(1)}, \ldots, h'_{\sigma(n)}),$$

which is the same convention as ([**?**] p.54)[5].

5. Appendix Four: The Homomorphism $\Phi_{\underline{x}} : G \longrightarrow \Sigma_n \int H$

Now suppose that $x_1, \ldots, x_n$ are coset representatives for $G/H = \{x_1H, \ldots, x_nH\}$. Therefore there is a homomorphism $\pi : G \longrightarrow \Sigma_n$ such that

$$gx_i = x_{\pi(g)(i)}h_i(g)$$

where $h_i(g) \in H$. Since

$$\begin{aligned} g(g'(x_i)) &= gx_{\pi(g')(i)}h_i(g') \\ &= x_{\pi(g)(\pi(g')(i))}h_{\pi(g')(i)}(g)h_i(g') \\ &= x_{(\pi(gg')(i)}h_{\pi(g')(i)}(g)h_i(g') \end{aligned}$$

we see that

$$h_i(gg') = h_{\pi(g')(i)}(g)h_i(g').$$

If we set

$$\Phi_{\underline{x}}(g) = (\pi(g), h_1(g), h_2(g), \ldots, h_n(g)) \in \Sigma_n \int H$$

[4]In the 3rd line of the first half of the verification one sees how permutations must be interpreted in order to make this convention for the wreath product work. A permutation $\sigma \in \Sigma_n$ acts on an ordered n-tuple by being applied to the suffices $1, 2, 3 \ldots$. For example applying σ to an n-tuple $(a_{Q(1)}, \ldots, a_{Q(n)})$, where Q is a function sending the set $\{1, \ldots, n\}$ to itself, we obtain $(a_{Q(\sigma(1))}, \ldots, a_{Q(\sigma(n))})$

[5]This is quite an eccentric convention in the sense that, in a universe of left actions, here the semi-direct product features Σ_n acting on H^n in the right!

we find that

$$\Phi_{\underline{x}}(gg')$$

$$= (\pi(gg'), h_1(gg'), h_2(gg'), \ldots, h_n(gg'))$$

$$= (\pi(g)\pi(g'), h_{\pi(g')(1)}(g)h_1(g'), h_{\pi(g')(2)}(g)h_2(g'), \ldots, h_{\pi(g')(n)}(g)h_n(g'))$$

$$= (\pi(g), h_1(g), \ldots, h_n(g)) \cdot (\pi(g'), h_1'(g'), \ldots, h_n'(g'))$$

$$= \Phi_{\underline{x}}(g)\Phi_{\underline{x}}(g')$$

so that

$$\Phi_{\underline{x}} : G \longrightarrow \Sigma_n \int H$$

is a homomorphism, depending on the choice of coset representatives.

If we change $x_1, \ldots, x_n$ to another set of coset representatives $y_1, \ldots, y_n$ then we may write

$$y_i = x_{\alpha(i)}\beta_i$$

where $\alpha \in \Sigma_n$ and $\beta_i \in H$. To obtain the relation between $\Phi_{\underline{x}}$ and $\Phi_{\underline{y}}$ we calculate

$$\begin{aligned} gy_i &= gx_{\alpha(i)}\beta_i \\ &= x_{\pi(g)(\alpha(i))}h_{\alpha(i)}(g)\beta_i \\ &= x_{\alpha(\alpha^{-1}(\pi(g)(\alpha(i))))}h_{\alpha(i)}(g)\beta_i \\ &= y_{\alpha^{-1}(\pi(g)(\alpha(i)))}\beta_{\alpha^{-1}(\pi(g)(\alpha(i)))}h_{\alpha(i)}(g)\beta_i. \end{aligned}$$

We have

$$(\alpha^{-1}, (\beta_{\alpha^{-1}(1)})^{-1}, \ldots, (\beta_{\alpha^{-1}(n)})^{-1})(\alpha, \beta_1, \ldots, \beta_n) = (1, 1, \ldots, 1)$$

so if we define

$$\mu = (\alpha, \beta_1, \ldots, \beta_n)$$

then

$$\mu^{-1}\Phi_{\underline{x}}(g)\mu$$

$$= (\alpha^{-1}, (\beta_{\alpha^{-1}(1)})^{-1}, \ldots)(\pi(g), h_1(g), \ldots, h_n(g))(\alpha, \beta_1, \ldots, \beta_n)$$

$$= (\alpha^{-1}, (\beta_{\alpha^{-1}(1)})^{-1}, \ldots)(\pi(g)\alpha, h_{\alpha(1)}(g)\beta_1, \ldots)$$

$$= (\alpha^{-1}\pi(g)\alpha, (\beta_{\alpha^{-1}(\pi(g)\alpha(1))})^{-1}h_{\alpha(1)}(g)\beta_1, \ldots)$$

so that

$$\Phi_{\underline{y}} = (\mu^{-1} - \mu) \cdot \Phi_{\underline{x}}.$$

Now suppose that A is a left H-module. Then $A^{\otimes n}$ is a left $\Sigma_n \int H$-module via the action

$$(1, h_1, \ldots, h_n)(a_1 \otimes \ldots \otimes a_n) = h_1(a_1) \otimes \ldots \otimes h_n(a_n)$$

$$(\sigma, 1, 1, \ldots, 1)(a_1 \otimes \ldots \otimes a_n) = a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)}$$

To verify that this does give an action we must show that

$$(\sigma, 1, \ldots, 1)(1, h_{\sigma(1)}, \ldots, h_{\sigma(n)}) \text{ and } (1, h_1, \ldots, h_n)(\sigma, 1, \ldots, 1)$$

act in the same manner.

However

$$(\sigma, 1, \ldots, 1)(1, h_{\sigma(1)}, \ldots, h_{\sigma(n)})(a_1 \otimes \ldots \otimes a_n)$$

$$= (\sigma, 1, \ldots, 1)(h_1(a_1) \otimes \ldots \otimes h_n(a_n)$$

$$= h_1(a_{\sigma^{-1}(1)}) \otimes \ldots \otimes h_n(a_{\sigma^{-1}(n)})$$

while

$$(1, h_{\sigma(1)}, \ldots, h_{\sigma(n)})(\sigma, 1, \ldots, 1)(a_1 \otimes \ldots \otimes a_n)$$

$$= (1, h_{\sigma(1)}, \ldots, h_{\sigma(n)})(a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)})$$

$$= h_1(a_{\sigma^{-1}(1)}) \otimes \ldots \otimes h_n(a_{\sigma^{-1}(n)}),$$

which agree, as required.

6. Appendix Five: The signed Σ_n-action on $C_*^{\otimes n}$

Now suppose that

$$C_* : \qquad \ldots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \ldots \xrightarrow{d} C_0 \longrightarrow 0$$

is a chain complex of H-modules.

Writing $|a|$ for the degree of a, define a differential D on $C_*^{\otimes n}$ by the formula

$$D(c_1 \otimes \ldots \otimes c_n) = \sum_{i=1}^{n} (-1)^{|c_1|+\ldots+|c_{i-1}|} c_1 \otimes \ldots \otimes c_{i-1} \otimes d(c_i) \otimes \ldots \otimes c_n.$$

By induction on n we see that D is a differential. For $n = 2$ we find that, since $dd = 0$,

$$D(D(c_1 \otimes c_2))$$

$$= D(d(c_1) \otimes c_2 + (-1)^{|c_1|} c_1 \otimes d(c_2))$$

$$= (-1)^{|c_1|-1} d(c_1) \otimes d(c_2) + (-1)^{|c_1|} d(c_1) \otimes d(c_2)$$

$$= 0.$$

Now on $C_*^{\otimes n}$

$$D(c_1 \otimes \ldots \otimes c_n) = d(c_1) \otimes (c_2 \otimes \ldots \otimes c_n) + (-1)^{|c_1|} c_1 \otimes D'(c_2 \otimes \ldots \otimes c_n)$$

where, by induction, D' is a differential and therefore from the case $n = 2$ we see that D is also a differential.

Also D commutes with the H^n-action on $C_*^{\otimes n}$.

Next we need the action of Σ_n on the complex $C_*^{\otimes n}$. If τ switches 1 and 2 and $n = 2$ then

$$\tau(c_1 \otimes c_2) = (-1)^{|c_1||c_2|} c_2 \otimes c_1$$

commutes with D since

$$\tau(D(c_1 \otimes c_2))$$

$$= \tau(d(c_1) \otimes c_2 + (-1)^{|c_1|} c_1 \otimes d(c_2))$$

$$= (-1)^{(|c_1|-1)|c_2|} c_2 \otimes d(c_1) + (-1)^{|c_1|+|c_1|(|c_2|-1)} d(c_2) \otimes c_1$$

while

$$D(\tau(c_1 \otimes c_2))$$

$$= D((-1)^{|c_1||c_2|} c_2 \otimes c_1)$$

$$= (-1)^{|c_1||c_2|} d(c_2) \otimes c_1 + (-1)^{|c_1||c_2|+|c_2|} c_2 \otimes d(c_1),$$

as required.

Next suppose that τ switches i_1 and $i_2 = i_1 + 1$. Set

$$\tau(c_1 \otimes \ldots \otimes c_{i_1} \otimes c_{i_2} \otimes \ldots \otimes c_n) = (-1)^{|c_{i_1}||c_{i_2}|}\ c_1 \otimes \ldots \otimes c_{i_2} \otimes c_{i_1} \otimes \ldots \otimes c_n.$$

The terms in $\tau(D(c_1 \otimes \ldots \otimes c_{i_1} \otimes c_{i_2} \otimes \ldots \otimes c_n))$ and $D(\tau(c_1 \otimes \ldots \otimes c_{i_1} \otimes c_{i_2} \otimes \ldots \otimes c_n))$ which do not involve $d(c_{i_1})$ or $d(c_{i_2})$ clearly coincide.

For the other terms $D(\tau(c_1 \otimes \ldots))$ contains

$$(-1)^{|c_{i_1}||c_{i_2}|+|c_1|+\ldots+|c_{i_1-1}|}\ \ldots d(c_{i_2}) \otimes c_{i_1} \otimes \ldots$$

$$+(-1)^{|c_{i_1}||c_{i_2}|+|c_1|+\ldots+|c_{i_1-1}|+|c_{i_2}|}\ \ldots c_{i_2} \otimes d(c_{i_1}) \otimes \ldots$$

whereas $\tau(D(c_1 \otimes \ldots))$ contains

$$(-1)^{|c_1|+\ldots+|c_{i_1-1}|+|c_{i_2}|(|c_{i_1}|-1)}\ \ldots c_{i_2} \otimes d(c_{i_1}) \otimes \ldots$$

$$(-1)^{|c_1|+\ldots+|c_{i_1-1}|+|c_{i_1}|+|c_{i_1}|(|c_{i_2}|-1)}\ \ldots d(c_{i_2}) \otimes c_{i_1} \otimes \ldots$$

which also coincide.

Next we need to check that the signed action gives a $\Sigma_n \int H$-action. For this we may take τ switching i_1 and $i_2 = i_1 + 1$. Then we find that

$$(1, h_1, \ldots, h_n)(\tau, 1, \ldots, 1)(c_1 \otimes \ldots \otimes c_n)$$

$$= (-1)^{|c_1||c_{i_2}|}(1, h_1, \ldots, h_n)(c_1 \otimes \ldots \otimes c_{i_1-1} \otimes c_{i_2} \otimes c_{i_1} \otimes c_{i_2+1} \otimes \ldots)$$

$$= (-1)^{|c_1||c_{i_2}|}(h_1(c_1) \otimes \ldots \otimes h_{i_1-1}(c_{i_1-1}) \otimes h_{i_1}(c_{i_2}) \otimes h_{i_2}(c_{i_1}) \otimes h_{i_2+1}(c_{i_2+1}) \otimes \ldots)$$

while

$$(\tau, h_1, \ldots, h_n)(c_1 \otimes \ldots \otimes c_n)$$

$$= \tau((h_1(c_1) \otimes \ldots \otimes h_{i_1-1}(c_{i_1-1}) \otimes h_{i_1}(c_{i_1}) \otimes h_{i_2}(c_{i_2}) \otimes h_{i_2+1}(c_{i_2+1}) \otimes \ldots))$$

$$= (-1)^{|c_1||c_{i_2}|}(h_1(c_1) \otimes \ldots \otimes h_{i_1-1}(c_{i_1-1}) \otimes h_{i_1}(c_{i_2}) \otimes h_{i_2}(c_{i_1}) \otimes h_{i_2+1}(c_{i_2+1}) \otimes \ldots)$$

which are equal, as required.

7. Appendix Six: The Monomial G-complex $\mathcal{M}_H^G(C_*)$

If A is an H-module we have the monomial G-module

$$\mathcal{M}_H^G(A) = \Phi_{\underline{x}}^*(A^{\otimes n})$$

(the notation is $\mathcal{M}_{H \to G}(A)$ in [**?**]).

Similarly, for chain complexes, we have

$$\mathcal{M}_H^G(C_*) = \Phi_{\underline{x}}^*(C_*^{\otimes n}).$$

If $\mathbb{Z}$ is a trivial H-module and $A = \mathbb{Z} \oplus B$ then

$$\mathcal{M}_H^G(A) = \mathbb{Z} \oplus \mathrm{Ind}_H^G(B) \oplus \ldots \oplus \mathcal{M}_H^G(B),$$

where the summands correspond to the number of copies of $\mathbb{Z}$ in the n-fold tensor product of $\mathbb{Z} \oplus B$.

8. Appendix Seven: The Norm Map

Let

$$C_* : \ 0 \longrightarrow C_0 \xrightarrow{d} C_1 \xrightarrow{d} \ldots \xrightarrow{d} C_{m-1} \xrightarrow{d} C_m \xrightarrow{d} \ldots$$

be an H-complex and for $\Sigma \subseteq \Sigma_n$ let W_* be a projective $\mathbb{Z}[\Sigma]$-resolution of $\mathbb{Z}$. Therefore W could be $\underline{B}_*\Sigma_n$, the bar-resolution of Σ_n.

Recall that the inhomogeneous bar resolution $\underline{B}_*G$ of G has degree m part $\underline{B}_m G$ given by the free left $\mathbb{Z}[G]$-module on m-tuples $[x_1|\ldots|x_m]$ for $m \geq 0$. The differential ∂ on the degree m part is the left $\mathbb{Z}[G]$-module homomorphism characterised by

$$\partial([x_1|\ldots|x_m])$$

$$= x_1[x_2|\ldots|x_m] + \sum_{i=1}^{m-1} (-1)^i [x_1|\ldots|x_i x_{i+1}|\ldots|x_m]$$

$$+(-1)^m [x_1|\ldots|x_{m-1}].$$

Write

$$D_* = \mathrm{Hom}_\Sigma(W_*, C_*^{\otimes n}).$$

Let us grade D_* in a cohomological way

$$D_t = \oplus_{q-m=t} \mathrm{Hom}_\Sigma(W_m, C_q^{\otimes n})$$

so that the total differential, d given by the formula below, will lower degree since $q - m = t$ goes to

$$q - 1 - m = q - (m+1) = t - 1.$$

The differentials are

$$d_W^* : \mathrm{Hom}_\Sigma(W_m, C_q^{\otimes n}) \longrightarrow \mathrm{Hom}_\Sigma(W_{m+1}, C_q^{\otimes n})$$

and

$$d_C : \mathrm{Hom}_\Sigma(W_m, C_q^{\otimes n}) \longrightarrow \mathrm{Hom}_\Sigma(W_m, C_{q+1}^{\otimes n}).$$

The formula for the total differential d is given by

$$d : D_t \longrightarrow D_{t-1}$$

is given by (if $q - m = t$ and $f : W_m \longrightarrow C_q^{\otimes n}$)

$$df = d_C f + (-1)^q f d_W.$$

This is a differential because

$$ddf = d_C d_C f + (-1)^{q+1} d_C f d_W + (-1)^q d_C f d_W \pm f d_W d_W = 0.$$

The homology of D_* depends only on the chain homotopy class of C_*.

Next we switch to the cohomological notation and write $H_{ev}^{**}(C_*)$ for

$$H_{ev}^{**}(C_*) = \prod_{r \geq 0} H^{2r}(C_*).$$

If C_* is a differential graded ring then $H_{ev}^{**}(C_*)$ inherits a graded ring structure.

Now set $C_* = \mathbb{Z}[T]$ with $\deg(T) = 1$ and having trivial differential and H-action. Then $C_*^{\otimes n} = \mathbb{Z}[T_1, T_2, \ldots, T_n]$ with $T_i = 1^{\otimes(i-1)} \otimes T \otimes 1^{\otimes(n-i)}$. We let Σ_n act by the signed permutations, as before, then

$$D_0 = \mathrm{Hom}_\Sigma(W_*, \mathbb{Z}[T_1, T_2, \ldots, T_n])$$

that is, W_m maps to the degree m part and

$$H^*(D_0) \cong H^*(\Sigma; \mathbb{Z}[T_1, T_2, \ldots, T_n]).$$

Also

$$H_{ev}^{**}(D_0) \cong H_{ev}^{**}(\Sigma; \mathbb{Z}[T_1, T_2, \ldots, T_n]).$$

Now suppose we have

$$\alpha = \alpha_0 + \alpha_2 + \ldots \in H_{ev}^{**}(C_*)$$

and let

$$a = a_0 + a_2 + \ldots$$

with $a_{2i} \in C_{2i}$ representing $\alpha_{2i} \in H^{2i}(C_*)$.

Define

$$F : \mathbb{Z}[T] \longrightarrow C_*$$

by $F(T^{2m}) = a_{2m}$ and $F(T^{2m+1}) = 0$. Therefore

$$F(\frac{1}{1-T^2}) = F(1) + F(T^2) + F(T^4) + \ldots = \alpha_0 + \alpha_2 + \ldots = \alpha.$$

Set

$$Y = \prod_1^n \frac{1}{(1-T_i^2)} \in \mathbb{Z}[[T_1, T_2, \ldots, T_n]]^\Sigma \cong H^0(\Sigma; \mathbb{Z}[[T_1, T_2, \ldots, T_n]]).$$

Define

$$1 \int_\Sigma \alpha = \mathrm{Hom}_\Sigma(1, F^{\otimes n})(Y) \in H^{**}_{ev}(\mathrm{Hom}_\Sigma(W_*, C_*^{\otimes n})).$$

This class is well-defined, natural in Σ, natural in C_*, associative and commutes with the external product.

If a represents α and $\epsilon : W \longrightarrow \mathbb{Z}$ is the augmentation then $\epsilon \int a$ represents $1 \int_\Sigma \alpha$ where

$$(\epsilon \int a)(w) = \epsilon(w)(a \otimes a \otimes \ldots \otimes a) \in \mathrm{Hom}_\Sigma(W_*, C_*^{\otimes n}).$$

Let $H, \Sigma \subseteq \Sigma_n$ etc be as above. Set

$$C_* = \mathrm{Hom}_H(X, A)$$

where X is a finite type projective H-resolution of $\mathbb{Z}$. We have a map, described in detail in Appendix Eight,

$$\mathcal{A}_1 : \mathrm{Hom}_\Sigma(W, \mathrm{Hom}_H(X, A)^{\otimes n}) \longrightarrow \mathrm{Hom}_{\Sigma \int H}(W \otimes X^{\otimes n}, A^{\otimes n})$$

and we may define

$$1 \int_\Sigma \alpha = \mathcal{A}_1^{**}(1 \int_\Sigma \alpha) \in H^{**}(\Sigma \int H, A^{\otimes n}).$$

This construction inherits naturality properties from the previous list. We have

$$1 \int_{\Sigma_n} \alpha \in H^{**}(\Sigma_n \int H, A^{\otimes n})$$

and

$$\Phi : G \longrightarrow \Sigma_n \int H$$

and therefore we may define

$$\mathcal{N}_H^G(\alpha) = \Phi^{**}(1 \int_{\Sigma_n} \alpha) \in H^{**}(G; \mathcal{M}_H^G(A)).$$

We are going to be interested in the case when $A = \mathbb{Z}$ with trivial H-action. So that $\mathcal{M}_H^G(\mathbb{Z}) \cong \mathbb{Z}$, the trivial G-module. Hence we have

$$\mathcal{N}_H^G(\alpha) = \Phi^{**}(1 \int_{\Sigma_n} \alpha) \in H^{**}(G; \mathbb{Z}).$$

9. Appendix Eight: The map $\mathcal{A}_1$

We have a map given in ([**?**] §5) (with $\Sigma = \Sigma_n$ in the notation of Appendix Seven)

$$\mathcal{A}_1 : \mathrm{Hom}_{\Sigma_n}(W, \mathrm{Hom}_H(X, A)^{\otimes n}) \longrightarrow \mathrm{Hom}_{\Sigma_n \int H}(W \otimes X^{\otimes n}, A^{\otimes n}).$$

In ([**?**] §3) the action of $\Sigma_n \int H$ on $A^{\otimes n}$ for a left H-module A is given by

$$(h_1, \ldots, h_n)(a_1 \otimes \ldots \otimes a_n) = h_1 a_1 \otimes \ldots \otimes h_n a_n$$

and

$$\sigma(a_1 \otimes \ldots \otimes a_n) = a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)}.$$

Let us check that this is a left-action. Clearly H^n acts by an left action. Also

$$\begin{aligned}
&(\sigma \cdot \sigma')(a_1 \otimes \ldots \otimes a_n) \\
&= a_{(\sigma\sigma')^{-1}(1)} \otimes \ldots \otimes a_{(\sigma\sigma')^{-1}(n)} \\
&= a_{\sigma'^{-1}(\sigma^{-1}(1))} \otimes \ldots \otimes a_{\sigma'^{-1}(\sigma^{-1}(n))} \\
&= \sigma(a_{\sigma'^{-1}(1)} \otimes \ldots \otimes a_{\sigma'^{-1}(n)}) \\
&= \sigma(\sigma'(a_1 \otimes \ldots \otimes a_n))
\end{aligned}$$

which is a left action by Σ_n.

Finally let us test the crucial relation

$$\begin{aligned}
&((\sigma^{-1}, 1, \ldots, 1) \cdot (1, h'_1, \ldots, h'_n)) \cdot (\sigma, 1, \ldots, 1) \\
&= (1, h'_{\sigma(1)}, \ldots, h'_{\sigma(n)}),
\end{aligned}$$

of ([**?**] p.54).

$$\begin{aligned}
&\sigma^{-1}((h'_1, \ldots, h'_n)(\sigma(a_1 \otimes \ldots \otimes a_n))) \\
&= \sigma^{-1}((h'_1, \ldots, h'_n)(a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)})) \\
&= \sigma^{-1}(h'_1 a_{\sigma^{-1}(1)} \otimes \ldots \otimes h'_n a_{\sigma^{-1}(n)}) \\
&= h'_{\sigma(1)} a_1 \otimes \ldots \otimes h'_{\sigma(n)} a_n \\
&= (h'_{\sigma(1)}, \ldots, h'_{\sigma(n)})(a_1 \otimes \ldots \otimes a_n)
\end{aligned}$$

as required.

When A is replaced by a chain complex of left H-modules, as in the definition of $\mathcal{A}_1$, the action by $\sigma \in \Sigma_n$ is given by the above action composed with the graded sign associated to σ.

Therefore the left action of $\Sigma_n \int H$ on $W \otimes X^{\otimes n}$ will be given by

$$(h_1, \ldots, h_n)(w \otimes x_1 \otimes \ldots \otimes x_n) = w \otimes h_1 a_1 \otimes \ldots \otimes h_n a_n$$

and

$$\sigma(w \otimes x_1 \otimes \ldots \otimes x_n) = SIGN(\sigma, \underline{x}) \sigma w \otimes x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(n)},$$

where $SIGN(\sigma, \underline{x}) = \pm 1$ is the sign depending on the permutation and the degrees of the x_i's.

Let us suppose that the H-module A is concentrated in degree zero and hence so is $A^{\otimes n}$. Therefore a homomorphism

$$\lambda \in \mathrm{Hom}_{\Sigma_n \int H}(W \otimes X^{\otimes n}, A^{\otimes n})$$

will satisfy the relations

$$(h_1 \otimes h_2 \otimes \ldots \otimes h_n)\lambda(w \otimes x_1 \otimes \ldots \otimes x_n)$$

$$= \lambda(w \otimes h_1 a_1 \otimes \ldots \otimes h_n a_n)$$

and

$$\sigma(\lambda(w \otimes x_1 \otimes \ldots \otimes x_n)$$

$$= \lambda(SIGN(\sigma, \underline{x}) \sigma w \otimes x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(n)}).$$

We have an isomorphism of (signed) left Σ_n-modules

$$\mu : \mathrm{Hom}_H(X, A)^{\otimes n} \xrightarrow{\cong} \mathrm{Hom}_{H^n}(X^{\otimes n}, A^{\otimes n})$$

given by

$$\mu(f_1 \otimes \ldots \otimes f_n)(x_1 \otimes \ldots \otimes x_n) = f_1(x_1) \otimes \ldots \otimes f_n(x_n).$$

With this identification

$$\mathcal{A}_1^{-1}(\lambda)(w)(x_1 \otimes \ldots \otimes x_n) = \lambda(w \otimes x_1 \otimes \ldots \otimes x_n) \in A^{\otimes n}$$

so that

$$\mathcal{A}_1^{-1}(\lambda)(\sigma' w)(x_{\sigma'^{-1}(1)} \otimes \ldots \otimes x_{\sigma'^{-1}(n)})$$

$$= \lambda(\sigma' w \otimes x_{\sigma'^{-1}(1)} \otimes \ldots \otimes x_{\sigma'^{-1}(n)})$$

$$= \sigma' \lambda(w \otimes x_1 \otimes \ldots \otimes x_n)$$

$$= \sigma' \mathcal{A}_1^{-1}(\lambda)(w)(x_1 \otimes \ldots \otimes x_n)$$

so that

$$\mathcal{A}_1^{-1}(\lambda) \in \mathrm{Hom}_{\Sigma_n}(W, \mathrm{Hom}_H(X, A)^{\otimes n}),$$

as required.

10. Appendix Nine: F_* when $n=2$ and $*\leq 4$

We shall need a $\Sigma_n \int H$-chain map of the form

$$F_* : \underline{B}_* \Sigma_n \int H \longrightarrow (\underline{B}_* H)^{\otimes n}.$$

Any two such maps covering the augmentations to $\mathbb{Z}$ are $\Sigma_n \int H$-chain homotopic.

Crucially we need F_4 and will obtain the formula for such a map by constructing F_0, F_1, F_2, F_3, F_4 inductively.

In this section I shall have a practice run in the case when $n = 2$. Even this is a rather gruesome calculation but since I have done it, I shall include the formulae - with a recommendation that reading it be skipped!

The formula found when $n = 2$ are

$$F_0([\ ]) = [\ ] \otimes [\ ],$$

$$F_1[(\sigma, h_1, h_2)] = [h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[\ ] + [\ ] \otimes [h_{\sigma^{-1}(2)}],$$

$$F_2[(\sigma, h_1, h_2)|(\sigma', h'_1, h'_2)]$$

$$= [h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]$$

$$+[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}] + [\ ] \otimes [h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}],$$

$$F_3([(\sigma, h_1, h_2)|(\sigma', h'_1, h'_2)|(\sigma'', h''_1, h''_2)])$$

$$= [h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]$$

$$+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1})(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]$$

$$+[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]$$

$$+[\ ] \otimes [h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}],$$

$$F_4([(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')])$$

$$= [h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}]$$

$$\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ]$$

$$+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]$$

$$\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]$$

$$\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+[h_{\sigma^{-1}(1)}]$$

$$\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+[\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}].$$

The details of the verification that $F_i d = dF_{i+1}$ in low dimensions when $n = 2$ are given below.

$$d(F_1([(\sigma,h_1,h_2)]))$$

$$= h_{\sigma^{-1}(1)}[\ ]\otimes h_{\sigma^{-1}(2)}[\ ] - [\ ]\otimes h_{\sigma^{-1}(2)}[\ ] + [\ ]\otimes h_{\sigma^{-1}(2)}[\ ] - [\ ]\otimes[\ ]$$

$$= F_0(d([(\sigma,h_1,h_2)])).$$

Since

$$d([(\sigma,h_1,h_2)|(\sigma',h_1',h_2')])$$

$$= \sigma((h_1,h_2)([(\sigma',h_1',h_2')])) - [\sigma\sigma',h_{\sigma'(1)}h_1',h_{\sigma'(2)}h_2'] + [(\sigma,h_1,h_2)]$$

we have

$$
\begin{aligned}
&F_1(d([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)]))\\
&=\sigma((h_1,h_2)([h'_{\sigma'^{-1}(1)}]\otimes h'_{\sigma'^{-1}(2)}[\ ]))\\
&\qquad+\sigma((h_1,h_2)([\ ]\otimes[h'_{\sigma'^{-1}(2)}]))\\
&\qquad-[h_{\sigma'(\sigma\sigma')^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma'(\sigma\sigma')^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]\\
&\qquad-[\ ]\otimes[h_{\sigma'(\sigma\sigma')^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}]\\
&\qquad+[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[\ ]+[\ ]\otimes[h_{\sigma^{-1}(2)}]\\
&=h_{\sigma^{-1}(1)}[h'_{\sigma'^{-1}\sigma^{-1}((1))}]\otimes h_{\sigma^{-1}(2)}h'_{\sigma'^{-1}\sigma^{-1}((2))}[\ ]\\
&\qquad+h_{\sigma^{-1}(1)}[\ ]\otimes h_{\sigma^{-1}(2)}[h'_{\sigma'^{-1}\sigma^{-1}((2))}])\\
&\qquad-[h_{\sigma'(\sigma\sigma')^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma'(\sigma\sigma')^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]\\
&\qquad-[\ ]\otimes[h_{\sigma'(\sigma\sigma')^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}]\\
&\qquad+[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[\ ]+[\ ]\otimes[h_{\sigma^{-1}(2)}]\\
&=h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]\\
&\qquad+h_{\sigma^{-1}(1)}[\ ]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}])\\
&\qquad-[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]\\
&\qquad-[\ ]\otimes[h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}]\\
&\qquad+[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[\ ]+[\ ]\otimes[h_{\sigma^{-1}(2)}]
\end{aligned}
$$

In order to verify the relation $dF_2 = F_1 d$ we have

$$d([h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ])$$

$$+d([h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}]) + d([\ ] \otimes [h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}])$$

$$= h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]$$

$$-[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]$$

$$+[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]$$

$$+h_{\sigma^{-1}(1)}[\ ] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}]) - [\ ] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}]$$

$$-[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ] + [h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[\ ]$$

$$+[\ ] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}] - [\ ] \otimes [h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}]$$

$$+[\ ] \otimes [h_{\sigma^{-1}(2)}]$$

$$= h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]$$

$$-[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]$$

$$+h_{\sigma^{-1}(1)}[\ ] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}])$$

$$+[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[\ ]$$

$$-[\ ] \otimes [h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}] + [\ ] \otimes [h_{\sigma^{-1}(2)}],$$

as required.

Next consider the differential

$$d([(\sigma, h_1, h_2)|(\sigma', h'_1, h'_2)|(\sigma'', h''_1, h''_2)])$$

$$= \sigma((h_1, h_2)((\sigma', h'_1, h'_2)|(\sigma'', h''_1, h''_2)]))$$

$$-[(\sigma\sigma', h_{\sigma'(1)}h'_1, h_{\sigma'(2)}h'_2)|(\sigma'', h''_1, h''_2)]$$

$$+[(\sigma, h_1, h_2)|(\sigma'\sigma'', h'_{\sigma''(1)}h''_1, h'_{\sigma''(2)}h''_2)]$$

$$-[(\sigma, h_1, h_2)|(\sigma', h'_1, h'_2)].$$

Now we apply F_2 to each of the four terms in $d([(\sigma, h_1, h_2)|(\sigma', h'_1, h'_2)|(\sigma'', h''_1, h''_2)])$.

The first term yields

$$F_2(\sigma((h_1,h_2)((\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')])))$$

$$= \sigma((h_1,h_2)(F_2((\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')])))$$

$$= \sigma((h_1,h_2)([h'_{\sigma'^{-1}(1)}|h''_{(\sigma'\sigma'')^{-1}(1)}] \otimes h'_{\sigma'^{-1}(2)}h''_{(\sigma'\sigma'')^{-1}(2)}[\ ])$$

$$+\sigma((h_1,h_2)([h'_{\sigma'^{-1}(1)}] \otimes h'_{\sigma'^{-1}(2)}[h''_{(\sigma'\sigma'')^{-1}(2)}])$$

$$+\sigma((h_1,h_2)([\ ] \otimes [h'_{\sigma'^{-1}(2)}|h''_{(\sigma'\sigma'')^{-1}(2)}])$$

$$= h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]$$

$$+h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]$$

$$+h_{\sigma^{-1}(1)}[\ ] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}].$$

The second term yields

$$-F_2([(\sigma\sigma',h_{\sigma'(1)}h_1',h_{\sigma'(2)}h_2')|(\sigma'',h_1'',h_2'')])$$

$$= -[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]$$

$$-[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]$$

$$-[\ ] \otimes [h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}|[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}].$$

The third term yields

$$F_2([(\sigma,h_1,h_2)|(\sigma'\sigma'',h'_{\sigma''(1)}h_1'',h'_{\sigma''(2)}h_2'')])$$

$$= [h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]$$

$$+[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]$$

$$+[\ ] \otimes [h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]$$

and the fourth term is equal to $-F_2[(\sigma,h_1,h_2)|(\sigma',h_1',h_2')]$, which is given by the formula mentioned earlier.

Collecting these terms together gives

$$
\begin{aligned}
&F_2(d([(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')]))\\
&= h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]\\
&\qquad\qquad +h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad\qquad +h_{\sigma^{-1}(1)}[\ ]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&-[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]\\
&\qquad\qquad -[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad\qquad -[\ ]\otimes[h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]\\
&\qquad\qquad +[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad\qquad +[\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad\qquad -[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]\\
&\qquad\qquad -[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}]\\
&\qquad\qquad -[\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}].
\end{aligned}
$$

The sum of the first, fourth and seventh terms in the above expression is equal to

$$
\begin{aligned}
&d([h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ])\\
&+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ].
\end{aligned}
$$

Therefore we obtain a simplification of the form

$$
\begin{aligned}
&F_2(d([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)]))\\
&-d([h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ])\\
&= \qquad h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad +h_{\sigma^{-1}(1)}[\ ]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad -[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad -[\ ]\otimes[h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad +[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad +[\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad -[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]\\
&\qquad -[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}]\\
&\qquad -[\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}]\\
&+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]
\end{aligned}
$$

In the simplification the fourth, sixth and ninth terms are equal to

$$
d([\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}])-[\ ]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]
$$

which yields a further simplification

$$
\begin{aligned}
&F_2(d([(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')]))\\
&-d([h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ])\\
&-d([\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}])\\
&= \qquad h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad +h_{\sigma^{-1}(1)}[\ ]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad -[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad +[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad -[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ]\\
&\qquad -[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}]\\
&+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]\\
&- \qquad [\ ]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]
\end{aligned}
$$

In the second simplification the second, fourth, sixth and eighth terms equal

$$d([h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}])+[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]$$

which yields a third simplification

$$F_2(d([(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')]))$$

$$-d([h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ])$$

$$-d([\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}])$$

$$-d([h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}])$$

$$\begin{aligned} = \quad & h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}] \\ & -[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}] \\ & -[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[\ ] \\ & +[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ] \\ & +[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}] \end{aligned}$$

$$= d([h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]),$$

which establishes $dF_3 = F_2 d$.

Recall that we defined

$$\begin{aligned} & F_4([(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')]) \\ & = [h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}] \\ & \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ] \\ & +[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \\ & \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\ & +[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \\ & \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\ & +[h_{\sigma^{-1}(1)}] \\ & \qquad \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\ & +[\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]. \end{aligned}$$

We want to verify that the differential applied to this expression is equal to

$$F_3(d([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)])).$$

We have

$$\begin{aligned}
&d([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)])\\
&=(\sigma(h_1,h_2)([(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]))\\
&-[(\sigma\sigma',h_{\sigma'(1)}h'_1,h_{\sigma'(2)}h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]\\
&+[(\sigma,h_1,h_2)|(\sigma'\sigma'',h'_{\sigma''(1)}h''_1,h'_{\sigma''(2)}h''_2)|(\sigma''',h'''_1,h'''_2)]\\
&-[(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma''\sigma''',h''_{\sigma'''(1)}h'''_1,h''_{\sigma'''(2)}h'''_2)]\\
&+[(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)].
\end{aligned}$$

The first term of

$$F_3(d([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]))$$

is equal to

$$\begin{aligned}
&(\sigma(h_1,h_2)(F_3([(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)])))\\
&=(\sigma(h_1,h_2)([h'_{(\sigma')^{-1}(1)}|h''_{(\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma'\sigma''\sigma''')^{-1}(1)}]\\
&\qquad\otimes h'_{(\sigma')^{-1}(2)}h''_{(\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma'\sigma''\sigma''')^{-1}(2)}[\ ]))\\
&+(\sigma(h_1,h_2)([h'_{(\sigma')^{-1}(1)}|h''_{(\sigma'\sigma'')^{-1}(1)}]\otimes h'_{(\sigma')^{-1}(2)}h''_{(\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma'\sigma''\sigma''')^{-1}(2)}]))\\
&\qquad+(\sigma(h_1,h_2)([h'_{(\sigma')^{-1}(1)}]\otimes h'_{(\sigma')^{-1}(2)}[h''_{(\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma'\sigma''\sigma''')^{-1}(2)}]))\\
&\qquad+(\sigma(h_1,h_2)([\ ]\otimes[h'_{(\sigma')^{-1}(2)}|h''_{(\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma'\sigma''\sigma''')^{-1}(2)}]))\\
&=h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}]\\
&\qquad\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ]\\
&+h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]\\
&\qquad+h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]\\
&\qquad+h_{\sigma^{-1}(1)}[\ ]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}].
\end{aligned}$$

The second term of

$$F_3(d([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]))$$

is equal to

$$
\begin{aligned}
&-F_3([(\sigma\sigma', h_{\sigma'(1)}h'_1, h_{\sigma'(2)}h'_2)|(\sigma'', h''_1, h''_2)|(\sigma''', h'''_1, h'''_2)])\\
&= -[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}]\\
&\qquad\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ]\\
&-[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1})(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]\\
&\qquad -[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1})(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]\\
&\qquad -[\ ] \otimes [h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]
\end{aligned}
$$

The third term of

$$F_3(d([(\sigma, h_1, h_2)|(\sigma', h'_1, h'_2)|(\sigma'', h''_1, h''_2)|(\sigma''', h'''_1, h'''_2)]))$$

is equal to

$$
\begin{aligned}
&+F_3([(\sigma, h_1, h_2)|(\sigma'\sigma'', h'_{\sigma''(1)}h''_1, h'_{\sigma''(2)}h''_2)|(\sigma''', h'''_1, h'''_2)])\\
&= [h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}]\\
&\qquad\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ]\\
&\qquad +[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\\
&\qquad\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]\\
&\qquad +[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]\\
&\qquad\qquad +[\ ] \otimes [h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}].
\end{aligned}
$$

The fourth term of

$$F_3(d([(\sigma, h_1, h_2)|(\sigma', h'_1, h'_2)|(\sigma'', h''_1, h''_2)|(\sigma''', h'''_1, h'''_2)]))$$

is equal to

$$
\begin{aligned}
&-F_3([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma''\sigma''',h''_{\sigma'''(1)}h'''_1,h''_{\sigma'''(2)}h'''_2)])\\
&=-[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}]\\
&\qquad\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ]\\
&\qquad-[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1})(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]\\
&\qquad-[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]\\
&\qquad-[\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}].
\end{aligned}
$$

The fifth term of

$$F_3(d([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]))$$

is equal to

$$
\begin{aligned}
&F_3([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)])\\
&=[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[\ ]\\
&\qquad+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma'^{-1})(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad+[h_{\sigma^{-1}(1)}]\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]\\
&\qquad+[\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}].
\end{aligned}
$$

The first term in

$$d(F_4([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]))$$

is equal to

$$
\begin{aligned}
&d([h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ]) \\
&= h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ] \\
&-[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ] \\
&+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ] \\
&-[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ] \\
&+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ].
\end{aligned}
$$

The second term in

$$d(F_4([(\sigma, h_1, h_2)|(\sigma', h'_1, h'_2)|(\sigma'', h''_1, h''_2)|(\sigma''', h'''_1, h'''_2)]))$$

is equal to

$$
\begin{aligned}
&d([h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]) \\
&= h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&-[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&-[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&-[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ] \\
&+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}] \\
&\qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}.[\ ]
\end{aligned}
$$

The third term in

$$d(F_4([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]))$$

is equal to

$$
\begin{aligned}
& d([h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \\
& \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]) \\
& = h_{\sigma^{-1}(1)}[h'_{(\sigma\sigma')^{-1}(1)}] \\
& \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
& -[h_{\sigma^{-1}(1)}h'_{(\sigma\sigma')^{-1}(1)}] \\
& \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
& +[h_{\sigma^{-1}(1)}] \\
& \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
& +[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \\
& \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
& -[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \\
& \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
& +[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \\
& \qquad \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}].
\end{aligned}
$$

The fourth term in

$$d(F_4([(\sigma, h_1, h_2)|(\sigma', h'_1, h'_2)|(\sigma'', h''_1, h''_2)|(\sigma''', h'''_1, h'''_2)]))$$

is equal to

$$
\begin{aligned}
&d([h_{\sigma^{-1}(1)}] \\
&\qquad\qquad \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]) \\
&= h_{\sigma^{-1}(1)}[\ ] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&-[\ ] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&-[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&+[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&-[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&+[h_{\sigma^{-1}(1)}] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}]
\end{aligned}
$$

The fifth term in

$$d(F_4([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]))$$

is equal to

$$
\begin{aligned}
&d([\ ] \otimes [h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]) \\
&= [\ ] \otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&-[\ ] \otimes [h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&+[\ ] \otimes [h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&-[\ ] \otimes [h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}] \\
&+[\ ] \otimes [h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}].
\end{aligned}
$$

The above laborious computation shows that

$$
\begin{aligned}
&d(F_4([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)])) \\
&= F_3(d([(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)])).
\end{aligned}
$$

The cancellation of terms is summarised in the following table.

F_3d	dF_4
1st term line 1	1st term line 1
1st term line 2	2nd term line 1
1st term line 3	3rd term line 1
1st term line 4	4th term line 1
2nd term line 1	1st term line 2
2nd term line 2	2nd term line 2
2nd term line 3	3rd term line 2
2nd term line 4	fifth term line 2
3rd term line 1	1st term line 3
3rd term line 2	2nd term line 3
3rd term line 3	4th term line 4
3rd term line 4	5th term line 3
4th term line 1	1st term line 4
4th term line 2	3rd term line 5
4th term line 3	4th term line 5
4th term line 4	5th term line 4
5th term line 1	2nd term line 6
5th term line 2	3rd term line 6
5th term line 3	4th term line 6
5th term line 4	5th term line 5

In addition there is further cancellation as follows:

dF_4	dF_4
1st term line 5	2nd term line 5
2nd term line 4	3rd term line 4
3rd term line 3	4th term line 3
4th term line 2	5th term line 1

This completes the verification that $dF_4 = F_3d$ when $n = 2$.

11. Appendix Ten: Examples of 4-cocycles

Example 11.1. Suppose that $\lambda : \underline{B}_2H \longrightarrow \mathbb{Z}$ is a 2-cocycle on H. Therefore

$$\lambda(h[h'|h'']) = h\lambda[h'|h'']$$

and

$$\lambda(d[h|h'|h'']) = \lambda[h'|h''] - \lambda[hh'|h''] + \lambda[h|h'h''] - \lambda[h|h'] = 0.$$

Now suppose that $n \geq 2$ and choose a pair of distinct integers $1 \leq i \neq j \leq n$. Define a function

$$\lambda_{i,j} : \underline{B}_4H^n \longrightarrow \mathbb{Z}$$

by the formula

$$\lambda_{i,j}((\hat{h}_1,\dots\hat{h}_n)[(h_1,\dots h_n)|(h'_1,\dots h'_n)|(h''_1,\dots h''_n)|(h'''_1,\dots h'''_n)])$$

$$= \lambda(\hat{h}_i[h_i|h'_i])\lambda(\hat{h}_j[h''_j|h'''_j])$$

$$= \lambda([h_i|h'_i])\lambda([h''_j|h'''_j]).$$

Also

$$\lambda_{i,j}d([(\hat{h}_1,\dots\hat{h}_n)|(h_1,\dots h_n)|(h'_1,\dots h'_n)|(h''_1,\dots h''_n)|(h'''_1,\dots h'''_n)])$$

$$= \lambda_{i,j}[(h_1,\dots h_n)|(h'_1,\dots h'_n)|(h''_1,\dots h''_n)|(h'''_1,\dots h'''_n)]$$

$$-\lambda_{i,j}[(\hat{h}_1h_1,\dots\hat{h}_nh_n)|(h'_1,\dots h'_n)|(h''_1,\dots h''_n)|(h'''_1,\dots h'''_n)]$$

$$+\lambda_{i,j}[(\hat{h}_1,\dots\hat{h}_n)|(h_1h'_1,\dots h_nh'_n)|(h''_1,\dots h''_n)|(h'''_1,\dots h'''_n)]$$

$$-\lambda_{i,j}[(\hat{h}_1,\dots\hat{h}_n)|(h_1,\dots h_n)|(h'_1h''_1,\dots h'_nh''_n)|(h'''_1,\dots h'''_n)]$$

$$+\lambda_{i,j}[(\hat{h}_1,\dots\hat{h}_n)|(h_1,\dots h_n)|(h'_1,\dots h'_n)|(h''_1h'''_1,\dots h''_nh'''_n)]$$

$$-\lambda_{i,j}[(\hat{h}_1,\dots\hat{h}_n)|(h_1,\dots h_n)|(h'_1,\dots h'_n)|(h''_1,\dots h''_n)]$$

$$= \lambda[h_i|(h'_i]\cdot\lambda[h''_j|(h'''_j] - \lambda[\hat{h}_ih_i|h'_i]\cdot\lambda[h''_j|(h'''_j]$$

$$+\lambda[\hat{h}_i|(h_ih'_i]\cdot\lambda[h''_j|(h'''_j] - \lambda[\hat{h}_i|(h_i]\cdot\lambda[h'_jh''_j|(h'''_j]$$

$$+\lambda[\hat{h}_i|h_i]\cdot\lambda[h'_j|h''_jh'''_j] - \lambda[\hat{h}_i|h_i]\cdot\lambda[h'_j|h''_j]$$

$$= \lambda(d[\hat{h}_i|h_i|h'_i])\lambda[h''_j|(h'''_j] + \lambda[\hat{h}_i|h_i]\lambda(d[h'|h''|h'''])$$

$$= 0.$$

Therefore for each $1 \leq i \neq j \leq n$ the map $\lambda_{i,j}$ is a cocycle in

$$\mathrm{Hom}_{\mathbb{Z}[H^n]}(\underline{B}_4H^n,\mathbb{Z}).$$

If $\pi_i : H^n \longrightarrow H$ is the i-th projection then the cohomology class

$$[\lambda_{i,j}] \in H^4(H^n;\mathbb{Z})$$

is equal to the cup-product $[\lambda\cdot\pi_i]\bigcup[\lambda\cdot\pi_j]$.

Example 11.2. $\Sigma_n \int H$ *when* $n = 2$

When $n = 2$ and λ is as in Example **??** we may form the composition

$$\underline{B}_4\Sigma_2\int H \xrightarrow{F_4} \underline{B}_*H^{\otimes 2} \xrightarrow{\lambda\otimes\lambda} \mathbb{Z}.$$

From the formula

$$F_4([(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')])$$

$$= [h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}]$$

$$\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}[\ ]$$

$$+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}|h''_{(\sigma\sigma'\sigma'')^{-1}(1)}]$$

$$\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}h''_{(\sigma\sigma'\sigma'')^{-1}(2)}[h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]$$

$$\otimes h_{\sigma^{-1}(2)}h'_{(\sigma\sigma')^{-1}(2)}[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+[h_{\sigma^{-1}(1)}]$$

$$\otimes h_{\sigma^{-1}(2)}[h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+[\ ]\otimes[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}|h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

we find that

$$[(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')]$$

$$\downarrow\ (\lambda\otimes\lambda)F_4$$

$$\lambda[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}].$$

Extend the map $(\lambda\otimes\lambda)F_4$ to $\tilde{F}_\lambda$ on all of $\underline{B}_4\Sigma_2\int H$ by the formula

$$\tilde{F}_\lambda((\hat\sigma,\hat h_1,\hat h_2)[(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')])$$

$$=(\lambda\otimes\lambda)F_4([(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')]).$$

Now let us examine how close this function comes to being a 4-cocycle in $\mathrm{Hom}_{\mathbb{Z}[\Sigma_2\int H]}(\underline{B}_4\Sigma_2\int H,\mathbb{Z})$.

We start with a 5-chain

$$[(\hat\sigma,\hat h_1,\hat h_2)|(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')]$$

whose boundary is

$$d([(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)])$$

$$= (\hat{\sigma},\hat{h}_1,\hat{h}_2)[(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]$$

$$-[(\hat{\sigma},\hat{h}_1,\hat{h}_2)(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]$$

$$+[(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]$$

$$-[(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]$$

$$+[(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)(\sigma''',h'''_1,h'''_2)]$$

$$-[(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)]$$

$$= (\hat{\sigma},\hat{h}_1,\hat{h}_2)[(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]$$

$$-[(\hat{\sigma}\sigma,\hat{h}_{\sigma(1)}h_1,\hat{h}_{\sigma(2)})h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]$$

$$+[(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma\sigma',h_{(\sigma')(1)}h'_1,h_{(\sigma')(2)})h'_2)|(\sigma'',h''_1,h''_2)|(\sigma''',h'''_1,h'''_2)]$$

$$-[(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma'\sigma'',h'_{\sigma''(1)}h''_1,h'_{\sigma''(2)}h''_2)|(\sigma''',h'''_1,h'''_2)]$$

$$+[(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma''\sigma''',h''_{\sigma'''(1)}h'''_1,h''_{\sigma'''(2)}h'''_2)]$$

$$-[(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma',h'_1,h'_2)|(\sigma'',h''_1,h''_2)]$$

Term by term we have

$$\tilde{F}_\lambda((\hat{\sigma},\hat{h}_1,\hat{h}_2)[(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')])$$

$$= \lambda[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\tilde{F}_\lambda([(\hat{\sigma}\sigma,\hat{h}_{\sigma(1)}h_1,\hat{h}_{\sigma(2)})h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')])$$

$$= -\lambda[\hat{h}_{\hat{\sigma}^{-1}(1))}h_{((\hat{\sigma}\sigma)^{-1}(1)}|h'_{(\hat{\sigma}\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\hat{\sigma}\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat{\sigma}\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+\tilde{F}_\lambda([(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma\sigma',h_{(\sigma')(1)}h_1',h_{(\sigma')(2)})h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')])$$

$$= +\lambda[\hat{h}_{\hat{\sigma}^{-1}(1)}|h_{(\hat{\sigma}\sigma)^{-1}(1)}h'_{(\hat{\sigma}\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\hat{\sigma}\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat{\sigma}\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\tilde{F}_\lambda([(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma'\sigma'',h'_{\sigma''(1)}h_1'',h'_{\sigma''(2)}h_2'')|(\sigma''',h_1''',h_2''')])$$

$$= -\lambda[\hat{h}_{\hat{\sigma}^{-1}(1)}|h_{(\hat{\sigma}\sigma)^{-1}(1)},]\cdot\lambda[h'_{(\hat{\sigma}\sigma\sigma')^{-1}(2)}h''_{(\hat{\sigma}\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat{\sigma}\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+\tilde{F}_\lambda([(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma''\sigma''',h''_{\sigma'''(1)}h_1''',h''_{\sigma'''(2)}h_2''')])$$

$$= +\lambda[\hat{h}_{\hat{\sigma}^{-1}(1)}|h_{(\hat{\sigma}\sigma)^{-1}(1)}]\cdot\lambda[h'_{(\hat{\sigma}\sigma\sigma')^{-1}(2)}|h''_{\hat{\sigma}\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\hat{\sigma}\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\tilde{F}_\lambda([(\hat{\sigma},\hat{h}_1,\hat{h}_2)|(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')])$$

$$= -\lambda[\hat{h}_{\hat{\sigma}^{-1}(1)}|h_{(\hat{\sigma}\sigma)^{-1}(1)}]\cdot\lambda[h'_{(\hat{\sigma}\sigma\sigma')^{-1}(2)}|h''_{(\hat{\sigma}\sigma\sigma'\sigma'')^{-1}(2)}].$$

Therefore

$$\tilde{F}_\lambda(d([(\hat\sigma,\hat h_1,\hat h_2)|(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')]))$$

$$= \lambda[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\lambda[\hat h_{\hat\sigma^{-1}(1))}h_{((\hat\sigma\sigma)^{-1}(1)}|h'_{(\hat\sigma\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\hat\sigma\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat\sigma\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+\lambda[\hat h_{\hat\sigma^{-1}(1)}|h_{(\hat\sigma\sigma)^{-1}(1)}h'_{(\hat\sigma\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\hat\sigma\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat\sigma\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\lambda[\hat h_{\hat\sigma^{-1}(1)}|h_{(\hat\sigma\sigma)^{-1}(1)},]\cdot\lambda[h'_{(\hat\sigma\sigma\sigma')^{-1}(2)}h''_{(\hat\sigma\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat\sigma\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+\lambda[\hat h_{\hat\sigma^{-1}(1)}|h_{(\hat\sigma\sigma)^{-1}(1)}]\cdot\lambda[h'_{(\hat\sigma\sigma\sigma')^{-1}(2)}|h''_{\hat\sigma\sigma\sigma'\sigma'')^{-1}(2)}h'''_{(\hat\sigma\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\lambda[\hat h_{\hat\sigma^{-1}(1)}|h_{(\hat\sigma\sigma)^{-1}(1)}]\cdot\lambda[h'_{(\hat\sigma\sigma\sigma')^{-1}(2)}|h''_{(\hat\sigma\sigma\sigma'\sigma'')^{-1}(2)}]$$

$$= \lambda[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\lambda[\hat h_{\hat\sigma^{-1}(1))}h_{((\hat\sigma\sigma)^{-1}(1)}|h'_{(\hat\sigma\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\hat\sigma\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat\sigma\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+\lambda[\hat h_{\hat\sigma^{-1}(1)}|h_{(\hat\sigma\sigma)^{-1}(1)}h'_{(\hat\sigma\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\hat\sigma\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat\sigma\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\lambda[\hat h_{\hat\sigma^{-1}(1)}|h_{(\hat\sigma\sigma)^{-1}(1)},]\cdot\lambda[h''_{(\hat\sigma\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat\sigma\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$= \lambda[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\lambda[h_{((\hat\sigma\sigma)^{-1}(1)}|h'_{(\hat\sigma\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\hat\sigma\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\hat\sigma\sigma\sigma'\sigma''\sigma''')^{-1}(2)}].$$

When $\hat\sigma = 1$ the above express is zero but when $\hat\sigma(1) = 2$ we obtain

$$\lambda[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\lambda[h_{((\sigma)^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}]\cdot\lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}].$$

so that the function which sends the 4-chain

$$(\hat\sigma,\hat h_1,\hat h_2)[(\sigma,h_1,h_2)|(\sigma',h_1',h_2')|(\sigma'',h_1'',h_2'')|(\sigma''',h_1''',h_2''')]$$

to the integer

$$\lambda[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}]\cdot\lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$+\lambda[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}]\cdot\lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}]$$

is a $\Sigma_2 \int H$-cocycle because its coboundary when $\hat{\sigma}$ is non-trivial is equal to

$$\lambda[h_{\sigma^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \cdot \lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

$$-\lambda[h_{((\sigma)^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}] \cdot \lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}]$$

$$+\lambda[h_{\sigma^{-1}(2)}|h'_{(\sigma\sigma')^{-1}(2)}] \cdot \lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(1)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(1)}]$$

$$-\lambda[h_{((\sigma)^{-1}(1)}|h'_{(\sigma\sigma')^{-1}(1)}] \cdot \lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(2)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(2)}]$$

which is zero.

Example 11.3. $\Sigma_n \int H$ *in general*

When $n \geq 2$ and λ is as in Example **??** we define a function

$$\tilde{c}_4 \in \mathrm{Hom}_{\mathbb{Z}[\Sigma_n \int H]}(\underline{B}_4 \Sigma_n \int H, \mathbb{Z})$$

by sending the 4-chain

$$z = (\hat{\sigma}, \hat{h}_1, \dots)[(\sigma, h_1, \dots)|(\sigma', h'_1, \dots)|(\sigma'', h''_1, \dots)|(\sigma''', h'''_1, \dots)]$$

to

$$\tilde{c}_4(z) = \sum_{1 \leq i \neq j \leq n} \lambda[h_{\sigma^{-1}(i)}|h'_{(\sigma\sigma')^{-1}(i)}] \cdot \lambda[h''_{(\sigma\sigma'\sigma'')^{-1}(j)}|h'''_{(\sigma\sigma'\sigma''\sigma''')^{-1}(j)}].$$

A computation of the coboundary of $\tilde{c}_4$ similar to that in the special case of Example **??** shows that $\tilde{c}_4$ is a 4-cocycle which represents a cohomology class

$$[\tilde{c}_4] \in H^4(\Sigma_n \int H; \mathbb{Z}).$$

12. Appendix Eleven: The Universal Coefficient Theorem (UCT)

In Example **??** I gave an explicit 4-cocycle for the group $\Sigma_n \int H$. In this Appendix we shall assume H is a finite group. In this case I just want to recall[6] explicitly how the Universal Coefficient Theorem is derived and what one must do to the 4-cocycle to obtain the corresponding homomorphism in $\mathrm{Hom}(H_3(\Sigma_n \int H; \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$.

Let C_* denote a chain complex of torsion free abelian groups. For each n we have a split exact sequence of finitely generated torsion-free abelian groups

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

where Z_n, B_n are cycles and boundaries in dimension n. We have a short exact sequence of chain complexes

$$0 \longrightarrow Z_* \longrightarrow C_* \longrightarrow B_{*-1} \longrightarrow 0$$

[6]Even though it is well-known!

where the differentials in the complexes to left and right are zero. Taking duals is exact, because the short exact sequences are split, so we obtain a short exact sequence of complexes

$$\mathrm{Hom}(B_{*-1}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(C_*, \mathbb{Z}) \longrightarrow \mathrm{Hom}(Z_*, \mathbb{Z}).$$

The resulting long exact sequence of cohomology looks like

$$\ldots \mathrm{Hom}(Z_{n-1}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(B_{n-1}, \mathbb{Z}) \longrightarrow H^n(C_*; \mathbb{Z}) \longrightarrow \mathrm{Hom}(Z_n, \mathbb{Z}) \longrightarrow \ldots.$$

We also have a short exact sequence of abelian groups

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n(C_*) \longrightarrow 0.$$

The boundary in the long exact cohomology sequence takes $f : Z_n \longrightarrow \mathbb{Z}$ and extends it to $\tilde{f} : C_n \longrightarrow \mathbb{Z}$ then composes with the boundary $d : C_{n+1} \longrightarrow C_n$ and observe that $\tilde{f} \cdot d$ factors through B_n. In fact the factorisation on B_n is just the restriction of f to B_n. Therefore the cokernel of

$$\mathrm{Hom}(Z_{n-1}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(B_{n-1}, \mathbb{Z})$$

is $\mathrm{Ext}(H_{n-1}(C_*), \mathbb{Z})$ since the B, Z, H short exact sequence is a projective resolution of $H_n(C_*)$. Similarly the kernel of

$$\mathrm{Hom}(Z_n, \mathbb{Z}) \longrightarrow \mathrm{Hom}(B_n, \mathbb{Z})$$

is $\mathrm{Hom}(H_n(C_*), \mathbb{Z})$.

This concludes the derivation of the UCT, which asserts the existence of a (non-naturally split) short exact sequence of the form

$$\mathrm{Ext}(H_{n-1}(C_*), \mathbb{Z}) \longrightarrow H^n(C_*; \mathbb{Z}) \longrightarrow \mathrm{Hom}(H_n(C_*), \mathbb{Z}).$$

In the case of group homology and cohomlogy when G is a finite group the UCT takes the form of an isomorphism, when $n \geq 1$,

$$\mathrm{Ext}(H_{n-1}(G; \mathbb{Z}), \mathbb{Z}) \xrightarrow{\cong} H^n(G; \mathbb{Z})$$

since $\mathrm{Hom}(H_n(G; \mathbb{Z}), \mathbb{Z})$ vanishes.

We shall be interested in finite group homology in dimension three where the isomorphism

$$\mathrm{Hom}(H_3(G; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}(H_3(G; \mathbb{Z}), \mathbb{Z})$$

is given by the boundary map in the long exact cohomology sequences of

$$0 \longrightarrow \mathrm{Hom}(C_*, \mathbb{Z}) \longrightarrow \mathrm{Hom}(C_*, \mathbb{Q}) \longrightarrow \mathrm{Hom}(C_*; \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

This boundary map is an isomorphism and its inverse starts with a representative 4-cocycle $f : C_4 \longrightarrow \mathbb{Z}$ such that $fd = 0$. Therefore the image of f in $\mathrm{Hom}(C_4, \mathbb{Q})$ is in the kernel of

$$d^* : \mathrm{Hom}(C_4, \mathbb{Q}) \longrightarrow \mathrm{Hom}(C_5, \mathbb{Q})$$

so that, because the middle complex is exact, there is $\hat{f} : C_3 \longrightarrow \mathbb{Q}$ such that $\hat{f}d = f$ mapping into the rational numbers. Therefore the image of $\hat{f}$ in $\mathrm{Hom}(C_3; \mathbb{Q}/\mathbb{Z})$ is zero when restricted to B_3 so that it induces

$$\tilde{f} : H_3(G; \mathbb{Z}) = Z_3/B_3 \longrightarrow \mathbb{Q}/\mathbb{Z}$$

which is the inverse image under the boundary map.

Since the contracting homotopy on $\underline{B}_3G$ sends $g[a|b|c]$ to $[g|a|b|c] \in \underline{B}_4G$ one finds that the formula for the map $H_3(G; \mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ corresponding to $\Phi^*\tilde{c}_2$ is represented by

$$[a|b|c] \mapsto \frac{1}{|G|} \sum_{g \in G} \tilde{c}_2[g|a|b|c] \in \mathbb{Q}/\mathbb{Z}.$$

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