GALOIS DESCENT OF REPRESENTATIONS

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Contents

Part 1. A_5 Descent Examples

1. Subgroups and elements of A_5 via $PGL_2\mathbb{F}_4$

1.1. Let A_5 denote the alternating group consisting of even permutations of the set with five elements and let $GL_2\mathbb{F}_4$ denote the group of 2×2 invertible matrices with entries in the field with four elements.

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In $GL_2\mathbb{F}_4$, if ξ is a cube root of unity in $\mathbb{F}_4$, then we have matrices

$$X_\xi = \begin{pmatrix} 0 & 1 \\ 1 & \xi \end{pmatrix}, X_\xi^2 = \begin{pmatrix} 0 & 1 \\ 1 & \xi \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & \xi \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ \xi & \xi \end{pmatrix}$$

$$X_\xi^3 = \begin{pmatrix} 0 & 1 \\ 1 & \xi \end{pmatrix}\begin{pmatrix} 1 & \xi \\ \xi & \xi \end{pmatrix} = \begin{pmatrix} \xi & \xi \\ \xi & 1 \end{pmatrix}$$

$$X_\xi^4 = \begin{pmatrix} 0 & 1 \\ 1 & \xi \end{pmatrix}\begin{pmatrix} \xi & \xi \\ \xi & 1 \end{pmatrix} = \begin{pmatrix} \xi & 1 \\ 1 & 0 \end{pmatrix}$$

$$X_\xi^5 = \begin{pmatrix} 0 & 1 \\ 1 & \xi \end{pmatrix}\begin{pmatrix} \xi & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This matrix gives a cyclic permutation of the projective line over $\mathbb{F}_4$. In fact, as we shall see from the elements described below, the projective general linear group $PGL_2\mathbb{F}_4 = GL_2\mathbb{F}_4/\mathbb{F}_4^*$ is isomorphic to A_5. Acting via right multiplication on row vectors, $PGL_2\mathbb{F}_4$ permutes via the points of the projective line

$$\mathbb{P}^1(\mathbb{F}_4) = \{(0,1),(1,0),(1,\xi),(\xi,1),(1,1)\}.$$

For example, X_ξ yields the 5-cycle $((0,1),(1,\xi),(1,1),(\xi,1),(1,0))$.

The 2-Sylow subgroup of A_5 is the Klein 4-group V_4 generated by the images in $PGL_2\mathbb{F}_4$ of the matrices A and B given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}$$

since $A^2 = I, B^2 = \xi \cdot I$ and

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix} = \begin{pmatrix} \xi & 1 \\ 1 & \xi \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = BA.$$

As even permutations of the projective line both A and B fix $(1,1)$ since

$$(1,1)A = (1,1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (1,1),$$

$$(1,1)B = (1,1)\begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix} = (\xi^2,\xi^2) = (1,1).$$

The 3-Sylow subgroup consists of the images of I, C, C^2 where

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \ C^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, C^3 = I.$$

Since the order of $PGL_2\mathbb{F}_4$ equals $\frac{(4^2-1)(4^2-4)}{3} = 60$ which is the order of A_5 and $PGL_2\mathbb{F}_4$ sits inside Σ_5 the above calculations with matrices show that $PGL_2\mathbb{F}_4 = A_5 \subset \Sigma_5$.

The subgroup A_4 has index five in A_5 and can be realised as the images of the matrices which fix $(1,1)$ in the projective line. Also setting

$$Y = \begin{pmatrix} 1 & \xi \\ \xi^2 & 0 \end{pmatrix} \text{ which satisfies } (1,1)Y = (1,1)\begin{pmatrix} 1 & \xi \\ \xi^2 & 0 \end{pmatrix} = (\xi, \xi) = (1,1)$$

and furthermore $Y^3 = I$. Finally Y normalises $V_4 = \langle A, B \rangle$ since we have

$$YAY^2 = \begin{pmatrix} 1 & \xi \\ \xi^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \xi \\ \xi^2 & 1 \end{pmatrix} = \begin{pmatrix} \xi & 1 \\ 0 & \xi^2 \end{pmatrix}\begin{pmatrix} 0 & \xi \\ \xi^2 & 1 \end{pmatrix} = \xi AB,$$

$$YBY^2 = \begin{pmatrix} 1 & \xi \\ \xi^2 & 0 \end{pmatrix}\begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}\begin{pmatrix} 0 & \xi \\ \xi^2 & 1 \end{pmatrix} = \begin{pmatrix} \xi & 0 \\ \xi^2 & 1 \end{pmatrix}\begin{pmatrix} 0 & \xi \\ \xi^2 & 1 \end{pmatrix} = \xi^2 A.$$

Hence

$$A_4 = \langle Y, A, B \rangle \subseteq A_5.$$

By Sylow's theorem $\langle Y \rangle$ and $\langle C \rangle$ are conjugate in A_5. Explicitly we have

$$\begin{pmatrix} 1 & 0 \\ \xi^2 & \xi^2 \end{pmatrix} C = \begin{pmatrix} 0 & 1 \\ \xi^2 & 0 \end{pmatrix} = Y \begin{pmatrix} 1 & 0 \\ \xi^2 & \xi^2 \end{pmatrix}.$$

One easily verifies that

$$Y = X_\xi^2 BCBX_\xi^3 \text{ and } X_\xi = C^2BC^2.$$

Since A_5 has order 60 its proper subgroups must have orders in the set $\{2, 3, 4, 5, 6, 10, 12, 20\}$. In fact there is no subgroup of order 20. For suppose that a subgroup contains $V_4 = \langle A, B \rangle$ and a 5-cycle. Then conjugating the elements of V_4 by the 5-cycle and multiplying the results by A, B, AB one finds that the subgroup must also contain a 3-cycle and hence equals A_5. Similarly, there is no subgroup of order 30.

The following table shows all the conjugacy class representatives of subgroups of A_5.

1.2. *Conjugacy classes of subgroups H of A_5*

H	Order	Generators	Number in conjugacy class
A_5	60	A, B, Y, X_ξ	1
A_4	12	A, B, Y	5
D_{10}	10	X_ξ, A	6
D_6	6	A, C	10
C_5	5	X_ξ	6
V_4	4	A, B	5
C_3	3	C	10
C_2	2	A	15
$\{1\}$	1	I	1

A simple argument using Sylow's theorems shows that each subgroup of A_5 is determined up to conjugacy by its order.

The classification of irreducible, finite-dimensional complex representations of $GL_2\mathbb{F}_4$ given in ([75] §3.2 p.89) shows that there are five irreducible representations of the quotient group $PGL_2\mathbb{F}_4 \cong A_5$. Therefore there are five conjugacy classes of elements of A_5 of orders $1, 2, 3, 5$ and 5. To see that there are two distinct conjugacy classes of order 5 observe that only one conjugacy class implies either that C_5 is normal in A_5 or there is a subgroup of order 20 or 30.

In the character table of A_5 given below the conjugacy classes of elements are labelled $1, 2, 3, 5^1$ and 5^2 and are represented by elements having orders $1, 2, 3, 5$ and 5, respectively. The α_i are real numbers given by $\alpha_1 = (1+\sqrt{5})/2$ and $\alpha_2 = (1-\sqrt{5})/2$.

1.3. *Character Table for* A_5

	1	2	3	5^1	5^2
1	1	1	1	1	1
$\nu_{3,1}$	3	-1	0	α_1	α_2
$\nu_{3,2}$	3	-1	0	α_2	α_1
ν_4	4	0	1	-1	-1
ν_5	5	1	-1	0	0

2. Complex irreducible representations of A_5

2.1. *The action of* $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$

Let σ denote the Frobenius automorphism of $\mathbb{F}_4$ given by $\sigma(z) = z^2$. Applying σ to the matrix entries given an involution on $GL_2\mathbb{F}_4$ and its quotient $PGL_2\mathbb{F}_4 \cong A_5$. Therefore, if ρ is a finite-dimensional complex irreducible representation of A_5 then so is $\sigma^*(\rho)$, the composition of ρ with the σ.

It is straightforward to verify that σ applied to the conjugacy class 5^1 gives 5^2 so that $\sigma^*(\nu_{3,1}) = \nu_{3,2}$. On the other hand, since σ^* preserves dimension, we must have

$$\sigma^*(1) = 1,\ \sigma^*(\nu_4) = \nu_4 \text{ and } \sigma^*(\nu_5) = \nu_5.$$

2.2. *Explicit models for ν_4 and ν_5*

The Borel subgroup of upper triangular matrices in $GL_2\mathbb{F}_4$ has order 36 so its image in $PGL_2\mathbb{F}_4$ has order 12 so is conjugate to A_4. Denote by B the image of the Borel subgroup in $PGL_2\mathbb{F}_4$ and also, when there is no confusion, the Borel subgroup of $GL_2\mathbb{F}_4$. This enables us to describe the irreducible representation ν_4 and ν_5 explicitly, following the description of irreducibles given in ([75] §3.2 p.89).

There is a short exact sequence representations of $GL_2\mathbb{F}_4$ of the form

$$0 \longrightarrow \nu_4 = S(1) \longrightarrow \mathrm{Ind}_B^{GL_2\mathbb{F}_4}(1) \longrightarrow L(1) \longrightarrow 0$$

where $L(1) = 1$, the one-dimensional trivial representation, and $S(1)$ is irreducible. Each representation in the short exact sequence factorises through $A_5 = PGL_2\mathbb{F}_4$ and $S(1)$ factorises through the irreducible representation ν_4.

If λ is a non-trivial character of $\mathbb{F}_4^*$ we know from ([75] §3.2p.89) that $\mathrm{Ind}_B^{GL_2\mathbb{F}_4}(\mathrm{Inf}_T^B(\lambda \otimes \lambda^2))$ is irreducible. Also $\lambda \otimes \lambda^2$ is trivial on the scalar matrices since $\lambda^3 = 1$ so that $\lambda \otimes \lambda^2$ factorises to give a non-trivial character which is conjugate to $\phi : B \cong A_4 \to A_4/V_4 \to \mathbb{C}^*$. This irreducible factorises through

$$\nu_5 = \mathrm{Ind}_B^{PGL_2\mathbb{F}_4}(\phi).$$

2.3. *Bases for ν_4 and ν_5*

If $\mu : B \to \mathbb{C}^*$ be a character. The standard basis for $\mathrm{Ind}_B^{PGL_2\mathbb{F}_4}(\mu)$ is

$$I \otimes_B 1,\ X_\xi \otimes_B 1,\ X_\xi^2 \otimes_B 1,\ X_\xi^3 \otimes_B 1,\ X_\xi^4 \otimes_B 1.$$

Define a basis V_1, W_1, W_2, W_3 for ν_4 by

$$V_1 = (I \otimes_B 1 - X_\xi^2 \otimes_B 1) - (I \otimes_B 1 - X_\xi^4 \otimes_B 1),$$

$$W_1 = I \otimes_B 1 - X_\xi \otimes_B 1,$$

$$W_2 = I \otimes_B 1 - X_\xi^3 \otimes_B 1,$$

$$W_3 = (I \otimes_B 1 - X_\xi^2 \otimes_B 1) + (I \otimes_B 1 - X_\xi^4 \otimes_B 1).$$

If $\xi_3 = e^{2\pi\sqrt{-1}/3}$ define a basis v_1, w_1, w_2, w_3, w_4 for ν_5 by

$$v = I \otimes_B 1 + X_\xi \otimes_B 1 + \xi_3 X_\xi^3 \otimes_B 1$$

$$w_1 = -\xi_3 I \otimes_B 1 - \xi_3 X_\xi \otimes_B 1 + 2\xi_3^2 X_\xi^3 \otimes_B 1 - X_\xi^4 \otimes_B 1 - X_\xi^2 \otimes_B 1$$

$$w_2 = (1 - \xi_3^2) I \otimes_B 1 + (\xi_3^2 - 1) X_\xi \otimes_B 1 - X_\xi^4 \otimes_B 1 + X_\xi^2 \otimes_B 1$$

$$w_3 = -\xi_3 I \otimes_B 1 - \xi_3 X_\xi \otimes_B 1 + 2\xi_3^2 X_\xi^3 \otimes_B 1 + 3X_\xi^4 \otimes_B 1 + 3X_\xi^2 \otimes_B 1$$

$$w_4 = (1 - \xi_3^2) I \otimes_B 1 + (\xi_3^2 - 1) X_\xi \otimes_B 1 + 3X_\xi^4 \otimes_B 1 - 3X_\xi^2 \otimes_B 1.$$

2.4. *The A_5-action on ν_4*

In terms of the ordered basis $\{V_1, W_1, W_2, W_3\}$ the matrices for the action of A, B, C, X_ξ on ν_4 are given by

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -1 & -\frac{1}{2} & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & -1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X_\xi = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & -1 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

2.5. *The A_5-action on ν_5*

In terms of the ordered basis $\{v_1, w_1, w_2, w_3, w_4\}$ the matrices for the action of A, B, C, X_ξ on ν_5 are given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{8}{3}\xi_3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\xi_3 - \xi_3^2}{3} \\ \frac{\xi_3^2}{3} & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \xi_3^2 - \xi_3 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} + \xi_3 & 0 & 0 \\ 0 & \frac{\xi_3}{2} - \frac{\xi_3^2}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\xi_3}{2} - \frac{\xi_3^2}{2} \\ 0 & 0 & 0 & \frac{1}{2} + \xi_3 & -\frac{1}{2} \end{pmatrix}$$

$$X_\xi = \begin{pmatrix} \frac{1}{3} & 0 & 0 & -\frac{4\xi_3}{3} & \frac{8}{3} + \frac{4\xi_3}{3} \\ 0 & \frac{1}{4} & \frac{1}{4} + \frac{\xi_3}{2} & -\frac{1}{4} - \frac{\xi_3}{2} & -\frac{1}{4} \\ 0 & \frac{1}{4} + \frac{\xi_3}{2} & -\frac{3}{4} & -\frac{1}{4} & \frac{\xi_3}{12} - \frac{\xi_3^2}{12} \\ -\frac{\xi_3^2}{6} & \frac{3\xi_3}{4} - \frac{3\xi_3^2}{4} & \frac{3}{4} & -\frac{1}{12} & \frac{\xi_3^2}{12} - \frac{\xi_3}{12} \\ \frac{\xi_3}{6} - \frac{1}{6} & \frac{3}{4} & \frac{\xi_3^2}{4} - \frac{\xi_3}{4} & -\frac{1}{12} - \frac{\xi_3}{6} & \frac{1}{4} \end{pmatrix}.$$

3. Semi-direct products

3.1. Following the notational conventions of ([73] p.36), if C acts on G via $\lambda : C \longrightarrow \mathrm{Aut}(G)$ then the semi-direct product $C \propto G$ is the group whose underlying set is $C \times G$ with multiplication given by

$$(c_1, g_1) \cdot (c_2, g_2) = (c_1 c_2, g_1 \lambda(c_1)(g_2)), \qquad c_i \in C, g_i \in G.$$

Let $\mathcal{H}$ denote a complex vector space and let $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})$ denote a representation of G on $\mathcal{H}$. For $c \in C$ denote by $c^*(\rho) : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})$ the representation of G given by the formula $c^*(\rho)(g)(h) = \lambda(c)(g)(h)$ for $c \in C, g \in G, h \in \mathcal{H}$.

Suppose that ρ is a representation for which Schur's Lemma holds; that is, $\mathrm{End}_{\mathbb{C}[G]}(\mathcal{H}) = \mathbb{C}$, the ring of scalar endomorphisms. Assume in addition that $c^*(\rho)$ and ρ are equivalent representations for each $c \in C$. Therefore for each $c \in C$ there exists

$$U_c \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})$$

such that, for all $c \in C, g \in G$,

$$c^*(\rho)(g) = \rho(\lambda(c)(g)) = U_c \cdot \rho(g) \cdot U_c^{-1} \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})$$

If $V_c \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})$ satisfies $U_c \cdot \rho(g) \cdot U_c^{-1} = V_c \cdot \rho(g) \cdot V_c^{-1}$ then, by the Schur Lemma condition, $U_c = V_c \in \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})/\mathbb{C}^* = \mathrm{ProjAut}_{\mathbb{C}}(\mathcal{H})$, the group of projective automorphisms of $\mathcal{H}$.

Proposition 3.2. Let G and C be as in §3.1. Let ρ be a representation of G for which Schur's Lemma holds. Assume in addition that $c^*(\rho)$ and ρ are equivalent representations for each $c \in C$. Then, in the notation of §3.1, there is a homomorphism of the form

$$\tilde{\rho} : C \propto G \longrightarrow \mathrm{ProjAut}_{\mathbb{C}}(\mathcal{H})$$

given by the formula $\tilde{\rho}(c, g) = \rho(g) U_c$.

ProofSince U_g is unique in the group of projective automorphisms we have

$U_gU_{g_1} = U_{gg_1}$ in this group and therefore

$$\begin{aligned}\tilde{\rho}(cc_1, g\lambda(c)(g_1)) &= \rho(g\lambda(c)(g_1))U_{cc_1} \\ &= \rho(g)\rho(\lambda(c)(g_1))U_cU_{c_1} \\ &= \rho(g)U_c\rho(g_1)U_c^{-1}U_cU_{c_1} \\ &= \rho(c)U_c\rho(g_1)U_{c_1} \\ &= \tilde{\rho}(c,g)\tilde{\rho}(c_1,g_1).\end{aligned}$$

□

Example 3.3. Let $G = PGL_2\mathbb{F}_4$ and let $C = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ generated by the involution given by σ, the Frobenius automorphism. From §2.1 we know that

$$\sigma^*(1) = 1, \ \sigma^*(\nu_4) = \nu_4 \text{ and } \sigma^*(\nu_5) = \nu_5.$$

(i) When $\rho = 1$, the trivial one-dimensional representation, then $U_\sigma = 1$ and the homomorphism of Proposition 3.2 is trivial. The trivial projective representation factors through each of the two one-dimensional representations $C \propto GL_2\mathbb{F}_4 \longrightarrow \mathbb{C}^*$ of the form

$$C \propto GL_2\mathbb{F}_4 \longrightarrow C \longrightarrow \mathbb{C}^*.$$

(ii) When $\rho = \nu_4$ we have $\mathcal{H} = \langle V_1, W_1, W_2, W_3\rangle$. Let U_σ be the linear involution on $\mathcal{H}$ given by

$$U_\sigma(V_1) = V_1, U_\sigma(W_1) = -W_1, U_\sigma(W_2) = -W_2, U_\sigma(W_3) = -W_3.$$

With this choice of U_σ and $U_1 = 1$ the projective homomorphism $\tilde{\rho}$ of Proposition 3.2 lifts to a representation

$$\tilde{\nu}_4 : \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \propto GL_2\mathbb{F}_4 \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})$$

given by $(c, g) \mapsto \nu_4(g)U_c$. This is easily verified using the relations $\sigma(A) = A, \sigma(C) = C, \sigma(X_\xi) = X_\xi AC, \sigma(B) = BA$ in $PGL_2\mathbb{F}_4$.

In terms of matrices σ acts on $\tilde{\nu}_4$ as

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The other lift of $\tilde{\rho}$ to a linear representation is the tensor product of $\tilde{\nu}_4$ with the non-trivial one-dimensional representation of the form

$$\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \propto GL_2\mathbb{F}_4 \longrightarrow \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \longrightarrow \mathbb{C}^*.$$

(iii) When $\rho = \nu_5$ we have $\mathcal{H} = \langle v, w_1, w_2, w_3, w_4\rangle$. Let U_σ be the linear involution on $\mathcal{H}$ given by

$$U_\sigma(v) = v, U_\sigma(w_1) = -w_1, U_\sigma(w_2) = -w_2, U_\sigma(w_3) = w_3, U_\sigma(w_4) = w_4.$$

With this choice of U_σ and $U_1 = 1$ the projective homomorphism $\tilde{\rho}$ of Proposition 3.2 lifts to a representation

$$\tilde{\nu}_5 : \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \propto GL_2\mathbb{F}_4 \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathcal{H})$$

given by $(c, g) \mapsto \nu_5(g)U_c$.

In terms of matrices σ acts on $\tilde{\nu}_5$ as

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The other linear lift of $\tilde{\rho}$ is constructed by tensoring with a non-trivial quadratic character, as in (ii).

3.4. *Conjugacy classes of subgroups H of* $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \propto PGL_2\mathbb{F}_4$

The following table shows the conjugacy classes of subgroups H which are not conjugate to a subgroup of $PGL_2\mathbb{F}_4 \cong A_5$.

H	Order	Generators	Number in conjugacy class
$\langle(\sigma, 1), A_5\rangle$	120	$(\sigma, 1), A, B, Y, X_\xi$	1
$\langle(\sigma, 1), A_4\rangle$	24	$(\sigma, 1), A, B, Y$	5
$C_4 \propto C_5$	20	$(\sigma, B), X_\xi$	6
$C_2 \times D_6$	12	$(\sigma, 1), A, C$	10
$\langle(\sigma, 1), V_4\rangle$	8	$(\sigma, 1), A, B$	15
D_6	6	$(\sigma, A), C$	10
$C_2 \times C_3$	6	$(\sigma, 1), C$	10
C_4	4	(σ, B)	15
$C_2 \times C_2$	4	$(\sigma, 1), A$	15
C_2	2	$(\sigma, 1)$	10

To determine the subgroups $J \subseteq C_2 \propto PGL_2\mathbb{F}_4$ up to conjugacy whose projection $J \subseteq C_2 \propto PGL_2\mathbb{F}_4 \longrightarrow C_2$ is non-trivial we consider the kernel of the projection. This a subgroup J' of index two in J which we may assume is one appearing in the table of §1.2. A laborious analysis of the possibilities yields the results of the above table.

In addition, let us record the action of the Frobenius on $Y \in PGL_2\mathbb{F}_4$

$$\sigma(Y) = AY^2A.$$

4. The Shintani correspondence for $GL_n\mathbb{F}_{q^d}$

4.1. Let C denote the cyclic group of order d given by the Galois group of $\mathbb{F}_{q^d}/\mathbb{F}_q$ generated by the Frobenius automorphism, σ. Let $\mathrm{Irr}(GL_n\mathbb{F}_{q^d})^C$ denote the set of finite-dimensional, irreducible complex representations ρ of $GL_n\mathbb{F}_{q^d}$ such that $\sigma^*(\rho)$ is equivalent to ρ. Let $\mathrm{Irr}(GL_n\mathbb{F}_q)$ denote the set of finite-dimensional, irreducible complex representations of $GL_n\mathbb{F}_q$.

The Shintani correspondence [69] is a bijection of the form

$$\mathrm{Sh} : \mathrm{Irr}(GL_n\mathbb{F}_{q^d})^C \stackrel{\cong}{\longrightarrow} \mathrm{Irr}(GL_n\mathbb{F}_q).$$

This correspondence is characterised in the following manner. Let $\chi_{\mathrm{Sh}(\rho)}$ denote the trace function of the irreducible representation $\mathrm{Sh}(\rho)$. Then there exists an irreducible linear representation $\tilde{\rho}$ of the semi-direct product $C \propto GL_n\mathbb{F}_{q^d}$ which is a lift of the projective homomorphism of Proposition 3.2. Let $\chi_{\tilde{\rho}}$ denote the trace function of $\tilde{\rho}$. This linear lift may be chosen in such a way that, for all $g \in GL_n\mathbb{F}_{q^d}$,

$$\chi_{\mathrm{Sh}(\rho)}((\sigma, g)^d) = \chi_{\tilde{\rho}}(\sigma, g).$$

The left side of this relation is interpreted in the following manner. The $C \propto GL_n\mathbb{F}_{q^d}$-conjugacy class of the element

$$(\sigma, g)^d = (1, g\sigma(g) \ldots \sigma^{d-1}(g))$$

intersects $GL_n\mathbb{F}_q$ in a unique $GL_n\mathbb{F}_q$-conjugacy class. The left side of the characterising relation denotes $\chi_{\mathrm{Sh}(\rho)}$ applied to any element of this $GL_n\mathbb{F}_q$-conjugacy class.

Example 4.2. $GL_2\mathbb{F}_4$
The group $GL_2\mathbb{F}_2$ is the dihedral group of order six whose irreducible representation consist of two one-dimensions, 1 and χ and a two-dimensional irreducible ν. Therefore the Shintani correspondence takes the form

$$\mathrm{Irr}(GL_2\mathbb{F}_4)^C = \{1, \nu_4, \nu_5\} \leftrightarrow \mathrm{Irr}(GL_2\mathbb{F}_2) = \{1, \chi, \nu\}.$$

Setting $g = 1$ in the characterising relation we find from Example 3.3(iii) that

$$\dim(\mathrm{Sh}(\nu_5)) = \chi_{\mathrm{Sh}(\nu_5)}((1, 1)) = \chi_{\tilde{\nu}_5}(\sigma, 1) = 1.$$

Since $\mathrm{Sh}(1) = 1$ we must have $\mathrm{Sh}(\nu_5) = \chi$ and $\mathrm{Sh}(\nu_4) = \nu$. This agrees with Example 3.3(ii) since $\chi_{\tilde{\nu}_4}(\sigma, 1) = \pm 2$ for the two choices of U_σ.

Part 2. Explicit Brauer Induction

5. The canonical homomorphism a_G

The homomorphism a_G is an explicit formula for Brauer's Induction Theorem, discovered by Robert Boltje [6]. The first such explicit formula (a derivation rather than a homomorphism) appeared in [74] and a topological formula for a_G, analogous to that of [74], was given by Peter Symonds [79]. The material of this section and its notation is taken from [75].

Definition 5.1. Let G be a finite group and let $R_+(G)$ denote the free abelian group on G-conjugacy classes of characters, $\phi : H \longrightarrow \mathbb{C}^*$, where $H \subseteq G$. We shall denote this character by (H, ϕ) and its G-conjugacy class by $(H, \phi)^G \in R_+(G)$.

If $J \subseteq G$ we define a restriction homomorphism $\mathrm{Res}_J^G : R_+(G) \longrightarrow R_+(J)$ by the double coset formula ([75] p.32)

$$\mathrm{Res}_J^G((H,\phi)^G) = \sum_{z \in J\backslash G/H} (J \cap zHz^{-1}, (z^{-1})^*(\phi))^J$$

where $(z^{-1})^*(\phi)(u) = \phi(z^{-1}uz) \in \mathbb{C}^*$. If $\pi : J \longrightarrow G$ is a surjection we define an inflation homomorphism $\pi^* : R_+(G) \longrightarrow R_+(J)$ by $\pi^*((H,\phi)^G) = (\pi^{-1}(H), \phi\pi)^J$. These maps make $R_+(-)$ into a contravariant functor from finite groups to abelian groups.

Define a homomorphism $b_G : R_+(G) \longrightarrow R(G)$ by $b_G((H,\phi)^G) = \mathrm{Ind}_H^G(\phi)$, the representation of G obtained from ϕ by induction. Then b_G is surjective because $b_G \cdot a_G = 1$.

5.2. *Axioms for a_G* The homomorphism a_G is uniquely characterised by functoriality and a normalisation property on one-dimensional characters of G.

(i) For $H \leq G$ the following diagram commutes.

$$\begin{array}{ccc} R(G) & \xrightarrow{a_G} & R_+(G) \\ {\scriptstyle Res_H^G}\downarrow & & \downarrow{\scriptstyle Res_H^G} \\ R(H) & \xrightarrow{a_H} & R_+(H) \end{array}$$

(ii) Let $\rho : G \longrightarrow GL_n(\mathbf{C})$ be a representation and suppose that

$$a_G(\rho) = \sum \alpha_{(H,\phi)^G} (H,\phi)^G \in R_+(G)$$

then $\alpha_{(G,\phi)^G} =< \rho, \phi >$ for each $(H,\phi)^G$ such that $H = G$. In particular, if ρ is one-dimensional then $a_G(\rho) = (G,\rho)^G$.

5.3. *The formula for $a_G(\rho)$*

The formula for $a_G(\rho)$ is given by ([75] Theorem 2.3.15 p. 48)

$$a_G(\rho) = \frac{1}{|G|} \sum_{(H,\phi)\leq(H',\phi') \text{ in } \mathcal{M}_G} |H| \mu^{\mathcal{M}_G}_{(H,\phi),(H',\phi')} \cdot < \phi', \mathrm{Res}^G_{H'}(\rho) >_{H'} \cdot (H,\phi)^G.$$

Here $< \phi', \mathrm{Res}^G_{H'}(\rho) >_{H'}$ is the Schur inner product of ϕ' and the restriction of ρ as representations of H' and $\mathcal{M}_G$ denotes the poset of pairs (*not* G-conjugacy classes of pairs) (H,ϕ). The Möbius function of the ordered pair $((H,\phi),(H',\phi'))$ in $\mathcal{M}_G$ is the integer defined by the alternating sum of the number chains in $\mathcal{M}_G$ from (H,ϕ) to (H',ϕ')

$\mu^{\mathcal{M}_G}_{(H,\phi),(H',\phi')}$

$= \sum_i (-1)^i \#\{\text{chains of length i with } (\mathrm{H}_0, \phi_0) = (\mathrm{H}, \phi), (\mathrm{H}_\mathrm{i}, \phi_\mathrm{i}) = (\mathrm{H}', \phi')\}.$

A chain of length i is a totally ordered subset of $\mathcal{M}_G$ of the form

$$(H_0, \phi_0) \overset{<}{\neq} (H_1, \phi_1) \overset{<}{\neq} \ldots \overset{<}{\neq} (H_i, \phi_i).$$

6. Explicit Brauer Induction data for $C_2 \propto PGL_2\mathbb{F}_4$

6.1. $\mathcal{M}_{C_2 \propto PGL_2\mathbb{F}_4}$

In order to compute the coefficients in the formula of §5.3 for $a_{PGL_2\mathbb{F}_4}(\nu_4)$, $a_{PGL_2\mathbb{F}_4}(\nu_5)$, $a_{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)\propto PGL_2\mathbb{F}_4}(\tilde{\nu}_4)$ and $\mathrm{a}_{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)\propto PGL_2\mathbb{F}_4}(\tilde{\nu}_5)$ we need to tabulate all the $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \propto PGL_2\mathbb{F}_4$-conjugacy classes of pairs $(H, \lambda) \in \mathcal{M}_{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)\propto PGL_2\mathbb{F}_4}$. This are given in the following table where $\lambda \sim \lambda'$ indicates conjugacy, $\xi_n = e^{2\pi\sqrt{-1}/n}$ and $\hat{H} = \mathrm{Hom}(H, \mathbb{C}^*)$, the group of characters of H.

H	$\hat{H}$	formulae
A_5	1	$-$
A_4	$1, \phi \sim \phi^2$	$\phi(Y) = \xi_3, \phi(A^iB^j) = 1$
D_{10}	$1, \phi$	$\phi(A) = -1, \phi(X_\xi) = 1$
D_6	$1, \phi$	$\phi(A) = -1, \phi(C) = 1$
C_5	$1, \phi \sim \phi^4 \sim \phi^2 \sim \phi^3$	$\phi(X_\xi) = \xi_5$
V_4	$1, \mu_1 \sim \mu_2 \sim \mu_3$	$\mu_1(A) = -1, \mu_1(B) = 1$
C_3	$1, \phi \sim \phi^2$	$\phi(C) = \xi_3$
C_2	$1, \phi$	$\phi(A) = -1$
$\{1\}$	1	$-$
$\langle(\sigma, 1)\rangle \cong C_2$	$1, \tau$	$\tau((\sigma, 1)) = -1$
$\langle(\sigma, 1), A\rangle \cong C_2 \times C_2$	$1, \tau, \phi \sim \tau\phi$	$\tau((\sigma, 1)) = -1 = \phi(A)$
C_4	$1, \phi \sim \phi^3, \phi^2$	$\phi((\sigma, B)) = \xi_4$
$\langle(\sigma, 1), C\rangle \cong C_2 \times C_3$	$1, \tau, \phi \sim \phi^2, \tau\phi \sim \tau\phi^2$	$\tau((\sigma, 1)) = -1, \phi(C) = \xi_3$
$\langle(\sigma, A), C\rangle \cong D_6$	$1, \tau$	$\tau((\sigma, A)) = -1, \tau(C) = 1$
$\langle(\sigma, 1), V_4\rangle$	$1, \tau, \mu, \tau\mu$	$\tau((\sigma, 1)) = -1 = \mu(A^iB)$
$\langle(\sigma, 1), A, C\rangle \cong C_2 \times D_6$	$1, \tau, \phi, \tau\phi$	$\tau((\sigma, 1)A^i) = -1$ $\phi((\sigma, 1)^iA) = -1$
$\langle(\sigma, B), X_\xi\rangle \cong C_4 \propto C_5$	$1, \tau, \tau^2, \tau^3$	$\tau((\sigma, B)) = \sqrt{-1}, \tau(X_\xi) = 1$
$\langle(\sigma, 1), A_4\rangle$	$1, \tau, \phi \sim \phi^2, \tau\phi \sim \tau\phi^2$	$\tau((\sigma, 1)) = -1, \phi(Y) = \xi_3,$ $\phi(A^iB^j) = 1$
$\langle(\sigma, 1), A_5\rangle$	$1, \tau$	$\tau((\sigma, 1)) = -1$

Note: (A_4, ϕ) and (A_4, ϕ^2) are conjugate in $C_2 \propto PGL_2\mathbb{F}_4$ (as is seen from the relation at the end of §3.4) but *not* in $PGL_2\mathbb{F}_4$. Similarly (C_5, ϕ^i) for $i = 1, 2, 3, 4$ are all conjugate in $C_2 \propto PGL_2\mathbb{F}_4$ (via (σ, B)) but only $(C_5, \phi^i) \sim (C_5, \phi^{4i})$ in $PGL_2\mathbb{F}_4$.

6.2. $\mathrm{Res}_H^{C_2\propto PGL_2\mathbb{F}_4}(\tilde{\nu}_4)_{ab}$ *and* $\mathrm{Res}_H^{C_2\propto PGL_2\mathbb{F}_4}(\tilde{\nu}_5)_{ab}$

In order to compute the Schur inner products which appear in the formula of §5.3 for $\rho = \tilde{\nu}_4$ and $\rho = \tilde{\nu}_5$ we need to know the multiplicities of each of characters of $\hat{H}$ which appear in the restriction of $\tilde{\nu}_4$ and $\tilde{\nu}_5$ to H. We denote the sum of these one-dimensional representations (with multiplicities) by $\mathrm{Res}_H^{C_2 \propto PGL_2\mathbb{F}_4}(\tilde{\nu}_4)_{ab}$ and $\mathrm{Res}_H^{C_2 \propto PGL_2\mathbb{F}_4}(\tilde{\nu}_5)_{ab}$. These "abelian parts" of the restrictions to H are given in the table below.

H	$\tilde{\nu}_4$	$\tilde{\nu}_5$
A_5	0	0
A_4	1	$\phi + \phi^2$
D_{10}	0	1
D_6	$1 + \phi$	1
C_5	$\phi + \phi^2 + \phi^3 + \phi^4$	$1 + \phi + \phi^2 + \phi^3 + \phi^4$
V_4	$1 + \mu_1 + \mu_2 + \mu_3$	$2 \cdot 1 + \mu_1 + \mu_2 + \mu_3$
C_3	$2 \cdot 1 + \phi + \phi^2$	$1 + 2\phi + 2\phi^2$
C_2	$2 \cdot 1 + 2 \cdot \phi$	$3 \cdot 1 + 2\phi$
$\{1\}$	$4 \cdot 1$	$5 \cdot 1$
$\langle(\sigma, 1)\rangle \cong C_2$	$1 \cdot 1 + 3 \cdot \tau$	$3 \cdot 1 + 2\tau$
$\langle(\sigma, 1), A\rangle \cong C_2 \times C_2$	$2 \cdot \tau + \phi + \tau\phi$	$2 \cdot 1 + \tau + \phi + \tau\phi$
C_4	$\phi + \phi^3 + \phi^2 + 1$	$1 + 2\phi^2 + \phi + \phi^3$
$\langle(\sigma, 1), C\rangle \cong C_2 \times C_3$	$1 + \tau + \tau\phi + \tau\phi^2$	$1 + \phi + \tau\phi + \phi^2 + \tau\phi^2$
$\langle(\sigma, A), C\rangle \cong D_6$	$2 \cdot \tau$	1
$\langle(\sigma, 1), V_4\rangle$	$\tau + \tau\mu$	$1 + \mu + \tau$
$\langle(\sigma, 1), A, C\rangle \cong C_2 \times D_6$	$\tau + \phi$	1
$\langle(\sigma, B), X_\xi\rangle \cong C_4 \propto C_5$	0	τ^2
$\langle(\sigma, 1), A_4\rangle$	τ	0
$\langle(\sigma, 1), A_5\rangle$	0	0

6.3. *$a_{PGL_2\mathbb{F}_4}(\nu_4)$ and $a_{PGL_2\mathbb{F}_4}(\nu_5)$*

Recalling that $A_5 \cong PGL_2\mathbb{F}_4$, a calculation of the formulae for $a_{PGL_2\mathbb{F}_4}(\nu_{3,1})$ may be found in ([75] p.50). Similar calculations using the data from the tables of §6.1 and §6.2 yield the following formulae:

$$\begin{aligned} a_{PGL_2\mathbb{F}_4}(\nu_4) \; &= (A_4,1)^{PGL_2\mathbb{F}_4} + (D_6,1)^{PGL_2\mathbb{F}_4} + (D_6,\phi)^{PGL_2\mathbb{F}_4} + (C_5,\phi)^{PGL_2\mathbb{F}_4} \\ &\quad +(C_5,\phi^2)^{PGL_2\mathbb{F}_4} - (C_3,1)^{PGL_2\mathbb{F}_4} + (C_3,\phi)^{PGL_2\mathbb{F}_4} \\ &\quad +(V_4,\mu_1)^{PGL_2\mathbb{F}_4} - (C_2,1)^{PGL_2\mathbb{F}_4} - (C_2,\phi)^{PGL_2\mathbb{F}_4} \end{aligned}$$

and

$$\begin{aligned} a_{PGL_2\mathbb{F}_4}(\nu_5) \; &= (A_4,\phi)^{PGL_2\mathbb{F}_4} + (A_4,\phi^2)^{PGL_2\mathbb{F}_4} + (D_{10},1)^{PGL_2\mathbb{F}_4} + (D_6,1)^{PGL_2\mathbb{F}_4} \\ &\quad +(C_5,\phi)^{PGL_2\mathbb{F}_4} + (C_5,\phi^2)^{PGL_2\mathbb{F}_4} \\ &\quad +(V_4,\mu_1)^{PGL_2\mathbb{F}_4} - 2(C_2,1)^{PGL_2\mathbb{F}_4} \end{aligned}$$

6.4. *$a_{C_2 \propto PGL_2\mathbb{F}_4}(\tilde{\nu}_4)$ and $a_{C_2 \propto PGL_2\mathbb{F}_4}(\tilde{\nu}_5)$*

Setting $G = C_2 \propto PGL_2\mathbb{F}_4$, the formula of §5.3 together with the tabulated data of §6.1 and §6.2 yield the following form for $a_G(\tilde{\nu}_5)$, which follows from the detailed calculations of Appendix §10:

$$a_G(\tilde{\nu}_5)$$

$$= (A_4, \phi)^G + (\langle(\sigma, B), X_\xi\rangle, \tau^2)^G + (\langle(\sigma, 1), A, C\rangle, 1)^G$$

$$+(\langle(\sigma, 1), V_4\rangle, 1)^G + (\langle(\sigma, 1), V_4\rangle, \tau)^G + (\langle(\sigma, 1), V_4\rangle, \mu)^G$$

$$+(\langle(\sigma, 1), C\rangle, \phi)^G + (\langle(\sigma, 1), C\rangle, \tau\phi)^G + (C_5, \phi)^G$$

$$-(V_4, 1)^G - (C_3, \phi)^G - (C_2, \phi)^G$$

$$+(\{1\}, 1)^G - (\langle(\sigma, 1)\rangle, 1)^G - (\langle(\sigma, 1)\rangle, \tau)^G - (\langle(\sigma, 1), A\rangle, 1)^G$$

$$+(\langle(\sigma, 1), A\rangle, \phi)^G + (C_4, \phi)^G - (C_4, \phi^2)^G.$$

Similarly for $\tilde{\nu}_4$ we obtain

$$a_G(\tilde{\nu}_4)$$

$$= (\langle(\sigma, 1), A_4\rangle, \tau)^G + (C_2 \times D_6, \tau)^G + (C_2 \times D_6, \phi)^G$$

$$+(C_5, \phi)^G + (\langle(\sigma, 1), V_4\rangle, \tau\mu)^G + (\langle(\sigma, 1), C\rangle, \tau\phi)^G$$

$$-(C_2, \phi)^G - (\langle(\sigma, 1), A\rangle, \tau)^G$$

$$+(C_4, \phi)^G - (\langle(\sigma, 1), C\rangle, \tau)^G.$$

Part 3. Galois descent algorithms

7. The weak descent algorithm

7.1. *Descent of representations*

In this section we are interested in the following construction. Suppose given a finite-dimensional irreducible, complex representation ρ of a group G which is invariant under under the action of a subgroup $C \subseteq \mathrm{Aut}(G)$. This gives rise to a projective representation

$$\tilde{\rho} : C \propto G \longrightarrow \mathrm{ProjAut}_{\mathbb{C}}(\mathcal{H})$$

where $\mathcal{H}$ is the underlying vector space of ρ.

Suppose that there is a linear lift of $\tilde{\rho}$ (which we shall also denote by $\tilde{\rho}$)

$$\tilde{\rho} : C \propto G \longrightarrow \mathrm{GL}(\mathcal{H}).$$

There are several choices of this linear lift but any two will differ by a twist by a one-dimensional representation of the form $\tau : C \propto G \longrightarrow C \longrightarrow \mathbb{C}^*$.

The same is true for $a_{C\propto G}(\tilde{\rho}) \in R_+(C \propto G)$ since $a_{C\propto G}$ commutes with twisting by one-dimensional representations.

Let G^C denote the subgroup of C-fixed points of G. Consider the endomorphism $\mathcal{E}$ of $R_+(C \propto G)$ which is given on the free generators by

$$\mathcal{E}(H,\phi)^{C\propto G} = \begin{cases} (H,\phi)^{C\propto G} & \text{if H is conjugate to H}' \subseteq \text{C} \times \text{G}^\text{C} \\ & \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we may write

$$\mathcal{E}(a_{C\propto G}(\tilde{\rho})) = \sum_{H \subseteq C \times G^C} \alpha_{(H,\phi)^{C\propto G}} \cdot (H,\phi)^{C\propto G},$$

which is well-defined up to twists by a one-dimensional representations τ. Therefore

$$\mathcal{E}(a_{C\propto G}(\tilde{\rho})) \in \mathrm{Im}(\mathrm{Ind}_{C\times G^C}^{C\propto G} : R_+(C \times G^C) \longrightarrow R_+(C \propto G)),$$

well-defined up to one-dimensional twists.

Similarly in terms of representations

$$b_{C\propto G}(\mathcal{E}(a_{C\propto G}(\tilde{\rho}))) \in \mathrm{Im}(\mathrm{Ind}_{C\times G^C}^{C\propto G} : R(C \times G^C) \longrightarrow R(C \propto G)),$$

well-defined up to one-dimensional twists.

We have a commutative diagram

$$\begin{array}{ccc} R(C \times G^C) & \xrightarrow[\mathrm{Ind}_{C\times G^C}^{C\propto G}]{} & R(C \propto G) \\ \Big\downarrow \mathrm{Res}_{G^C}^{C\times G^C} & & \Big\downarrow \mathrm{Res}_{G}^{C\propto G} \\ R(G^C) & \xrightarrow{\mathrm{Ind}_{G^C}^{G}} & R(G) \end{array}$$

so that

$$\mathrm{Res}_G^{C\propto G}(b_{C\propto G}(\mathcal{E}(a_{C\propto G}(\tilde{\rho})))) \in \mathrm{Im}(\mathrm{Ind}_{G^C}^{G} : R(G^C) \longrightarrow R(G))$$

is a well-defined element depending only on ρ.

For example, if C is cyclic then the linear lift always exists.

Finally, if $\mathrm{Ind}_{G^C}^{G}$ is injective, we have a construction of the form

$$\mathrm{Irr}(G)^C \longrightarrow R(G^C).$$

The following result, proved in Appendix §11, will be required in Example 7.3.

Lemma 7.2. (i) The induction homomorphism

$$\mathrm{Ind}_{D_6}^{PGL_2\mathbb{F}_4} : R(D_6) \longrightarrow R(PGL_2\mathbb{F}_4)$$

is injective.

(ii) The kernel of the induction homomorphism

$$\mathrm{Ind}_{C_2\times D_6}^{C_2\propto PGL_2\mathbb{F}_4} : R(C_2 \times D_6) \longrightarrow R(C_2 \propto PGL_2\mathbb{F}_4)$$

is equal to $\langle(1-\tau)\otimes(1+2\chi+3\nu)\rangle$ where ν is the 2-dimensional irreducible representation of D_6 and τ, χ are the non-trivial quadratic characters of C_2, D_6, respectively.

Example 7.3. *$G = GL_2\mathbb{F}_4$ and $C = \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$*

In this example $G^C = GL_2\mathbb{F}_2 = PGL_2\mathbb{F}_2 \cong D_6$. By Lemma 7.2

$$\mathrm{Ind}_{PGL_2\mathbb{F}_2}^{PGL_2\mathbb{F}_4} : R(PGL_2\mathbb{F}_2) \longrightarrow R(PGL_2\mathbb{F}_4)$$

is injective and so is the analogous homomorphism for GL_2.

From the formulae of §6.4 one finds that

$$b_{C\propto G}(\mathcal{E}(a_{C_2\propto G}(\tilde{\nu}_5))) = \mathrm{Ind}_{C_2\times C^C}^{C_2\propto G}(-\tau\otimes\chi - (1+\tau)\otimes\nu)$$

and

$$b_{C\propto G}(\mathcal{E}(a_{C_2\propto G}(\tilde{\nu}_4))) = \mathrm{Ind}_{C_2\times C^C}^{C_2\propto G}(-(1+\tau)\otimes\nu - \tau\otimes(1+2\chi)).$$

These elements are only determined by ν_4 and ν_5 up to tensoring with τ so to obtain elements determined which are uniquely by ν_4 and ν_5 we should form

$$(1+\tau)\cdot b_{C\propto G}(\mathcal{E}(a_{C_2\propto G}(\tilde{\nu}_5))) = \mathrm{Ind}_{C_2\times C^C}^{C_2\propto G}(-(1+\tau)\otimes(\chi+2\otimes\nu)$$

and

$$(1+\tau)\cdot b_{C\propto G}(\mathcal{E}(a_{C_2\propto G}(\tilde{\nu}_4))) = \mathrm{Ind}_{C_2\times C^C}^{C_2\propto G}(-(1+\tau)\otimes(2\nu+1+2\chi)).$$

Note that, by Lemma 7.2(ii), the homomorphism

$$(1+\tau)\cdot\mathrm{Ind}_{C_2\times D_6}^{C_2\propto PGL_2\mathbb{F}_4} : R(C_2\times D_6) \longrightarrow R(C_2\propto PGL_2\mathbb{F}_4)$$

is injective.

Remark 7.4. If we were to apply a similar algorithm - deleting terms which are not subconjugate to $PGL_2\mathbb{F}_2$ - to $a_{PGL_2\mathbb{F}_4}(\nu_4)$ and $a_{PGL_2\mathbb{F}_4}(\nu_5)$ we would obtain the following results. The irreducible ν_5 would yield $-1-2\nu$ and for ν_4 we would obtain $-1-\chi-\nu$. Applying $(1+\tau)\cdot\mathrm{Ind}_{C_2\times D_6}^{C_2\propto PGL_2\mathbb{F}_4}$ to these elements yields

$$(1+\tau)\otimes(-1-2\nu) \text{ and } (1+\tau)\otimes(-1-\chi-\nu),$$

respectively.

Subtracting these elements from the elements of §7.3 yields

$$\nu_5 \mapsto -(1+\tau)\otimes(\chi-1) \text{ and } \nu_4 \mapsto -(1+\tau)\otimes(\nu+\chi).$$

It is interesting to compare these values with the Shintani correspondence of §4.2.

8. The strong descent algorithm

8.1. *Monomial complexes*

This section is merely a sketch and is suitable only for those familiar with the monmial complexes and monomal resolutions of [7] (see also [77]). In the derived category of monomial complexes there exists a unique monomial resolution for $\tilde{\rho}$ which possesses an Euler characteristic equal to $a_{C \propto G}(\tilde{\rho}) \in R_+(C \propto G)$. The Lines of the form $\underline{\mathrm{Ind}}_H^{C \propto G}(\phi)$ with H not subconjugate to $C \times G^C$ form a sub-monomial complex. The quotient monomial complex has an Euler characteristic in

$$\mathrm{Im}(\mathrm{Ind}_{C \times G^C}^{C \propto G} : R_+(C \times G^C) \longrightarrow R_+(C \propto G))$$

which is the one featured in §7.1 and Example 7.2.

Since I have not computed the monomial resolution in this paper I cannot give here the monomial complex computation which is analogous to §7.

Part 4. Appendices

9. Appendix A: Subgroups of $C_2 \propto A_5$

9.1. The binary isocahedral group B_{12} (see [61] pp.111-112) has order 120 and has $A_5 \cong PGL_2\mathbb{F}_4$ as a quotient in a central extension

$$\{\pm 1\} \longrightarrow B_{120} \longrightarrow A_5.$$

The maximal subgroups of B_{120} are:

(ii) 6 conjugate copies of the quaternion group Q_{20},
(ii) 5 conjugate copies of the binary tetrahedral group B_{24} and
(iii) 10 conjugate copies of the quaternion group Q_{12}.

Each of these maximal subgroups has image in A_5 a maximal subgroup given by dividing out by a central element of order two. Therefore the quotients of Q_{20} and Q_{12} must be dihedral groups D_{10} and D_6. The image of B_{24} is also easy to describe up to conjugacy because any subgroup of order 12 in A_5 will suffice. By §1.1 $PGL_2\mathbb{F}_4$ is generated by X_ξ of order 5, Y of order 3 and $V_4 = \langle A, B \rangle$ with Y normalising V_4. Hence up to conjugacy the image of B_{24} is $\langle Y, A, B \rangle$.

A copy of $D_{10} = \langle X_\xi, AB \rangle$ since $ABX_\xi AB = X_\xi^4$ and a copy of $D_6 = \langle A, C \rangle$ in the notation of §1.2.

Next I shall list the subgroups up to conjugacy in the semi-direct product $C_2 \propto A_5$.

To determine the subgroups $J \subseteq C_2 \propto PGL_2\mathbb{F}_4$ up to conjugacy we have those for which $J \subseteq PGL_2\mathbb{F}_4$, which are listed in §1.2. For J whose projection $J \subseteq C_2 \propto PGL_2\mathbb{F}_4 \longrightarrow C_2$ is non-trivial we have a subgroup of index two $J' \subset J$ which we may assume is one of the list of §1.2. Furthermore there is $(\sigma, X) \in C_2 \propto PGL_2\mathbb{F}_4$ which normalises J' and for which $(\sigma, X)(\sigma, X) = (1, X\sigma(X)) \in J'$.

From the table of §1.2 we know that the list of subgroups of A_5 up to conjugacy J' is equal to one of the following:

$$\{1\}, C_2 = \langle A \rangle, C_3 = \langle C \rangle, V_4 = \langle A, B \rangle, C_5 = \langle X_\xi \rangle, D_6 = \langle A, C \rangle,$$

$$D_{10} = \langle X_\xi, AB \rangle, A_4 = \langle Y, A, B \rangle, A_5.$$

Case: $J' = \{1\}$:

This case means that $J = C_2$ generated by (σ, X) such that $1 = (\sigma, X)^2 = (1, X\sigma(X))$. Since $X \in PGL_2\mathbb{F}_4$ we may suppose that X is represented by

$$\begin{pmatrix} 1 & \beta \\ \gamma & \delta \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \beta \\ \gamma & 1 \end{pmatrix}.$$

In the second case $\beta = 0$, which is impossible. In the first case

$$I = \begin{pmatrix} 1 & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & \sigma(\beta) \\ \sigma(\gamma) & \sigma(\delta) \end{pmatrix} = \begin{pmatrix} 1 + \beta\sigma(\gamma) & \sigma(\beta) + \beta\sigma(\delta) \\ \gamma + \delta\sigma(\gamma) & \gamma\sigma(\beta) + \delta\sigma(\delta) \end{pmatrix}$$

so that either $\beta = 0 \neq \gamma$, $\beta \neq 0 = \gamma$ or $\beta = 0 = \gamma$ with $\delta \neq 0$ in all three cases. We shall show that in each of these cases $\sigma, X)$ is conjugate to $(\sigma, 1)$ in $C_2 \propto PGL_2\mathbb{F}_4$. Conjugating by $(\sigma, 1)$ if necessary we may suppose that $\delta = 1, \xi$ and transposing X we may suppose that $\gamma = 0$. Of these possibilities the non-identity$X$'s are

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}, \begin{pmatrix} 1 & \xi^2 \\ 0 & \xi \end{pmatrix}.$$

Conjugating (σ, X) in these cases, respectively, by

$$(1, \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}), (1, \begin{pmatrix} 1 & \xi \\ 0 & \xi \end{pmatrix}) \text{ and } (1, \begin{pmatrix} 0 & \xi \\ 1 & \xi^2 \end{pmatrix})$$

yields $(\sigma, 1)$. Therefore any $J = C_2 \subset C_2 \propto A_5$ which is not contained in A_5 is conjugate to $C_2 = \langle(\sigma, 1)\rangle$.

Case: $J' = C_2 = \langle A \rangle$:

In this case $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and there is (σ, X) which together with $(1, A)$ generates J, which must be abelian. Hence $AX = XA$ which, $PGL_2\mathbb{F}_4$, implies that X is the identity (and $J = C_2 \times C_2 = \langle(\sigma, 1), (1, A)\rangle$) or

$$X = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$$

with $b = \xi, \xi^2$. In both cases $(\sigma, X)^2 = (1, A)$. Finally, in $C_2 \propto PGL_2\mathbb{F}_4$,

$$(\sigma, \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix})(1, A) = (\sigma, \begin{pmatrix} 1 & \xi^2 \\ \xi^2 & 1 \end{pmatrix}).$$

Therefore up to conjugation there are two types of $J \subset C_2 \propto A_5$ which are not contained in A_5 and they are

$$(i)\ J = C_2 \times C_2 = \langle(\sigma, 1), (1, A)\rangle$$

or

$$(ii)\ J = C_4 = \langle(1, A), (\sigma, \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}\rangle = \langle(\sigma, \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}\rangle.$$

Case: $J' = \langle C\rangle$

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ C^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\ C^3 = I.$$

We must have $(\sigma, X) \in J$, $X\sigma(X) \in \langle C\rangle$ and (σ, X) normalises $\langle C\rangle$. Hence

$$(\sigma, X)(1, C)(\sigma, \sigma(X)^{-1}) = (1, XCX^{-1}) \in \langle C\rangle.$$

But from the table in §1.2 we know that the normaliser of $\langle C\rangle$ is $D_6\langle A, C\rangle$ so without loss of generality we may take $X = 1, A$ and we obtain either

$$(i)\quad J = \langle(\sigma, 1), C\rangle \cong C_2 \times C_3 \text{ or } (ii)\quad J = \langle(\sigma, A), C\rangle \cong D_6.$$

$J' = V_4 = \langle A, B\rangle$

(σ, X) normalises V_4 and $X\sigma(X) \in V_4$. We have $\mathrm{Aut}(V_4) \cong \Sigma_3$, the permutations of $\{A, B, AB\}$. Since

$$(\sigma, X)(\sigma, X)(1, A)(\sigma, \sigma(X)^{-1})(\sigma, \sigma(X)^{-1}) = (1, A)$$

conjugation by (σ, X) on V_4 must be trivial or of order two.

We first show that, up to conjugation in $C_2 \propto PGL_2\mathbb{F}_4$, either (a) (σ, X) fixes all of V_4 or (b) fixes A and switches B, AB. Let $Y = \begin{pmatrix} 1 & \xi \\ \xi^2 & 0 \end{pmatrix}$ be as in §1.1. Then $Y\sigma(Y) = B \in V_4$, $\mathrm{YAY}^{-1} = AB, YBY^{-1} = A$. Therefore

$$(1, Y^{-1})(\sigma, X)(1, Y) = (\sigma, Y^{-1}X\sigma(Y)) = (\sigma, Y^{-1}XYB)$$

and $Y^{-1}X\sigma(Y)\sigma(Y)^{-1}\sigma(X)Y = Y^{-1}X\sigma(X)Y \in V_4$ so conjugating by $(1, Y)^{\pm 1}$ transforms from any case not in (a) or (b) to case (b) above.

Case (a) does not occur. It would imply that $AX = XA, BX = X\sigma(B)$ in $PGL_2\mathbb{F}_4$. If X is presented in $GL_2\mathbb{F}_4$ by $X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ the first condition implies

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \lambda \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}$$

which implies $\lambda^2 = 1$ so $\lambda = 1$ and $X = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$. Therefore

$$\begin{pmatrix} \alpha + \xi\beta & \beta + \xi\alpha \\ \\ \beta + \xi\alpha & \alpha + \xi\beta \end{pmatrix} = BX = \mu X\sigma(B) = \mu \begin{pmatrix} \alpha + \xi^2\beta & \beta + \xi^2\alpha \\ \\ \beta + \xi^2\alpha & \alpha + \xi^2\beta \end{pmatrix}.$$

and adding components from the upper rows yields $(\alpha+\beta)\xi^2 = \mu\xi(\alpha+\beta)$. Either $\alpha + \beta = 0$ or $\mu = \xi$ which implies $\alpha + \xi\beta = \xi\alpha + \beta$, both of which imply the impossible condition that $\alpha = \beta$.

Now let us consider Case (b), which implies that $AX = XA, ABX = X\sigma(B)$ in $PGL_2\mathbb{F}_4$ so that X has the same form as in case (a).

If $\beta = 0$ then $X = I$ which is consistent since $(\sigma, 1)$ normalises V_4. Similarly if $\alpha = 0$ then $X = A$ which is consistent since (σ, A) also normalises V_4 and in both cases the group is $\langle(\sigma, 1), V_4\rangle$. We are left with $X = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$. Since $\beta = 0, 1$ have been dealt with we have $\beta = \xi, \xi^2$ which give $X\sigma(X) = A$ and $X = B$ or $X = \sigma(B) = AB$. Thus in all cases the group, up to conjugation in $C_2 \propto PGL_2\mathbb{F}_4$, is

$$J = \langle(\sigma, 1), V_4\rangle = \langle(\sigma, 1), A, B\rangle.$$

<u>Case: $J' = C_5 = \langle X_\xi\rangle$</u>

We must have (σ, Z) normalising C_5 and $Z\sigma(Z) \in C_5$. Since $AX_\xi A^{-1} = AX_\xi A = X_\xi^4$ we may conjugate by $(1, A)$, which replaces Z by AZA, and therefore we may assume $Z\sigma(Z) = 1, X_\xi, X_\xi^2 \in C_5$.

When $Z\sigma(Z) = 1$ then (σZ) has order two and is conjugate to $(\sigma, 1)$. Therefore, to show that this case does not occur one lists the elements of each of the six C_5's in $PGL_2\mathbb{F}_4$ and observes that the Frobenius automorphism does not preserve any of them.

If $Z\sigma(Z) = \lambda X_\xi^i$ with $i = 1, 2, 3, 4$ we may conjugate by A and therefore assume that $i = 1, 2$. One finds that, for $i = 1, 2$,

$$Z = \sigma(\lambda)\lambda^{-1}X_\xi^{-i}Z\sigma(X_\xi^i).$$

By inspecting the cases one finds that this relation is impossible also. Therefore there are no cases in which $J' = C_5$.

<u>Case: $J' = D_6 = \langle A, C\rangle$</u>

$$A = \begin{pmatrix} 0 & 1 \\ \\ 1 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 \\ \\ 1 & 1 \end{pmatrix},\ C^2 = \begin{pmatrix} 1 & 1 \\ \\ 1 & 0 \end{pmatrix}, C^3 = I.$$

Hence one possibility is

$$J = \langle(\sigma, 1), A, C\rangle \cong C_2 \times D_6.$$

Otherwise we need (σ, X) which normalises D_6 and satisfies $(\sigma, X)^2 \in D_6$. Since $\langle C\rangle$ is a characteristic subgroup of D_6 we must have (σ, X) which normalises $\langle C\rangle$ so that up to conjugation $X = 1, A$. However we also have

$$J = \langle(\sigma, X), A, C\rangle = \langle(\sigma, 1), A, C\rangle \cong C_2 \times D_6.$$

Therefore, up to conjugation

$$J = \langle(\sigma, 1), A, C\rangle \cong C_2 \times D_6.$$

$\underline{J' = D_{10} = \langle X_\xi, A\rangle}$

For this case we need to analyse (σ, Z)'s which normalise D_{10} - and hence C_5 - such that $(\sigma, Z)^2 = (1, Z\sigma(Z)) \in D_{10}$. The analysis of the C_5 case shows that we must have $Z\sigma(Z) = \lambda X_\xi^i A$ with $0 \le i \le 4$. When $i = 0$ we know that (σ, Z) is conjugate to (σ, B) and any inspection similar to the $J' = C_5$ case shows that

$$J = C_4 \propto C_5 = \langle(\sigma, B), X_\xi\rangle.$$

Conjugating by A implies that the other cases are $i = 1, 2$ which are eliminated by an inspection similar to the $J' = C_5$ case.

$\underline{J' = A_4 = \langle Y, A, B\rangle}$

$$Y = \begin{pmatrix} 1 & \xi \\ \xi^2 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}, AB = \begin{pmatrix} \xi & 1 \\ 1 & \xi \end{pmatrix}.$$

Firstly we observe that $\sigma(A) = A$ and

$$\sigma(B) = AB \in PGL_2\mathbb{F}_4, \sigma(AB) = B \in PGL_2\mathbb{F}_4.$$

Therefore if (σ, X) normalises A_4 it must also normalise $V_4 = \langle A, B\rangle$. However

$$(\sigma, X)(1, A)(\sigma, \sigma(X)^{-1}) = (1, XAX^{-1})$$

$$(\sigma, X)(1, B)(\sigma, \sigma(X)^{-1}) = (1, XABX^{-1})$$

$$(\sigma, X)(1, AB)(\sigma, \sigma(X)^{-1}) = (1, XBX^{-1})$$

so that X normalises V_4. Therefore $X \in A_4 \subset PGL_2\mathbb{F}_4$. Therefore $\langle(\sigma, X), A_4\rangle = \langle(\sigma, 1), A_4\rangle$. The only question is whether $(\sigma, 1)$ normalises A_4. However

$$\sigma(Y) = \begin{pmatrix} 1 & \xi^2 \\ \xi & 0 \end{pmatrix} = A \begin{pmatrix} 0 & \xi \\ \xi^2 & 1 \end{pmatrix} A = AY^2A.$$

Therefore σ normalises A_4 and we obtain

$$J = C_2 \propto A_4 = \langle(\sigma, 1), Y, A, B\rangle.$$

$\underline{J' = A_5}$

In this case

$$J = \langle(\sigma, 1), A_5\rangle.$$

10. $a_{C_2 \propto PGL_2\mathbb{F}_4}(\tilde{\nu}_4)$ AND $a_{C_2 \propto PGL_2\mathbb{F}_4}(\tilde{\nu}_5)$

10.1. In §5.3 the coefficient of $(H,\phi)^G$ in $a_G(\rho)$ is zero unless $\rho^{H,\phi)}$ is non-zero. Therefore the general forms of $a_{C_2 \propto PGL_2\mathbb{F}_4}(\tilde{\nu}_i)$ for $i = 4, 5$ are the following, in which $G = C_2 \propto PGL_2\mathbb{F}_4$.

Here we have used the fact that if (H,ϕ) is maximal amongst pairs with $\rho^{H,\phi)}$ is non-zero then the coefficient of $(H,\phi)^G$ in $a_G(\rho)$ is equal to

$$\frac{1}{N_G(H,\phi) : H]} \cdot \dim_{\mathbb{C}}(\rho^{(H,\psi)}).$$

where $N_G(H,\phi)$ is the normaliser of (H,ϕ). For example, for $\tilde{\nu}_4$ and $(C_5,\phi) < (H,\psi)$ then $\dim_{\mathbb{C}}(\tilde{\nu}_4^{(H,\psi)}) = 0$. Also $C_4 \propto C_5$ normalises C_5 but does not normalise (C_5,ϕ) since $(\sigma, B)^*(\phi) = \phi^2$. Therefore $N_G((C_5,\phi)) = C_5$ and the coefficient of $(C_5,\phi)^G = (C_5,\phi^i)^G$ for $1 \leq i \leq 4$ equals 1.

Having recorded the general form of the formulae in §10.2 our first step will be to use the functoriality of a_G of §5.2 together with the formulae for $a_{PGL_2\mathbb{F}_4}(\nu_i)$ in §6.3 to simplify the expressions.

$$\begin{aligned}
& a_G(\tilde{\nu}_5) \\
& = (A_4,\phi)^G + (\langle(\sigma,B),X_\xi\rangle,\tau^2)^G + (\langle(\sigma,1),A,C\rangle,1)^G \\
& \quad +(\langle(\sigma,1),V_4\rangle,1)^G + (\langle(\sigma,1),V_4\rangle,\tau)^G + (\langle(\sigma,1),V_4\rangle,\mu)^G \\
& \quad +(\langle(\sigma,1),C\rangle,\phi)^G + (\langle(\sigma,1),C\rangle,\tau\phi)^G \\
& \quad +a_1(D_{10},1)^G + a_2(D_6,1)^G + a_3(C_5,1)^G + a_4(C_5,\phi)^G \\
& \quad +a_5(V_4,1)^G + a_6(V_4,\mu_1)^G + a_7(C_3,1)^G + a_8(C_3,\phi)^G + a_9(C_2,1)^G + a_{10}(C_2,\phi)^G \\
& \quad +a_{11}(\{1\},1)^G + a_{12}(\langle(\sigma,1)\rangle,1)^G + a_{13}(\langle(\sigma,1)\rangle,\tau)^G + a_{14}(\langle(\sigma,1),A\rangle,1)^G \\
& \quad +a_{15}(\langle(\sigma,1),A\rangle,\tau)^G + a_{16}(\langle(\sigma,1),A\rangle,\phi)^G + a_{17}(C_4,1)^G + a_{18}(C_4,\phi)^G \\
& \quad +a_{19}(C_4,\phi^2)^G + a_{20}(\langle(\sigma,1),C\rangle,1)^G
\end{aligned}$$

and

$$
\begin{aligned}
& a_G(\tilde{\nu}_4) \\
& = (\langle(\sigma,1),A_4\rangle,\tau)^G + (C_2\times D_6,\tau)^G + (C_2\times D_6,\phi)^G \\
& \quad +(C_5,\phi)^G + (\langle(\sigma,1),V_4\rangle,\tau\mu)^G + (\langle(\sigma,1),C\rangle,\tau\phi)^G \\
& \quad +b_1(A_4,1)^G + b_2(D_6,1)^G + b_3(D_6,\phi)^G + b_4(V_4,1)^G + b_5(V_4,\mu_1)^G \\
& \quad +b_6(C_3,1)^G + b_7(C_3,\phi)^G + b_8(C_2,1)^G + b_9(C_2,\phi)^G \\
& \quad +b_{10}(\{1\},1)^G + b_{11}(\langle(\sigma,1)\rangle,1)^G + b_{12}(\langle(\sigma,1)\rangle,\tau)^G + b_{13}(\langle(\sigma,1),A\rangle,\tau)^G \\
& \quad\quad +b_{14}(\langle(\sigma,1),A\rangle,\phi)^G + b_{15}(C_4,1)^G + b_{16}(C_4,\phi)^G + b_{17}(C_4,\phi^2)^G \\
& \quad +b_{18}(\langle(\sigma,1),C\rangle,1)^G + b_{19}(\langle(\sigma,1),C\rangle,\tau)^G \\
& \quad\quad +b_{20}(\langle(\sigma,A),C\rangle,\tau)^G + b_{21}(\langle(\sigma,1),V_4\rangle,\tau)^G
\end{aligned}
$$

10.2. $\mathrm{Res}^{C_2\propto PGL_2\mathbb{F}_4}_{PGL_2\mathbb{F}_4} a_{C_2\propto PGL_2\mathbb{F}_4}(\tilde{\nu}_i) = a_{PGL_2\mathbb{F}_4}(\nu_i)$

The restriction homomorphism $R_+(C_2 \propto PGL_2\mathbb{F}_4) \longrightarrow R_+(PGL_2\mathbb{F}_4)$ is given by

$$
\begin{aligned}
& \mathrm{Res}^{C_2\propto PGL_2\mathbb{F}_4}_{PGL_2\mathbb{F}_4}((H,\lambda)^G) \\
& = \textstyle\sum_{z\in PGL_2\mathbb{F}_4\backslash C_2\propto PGL_2\mathbb{F}_4/H}\ (PGL_2\mathbb{F}_4\cap zHz^{-1},(z^{-1})^*(\lambda))^{PGL_2\mathbb{F}_4} \\
& = \textstyle\sum_{z=1,(\sigma,1)\in PGL_2\mathbb{F}_4\backslash C_2\propto PGL_2\mathbb{F}_4/H}\ (PGL_2\mathbb{F}_4\cap zHz^{-1},(z^{-1})^*(\lambda))^{PGL_2\mathbb{F}_4} \\
& = \begin{cases} (PGL_2\mathbb{F}_4\cap H,\lambda)^{PGL_2\mathbb{F}_4} & \text{if } (\sigma,Z)\in H, \\ \textstyle\sum_{g=1,(\sigma,Z)}\ (PGL_2\mathbb{F}_4\cap g(H),g(\lambda))^{PGL_2\mathbb{F}_4} & \text{if } (\sigma,Z)\notin H. \end{cases}
\end{aligned}
$$

Therefore, if $G' = A_5 = PGL_2\mathbb{F}_4$,

$$
\begin{aligned}
&\mathrm{Res}^{C_2 \propto PGL_2\mathbb{F}_4}_{PGL_2\mathbb{F}_4} a_{C_2 \propto PGL_2\mathbb{F}_4}(\tilde{\nu}_5) \\
&= (A_4, \phi)^{G'} + (A_4, \phi^2)^{G'} + (D_{10}, 1)^{G'} + (D_6, 1)^{G'} \\
&\quad +2(V_4, 1)^{G'} + (V_4, \mu_1)^{G'} + (C_3, \phi)^{G'} + (C_3, \phi)^{G'} \\
&\quad +2a_1(D_{10}, 1)^{G'} + 2a_2(D_6, 1)^{G'} + 2a_3(C_5, 1)^{G'} + a_4(C_5, \phi)^{G'} + a_4(C_5, \phi^2)^{G'} \\
&\quad +2a_5(V_4, 1)^{G'} + 2a_6(V_4, \mu_1)^{G'} + 2a_7(C_3, 1)^{G'} \\
&\quad +2a_8(C_3, \phi)^{G'} + 2a_9(C_2, 1)^{G'} + 2a_{10}(C_2, \phi)^{G'} \\
&\quad +2a_{11}(\{1\}, 1)^{G'} + a_{12}(\{1\}, 1)^{G'} + a_{13}(\{1\}, 1)^{G'} + a_{14}(C_2, 1)^{G'} \\
&\quad +a_{15}(C_2, 1)^{G'} + a_{16}(C_2, \phi)^{G'} + a_{17}(C_2, 1)^{G'} + a_{18}(C_2, \phi)^{G'} \\
&\quad +a_{19}(C_2, 1)^{G'} + a_{20}(C_3, 1)^{G'} \\
&= a_{PGL_2\mathbb{F}_4}(\nu_5).
\end{aligned}
$$

Therefore, comparing with the formulae of §6.3 we obtain the following coefficient relations.

$$
\begin{aligned}
&1 + 2a_1 = 1, 1 + 2a_2 = 1, 2a_3 = 0, a_4 = 1, 2 + 2a_5 = 0, \\
&1 + 2a_6 = 1, 2a_7 + a_{20} = 0, 2a_8 + 2 = 0, \\
&2a_9 + a_{14} + a_{15} + a_{17} + a_{19} = -2 \\
&2a_{10} + a_{16} + a_{18} = 0, 2a_{11} + a_{12} + a_{13} = 0.
\end{aligned}
$$

$$\mathrm{Res}^{C_2 \propto PGL_2\mathbb{F}_4}_{PGL_2\mathbb{F}_4} a_{C_2 \propto PGL_2\mathbb{F}_4}(\tilde{\nu}_4)$$

$$= (A_4, 1)^{G'} + (D_6, 1)^{G'} + (D_6, \phi)^{G'} + (C_5, \phi)^{G'}$$

$$+(C_5, \phi^2)^{G'} + (V_4, \mu_1)^{G'} + (C_3, \phi)^{G'}$$

$$+2b_1(A_4, 1)^{G'} + 2b_2(D_6, 1)^{G'} + 2b_3(D_6, \phi)^{G'} + 2b_4(V_4, 1)^{G'} + 2b_5(V_4, \mu_1)^{G'}$$

$$+2b_6(C_3, 1)^{G'} + 2b_7(C_3, \phi)^{G'} + 2b_8(C_2, 1)^{G'} + 2b_9(C_2, \phi)^{G'}$$

$$+2b_{10}(\{1\}, 1)^{G'} + b_{11}(\{1\}, 1)^{G'} + b_{12}(\{1\}, 1)^{G'} + b_{13}(C_2, 1)^{G'}$$

$$+b_{14}(C_2, \phi)^{G'} + b_{15}(C_2, 1)^{G'} + b_{16}(C_2, \phi)^{G'} + b_{17}(C_2, 1)^{G'}$$

$$+b_{18}(C_3, 1)^{G'} + b_{19}(C_3, 1)^{G'}$$

$$+b_{20}(C_3, 1)^{G'} + b_{21}(V_4, 1)^{G'}$$

$$= a_{PGL_2\mathbb{F}_4}(\nu_4).$$

Therefore, comparing with the formulae of §6.3 we obtain the following coefficient relations.

$$1 + 2b_1 = 1, 1 + 2b_2 = 1, 1 + 2b_3 = 1, 2b_4 + b_{21} = 0$$

$$1 + 2b_5 = 1, 2b_6 + b_{18} + b_{19} + b_{20} = -1, 2b_7 + 1 = 1$$

$$2b_8 + b_{13} + b_{15} + b_{17} = -1, 2b_9 + b_{14} + b_{16} = -1$$

$$2b_{10} + b_{11} + b_{12} = 0$$

10.3. *The simplified formulae*

The simplified formulae become:

$a_G(\tilde{\nu}_5)$

$$= (A_4,\phi)^G + (\langle(\sigma,B),X_\xi\rangle,\tau^2)^G + (\langle(\sigma,1),A,C\rangle,1)^G$$

$$+(\langle(\sigma,1),V_4\rangle,1)^G + (\langle(\sigma,1),V_4\rangle,\tau)^G + (\langle(\sigma,1),V_4\rangle,\mu)^G$$

$$+(\langle(\sigma,1),C\rangle,\phi)^G + (\langle(\sigma,1),C\rangle,\tau\phi)^G$$

$$+(C_5,\phi)^G - (V_4,1)^G + a_7(C_3,1)^G - (C_3,\phi)^G + a_9(C_2,1)^G + a_{10}(C_2,\phi)^G$$

$$+a_{11}(\{1\},1)^G + a_{12}(\langle(\sigma,1)\rangle,1)^G - (a_{12}+2a_{11})(\langle(\sigma,1)\rangle,\tau)^G + a_{14}(\langle(\sigma,1),A\rangle,1)^G$$

$$+a_{15}(\langle(\sigma,1),A\rangle,\tau)^G + a_{16}(\langle(\sigma,1),A\rangle,\phi)^G + a_{17}(C_4,1)^G - (a_{16}+2a_{10})(C_4,\phi)^G$$

$$-(2+a_{14}+a_{15}+a_{17}+2a_9)(C_4,\phi^2)^G - 2a_7(\langle(\sigma,1),C\rangle,1)^G$$

and

$a_G(\tilde{\nu}_4)$

$$= (\langle(\sigma,1),A_4\rangle,\tau)^G + (C_2\times D_6,\tau)^G + (C_2\times D_6,\phi)^G$$

$$+(C_5,\phi)^G + (\langle(\sigma,1),V_4\rangle,\tau\mu)^G + (\langle(\sigma,1),C\rangle,\tau\phi)^G + b_4(V_4,1)^G$$

$$+b_6(C_3,1)^G + b_8(C_2,1)^G + b_9(C_2,\phi)^G$$

$$+b_{10}(\{1\},1)^G + b_{11}(\langle(\sigma,1)\rangle,1)^G - (b_{11}+2b_{10})(\langle(\sigma,1)\rangle,\tau)^G + b_{13}(\langle(\sigma,1),A\rangle,\tau)^G$$

$$+b_{14}(\langle(\sigma,1),A\rangle,\phi)^G + b_{15}(C_4,1)^G - (1+b_{14}+2b_9)(C_4,\phi)^G$$

$$-(1+b_{13}+b_{15}+2b_8)(C_4,\phi^2)^G + b_{18}(\langle(\sigma,1),C\rangle,1)^G + b_{19}(\langle(\sigma,1),C\rangle,\tau)^G$$

$$-(1+b_{18}+b_{19}+2b_6)(\langle(\sigma,A),C\rangle,\tau)^G - 2b_4(\langle(\sigma,1),V_4\rangle,\tau)^G$$

10.4. *Evaluation of a_7*

We list the contributions from 0-chains, 1-chains etc to the formula of §5.3, the formulae should be self-explanatory.

0-chains:

$$(C_3,1):\ 10\times\frac{3}{120}\times <1,\mathrm{Res}(\tilde{\nu}_5)>= 1/4.$$

1-chains:

$$(C_3,1) < (D_6,1): \ 10 \times \tfrac{3}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_5) >= 1/4,$$

$$(C_3,1) < (C_2 \times C_3,1): \ 10 \times \tfrac{3}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_5) >= 1/4,$$

$$(C_3,1) < (C_2 \times D_6,1): \ 10 \times \tfrac{3}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_5) >= 1/4,$$

$$(C_3,1) < (\langle(\sigma,A),C\rangle,1): \ 10 \times \tfrac{3}{120} \times 1 = 1/4.$$

2-chains:

$$(C_3,1) < (D_6,1) < (C_2 \times D_6,1): \ 10 \times \tfrac{3}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_5) >= 1/4,$$

$$(C_3,1) < (C_2 \times C_3,1) < (C_2 \times D_6,1): \ 10 \times \tfrac{3}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_5) >= 1/4,$$

$$(C_3,1) < (\langle(\sigma,A),C\rangle,1) < (C_2 \times D_6,1): \ 10 \times \tfrac{3}{120} \times 1 = 1/4.$$

Therefore $a_7 = 1/4 - 1 + 3/4 = 0$.

10.5. *Evaluation of* b_6

0-chains:

$$(C_3,1): \ 10 \times \frac{3}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_4) >= 1/2.$$

1-chains:

$$(C_3,1) < (D_6,1): \ 10 \times \tfrac{3}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_4) >= 1/4,$$

$$(C_3,1) < (D_6,\phi): \ 10 \times \tfrac{3}{120} \times < \phi, \mathrm{Res}(\tilde{\nu}_4) >= 1/4,$$

$$(C_3,1) < (C_2 \times C_3,1): \ 10 \times \tfrac{3}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_4) >= 1/4,$$

$$(C_3,1) < (C_2 \times C_3,\tau): \ 10 \times \tfrac{3}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/4,$$

$$(C_3,1) < (\langle(\sigma,A),C\rangle,\tau): \ 10 \times \tfrac{3}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/2.$$

$$(C_3,1) < (C_2 \times D_6,\tau): \ 10 \times \tfrac{3}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/4,$$

$$(C_3,1) < (C_2 \times D_6,\phi): \ 10 \times \tfrac{3}{120} \times < \phi, \mathrm{Res}(\tilde{\nu}_4) >= 1/4,$$

$$(C_3,1) < (A_4,1): \ 5 \times 4 \times \tfrac{3}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_4) >= 1/2$$

$$(C_3,1) < (\langle(\sigma,1),A_4\rangle,\tau): \ 5 \times 4 \times \tfrac{3}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/2$$

2-chains:

$$(C_3,1) < (D_6,1) < (C_2\times D_6,\tau):\ 10\times\tfrac{3}{120}\times <\tau,\mathrm{Res}(\tilde\nu_4)>= 1/4$$

$$(C_3,1) < (D_6,\phi) < (C_2\times D_6,\phi):\ 10\times\tfrac{3}{120}\times <\phi,\mathrm{Res}(\tilde\nu_4)>= 1/4$$

$$(C_3,1) < (C_2\times C_3,\tau) < (C_2\times D_6,\tau):\ 10\times\tfrac{3}{120}\times <\tau,\mathrm{Res}(\tilde\nu_4)>= 1/4$$

$$(C_3,1) < (C_2\times C_3,1) < (C_2\times D_6,\phi):\ 10\times\tfrac{3}{120}\times <\phi,\mathrm{Res}(\tilde\nu_4)>= 1/4$$

$$(C_3,1) < (\langle(\sigma,A),C\rangle,\tau) < (C_2\times D_6,\tau):\ 10\times\tfrac{3}{120}\times <\tau,\mathrm{Res}(\tilde\nu_4)>= 1/4$$

$$(C_3,1) < (\langle(\sigma,A),C\rangle,\tau) < (C_2\times D_6,\phi):\ 10\times\tfrac{3}{120}\times <\phi,\mathrm{Res}(\tilde\nu_4)>= 1/4$$

$$(C_3,1) < (A_4,1) < (\langle(\sigma,1),A_4\rangle,\tau):\ 5\times 4\times\tfrac{3}{120}\times <\tau,\mathrm{Res}(\tilde\nu_4)>= 1/2$$

$$(C_3,1) < (C_2\times C_3,\tau) < (\langle(\sigma,1),A_4\rangle,\tau):\ 5\times 4\times\tfrac{3}{120}\times <\tau,\mathrm{Res}(\tilde\nu_4)>= 1/2$$

Therefore $b_6 = 1/2 - 3 + 5/2 = 0$.

10.6. *Evaluation of* b_{18} *and* b_{19}

0-chains:

$$(C_2\times C_3,1):\ 10\times\tfrac{6}{120}\times <1,\mathrm{Res}(\tilde\nu_4)>= 1/2$$

$$(C_2\times C_3,\tau):\ 10\times\tfrac{6}{120}\times <\tau,\mathrm{Res}(\tilde\nu_4)>= 1/2$$

1-chains:

$$(C_2\times C_3,1) < (C_2\times D_6,\phi):\ 10\times\tfrac{6}{120}\times <\phi,\mathrm{Res}(\tilde\nu_4)>= 1/2$$

$$(C_2\times C_3,\tau) < (C_2\times D_6,\tau):\ 10\times\tfrac{6}{120}\times <\tau,\mathrm{Res}(\tilde\nu_4)>= 1/2$$

$$(C_2\times C_3,\tau) < (\langle(\sigma,1),A_4\rangle,\tau):\ 5\times 4\ \times\tfrac{6}{120}\times <\tau,\mathrm{Res}(\tilde\nu_4)>= 1.$$

Therefore $b_{18} = 1/2 - 1/2 = 0$ and $b_{19} = 1/2 - 3/2 = -1$.

10.7. *Evaluation of* a_{14}*,* a_{15} *and* a_{16}

0-chains:

$$(C_2\times C_2,1):\ 15\times\tfrac{4}{120}\times <1,\mathrm{Res}(\tilde\nu_5)>= 1$$

$$(C_2\times C_2,\tau):\ 15\times\tfrac{4}{120}\times <\tau,\mathrm{Res}(\tilde\nu_5)>= 1/2$$

$$(C_2\times C_2,\phi):\ 15\times\tfrac{4}{120}\times <\phi,\mathrm{Res}(\tilde\nu_5)>= 1/2$$

$$(C_2\times C_2,\tau\phi):\ 15\times\tfrac{4}{120}\times <\tau\phi,\mathrm{Res}(\tilde\nu_5)>= 1/2$$

1-chains:

$$(C_2 \times C_2, 1) < (C_2 \propto V_4, 1): \ 15 \times \tfrac{4}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_5) >= 1/2$$

$$(C_2 \times C_2, 1) < (C_2 \propto V_4, \mu): \ 15 \times \tfrac{4}{120} \times < \mu, \mathrm{Res}(\tilde{\nu}_5) >= 1/2$$

$$(C_2 \times C_2, \tau) < (C_2 \propto V_4, \tau): \ 15 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_5) >= 1/2$$

$$(C_2 \times C_2, 1) < (C_2 \times D_6, 1): \ 10 \times 3 \times \tfrac{4}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_5) >= 1$$

Therefore $a_{14} = 1 - 2 = -1$, $a_{15} = 1/2 - 1/2 = 0$ and $a_{16} = 1/2 + 1/2 = 1$.

10.8. *Evaluation of* b_{13} *and* b_{14}

0-chains:

$$(C_2 \times C_2, \tau): \ 15 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1$$

$$(C_2 \times C_2, \phi): \ 15 \times \tfrac{4}{120} \times < \phi, \mathrm{Res}(\tilde{\nu}_4) >= 1/2$$

$$(C_2 \times C_2, \tau\phi): \ 15 \times \tfrac{4}{120} \times < \tau\phi, \mathrm{Res}(\tilde{\nu}_4) >= 1/2$$

1-chains:

$$(C_2 \times C_2, \tau) < (C_2 \propto V_4, \tau\mu): \ 15 \times \tfrac{4}{120} \times < \tau\mu, \mathrm{Res}(\tilde{\nu}_4) >= 1/2$$

$$(C_2 \times C_2, \tau) < (C_2 \propto V_4, \tau): \ 15 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/2$$

$$(C_2 \times C_2, \phi) < (C_2 \times D_6, \phi): \ 10 \times 3 \times \tfrac{4}{120} \times < \phi, \mathrm{Res}(\tilde{\nu}_4) >= 1$$

$$(C_2 \times C_2, \tau) < (C_2 \times D_6, \tau): \ 10 \times 3 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1$$

$$(C_2 \times C_2, \tau) < (C_2 \propto A_4, \tau): \ 5 \times 3 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/2$$

2-chains:

$$(C_2 \times C_2, \tau) < (C_2 \propto V_4, \tau) < (C_2 \propto A_4, \tau): \ 5 \times 3 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/2$$

Therefore $b_{13} = 1 - (1/2 + 1/2 + 1 + 1/2) + 1/2 = -1$ and $b_{14} = 1/2 + 1/2 - 1 = 0$.

10.9. *Evaluation of* a_{17}, a_{18} *and* a_{19}

0-chains:

$$(C_4,1):\ 15\times\tfrac{4}{120}\times<1,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(C_4,\phi):\ 15\times\tfrac{4}{120}\times<\phi,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(C_4,\phi^2):\ 15\times\tfrac{4}{120}\times<\phi^2,\mathrm{Res}(\tilde{\nu}_5)>=1$$

$$(C_4,\phi^3):\ 15\times\tfrac{4}{120}\times<\phi^3,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

1-chains:

$$(C_4,1)<(C_2\propto V_4,1):\ 15\times\tfrac{4}{120}\times<1,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(C_4,\phi^2)<(C_2\propto V_4,\tau):\ 15\times\tfrac{4}{120}\times<\tau,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(C_4,\phi^2)<(C_2\propto V_4,\mu):\ 15\times\tfrac{4}{120}\times<\mu,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(C_4,\phi^2)<(C_4\propto C_5,\tau^2):\ 6\times 5\times\tfrac{4}{120}\times<\tau^2,\mathrm{Res}(\tilde{\nu}_5)>=1$$

Therefore $a_{17}=1/2-1/2=0$, $a_{18}=1/2+1/2=1$ and $a_{19}=1-(1/2+1/2+1)=-1$.

10.10. *Evaluation of* b_{15}, b_{16} *and* b_{17}

0-chains:

$$(C_4,1):\ 15\times\tfrac{4}{120}\times<1,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(C_4,\phi):\ 15\times\tfrac{4}{120}\times<\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(C_4,\phi^2):\ 15\times\tfrac{4}{120}\times<\phi^2,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(C_4,\phi^3):\ 15\times\tfrac{4}{120}\times<\phi^3,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

1-chains:

$$(C_4,1)<(C_2\propto V_4,\tau\mu):\ 15\times\tfrac{4}{120}\times<1,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(C_4,\phi^2)<(C_2\propto V_4,\tau):\ 15\times\tfrac{4}{120}\times<\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(C_4,\phi^2)<(C_2\propto A_4,\tau):\ 5\times\tfrac{4}{120}\times<\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

2-chains:

$$(C_4,\phi^2)<(C_2\propto V_4,\tau)<(C_2\propto A_4,\tau):\ 5\times\tfrac{4}{120}\times<\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

Therefore $b_{15} = 1/2 - 1/2 = 0$, $b_{16} = 1/2 + 1/2 = 1$ and $b_{17} = 1/2 - (1/2 + 1/6) + 1/6 = 0$.

10.11. *Evaluation of* a_{12} *and* a_{13}

0-chains:

$$(\langle(\sigma,1)\rangle,1):\ 10\times\tfrac{2}{120}\times<1,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,\phi):\ 10\times\tfrac{2}{120}\times<\phi,\mathrm{Res}(\tilde{\nu}_5)>=1/3$$

1-chains:

$$(\langle(\sigma,1)\rangle,1)<(C_2\times C_2,1):\ 15\times 2\times\tfrac{2}{120}\times<1,\mathrm{Res}(\tilde{\nu}_5)>=1$$

$$(\langle(\sigma,1)\rangle,\phi)<(C_2\times C_2,\tau):\ 15\times 2\times\tfrac{2}{120}\times<\tau,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,1)<(C_2\times C_2,\phi):\ 15\times 2\times\tfrac{2}{120}\times<\phi,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,\phi)<(C_2\times C_2,\tau\phi):\ 15\times 2\times\tfrac{2}{120}\times<\tau\phi,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,1)<(C_2\times C_3,1):\ 10\times\tfrac{2}{120}\times<1,\mathrm{Res}(\tilde{\nu}_5)>=1/6$$

$$(\langle(\sigma,1)\rangle,1)<(C_2\times C_3,\phi):\ 10\times\tfrac{2}{120}\times<\phi,\mathrm{Res}(\tilde{\nu}_5)>=1/6$$

$$(\langle(\sigma,1)\rangle,\phi)<(C_2\times C_3,\tau\phi):\ 10\times\tfrac{2}{120}\times<\tau\phi,\mathrm{Res}(\tilde{\nu}_5)>=1/6$$

$$(\langle(\sigma,1)\rangle,1)<(C_2\times C_3,\phi^2):\ 10\times\tfrac{2}{120}\times<\phi^2,\mathrm{Res}(\tilde{\nu}_5)>=1/6$$

$$(\langle(\sigma,1)\rangle,\phi)<(C_2\times C_3,\tau\phi^2):\ 10\times\tfrac{2}{120}\times<\tau\phi^2,\mathrm{Res}(\tilde{\nu}_5)>=1/6$$

$$(\langle(\sigma,A)\rangle,1)<(\langle(\sigma,A),C\rangle,1):\ 10\times 3\times\tfrac{2}{120}\times<1,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$
$$\langle B((\sigma,1)B\rangle=\langle(\sigma,A)\rangle,\langle C((\sigma,A)C^2\rangle=\langle(\sigma,AC)\rangle$$

$$(\langle(\sigma,1)\rangle,1)<(\langle(\sigma,1),V_4\rangle,1):\ 15\times 2\times\tfrac{2}{120}\times<1,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,1)<(\langle(\sigma,1),V_4\rangle,\mu):\ 15\times 2\times\tfrac{2}{120}\times<\mu,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,\phi)<(\langle(\sigma,1),V_4\rangle,\tau):\ 15\times 2\times\tfrac{2}{120}\times<\tau,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,1)<(C_2\times D_6,1):\ 10\times 4\times\tfrac{2}{120}\times<1,\mathrm{Res}(\tilde{\nu}_5)>=2/3$$

2-chains:

$$(\langle(\sigma,1)\rangle,1) < (C_2\times C_2,1) < (\langle(\sigma,1),V_4\rangle,1): \\ 15\times 2\times \tfrac{2}{120}\times <1,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,\phi) < (C_2\times C_2,\tau) < (\langle(\sigma,1),V_4\rangle,\tau): \\ 15\times 2\times \tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,1) < (C_2\times C_2,1) < (\langle(\sigma,1),V_4\rangle,\mu) \\ 15\times 2\times \tfrac{2}{120}\times <\mu,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

$$(\langle(\sigma,1)\rangle,1) < (C_2\times C_2,1) < (C_2\times D_6,1): \\ 10\times 3\times 2\times \tfrac{2}{120}\times <1,\mathrm{Res}(\tilde{\nu}_5)>=1$$

$$(\langle(\sigma,1)\rangle,1) < (C_2\times C_3,1) < (C_2\times D_6,1): \\ 10\times \tfrac{2}{120}\times <1,\mathrm{Res}(\tilde{\nu}_5)>=1/6$$

$$(\langle(\sigma,A)\rangle,1) < (\langle(\sigma,A),C\rangle,1) < (C_2\times D_6,1): \\ 10\times 3\times \tfrac{2}{120}\times <1,\mathrm{Res}(\tilde{\nu}_5)>=1/2$$

Therefore

$$\begin{aligned} a_{12} &= 1/2-(1+1/2+1/6+1/6+1/6+1/2+1/2+1/2+2/3) \\ &\quad +(1/2+1/2+1+1/6+1/2) \\ &= 1/2-(3+1/2+2/3)+(2+2/3) \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} a_{13} &= 1/3-(1/2+1/2+1/6+1/6+1/2)+(1/2) \\ &= 1/3-(1+1/3+1/2)+(1/2) \\ &= -1. \end{aligned}$$

10.12. *Evaluation of* b_{11} *and* b_{12}

0-chains:

$$(\langle(\sigma,1)\rangle,1):\ 10\times\tfrac{2}{120}\times <1,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

$$(\langle(\sigma,1)\rangle,\tau):\ 10\times\tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

1-chains:

$$(\langle(\sigma,1)\rangle,\tau)<(C_2\times C_2,\tau):\ 15\times 2\times\tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1$$

$$(\langle(\sigma,1)\rangle,1)<(C_2\times C_2,\phi):\ 15\times\tfrac{2}{120}\times <\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/4$$

$$(\langle(\sigma,1)\rangle,\tau)<(C_2\times C_2,\phi):\ 15\times\tfrac{2}{120}\times <\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/4$$

$$(\langle(\sigma,A)\rangle,1)<(C_2\times C_2,\tau\phi):\ 15\times\tfrac{2}{120}\times <\tau\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/4$$

$$(\langle(\sigma,1)\rangle,\tau)<(C_2\times C_2,\tau\phi):\ 15\times\tfrac{2}{120}\times <\tau\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/4$$

$$(\langle(\sigma,1)\rangle,1)<(C_2\times C_3,1):\ 10\times\tfrac{2}{120}\times <1,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

$$(\langle(\sigma,1)\rangle,\tau)<(C_2\times C_3,\tau):\ 10\times\tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

$$(\langle(\sigma,1)\rangle,\tau)<(C_2\times C_3,\tau\phi):\ 10\times\tfrac{2}{120}\times <\tau\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

$$(\langle(\sigma,1)\rangle,\tau)<(C_2\times C_3,\tau\phi^2):\ 10\times\tfrac{2}{120}\times <\tau\phi^2,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

$$(\langle(\sigma,A)\rangle,\tau)<(\langle(\sigma,A),C\rangle,\tau):\ 10\times 3\times\tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1$$
$$\langle B((\sigma,1)B\rangle=\langle(\sigma,A)\rangle,\langle C((\sigma,A)C^2\rangle=\langle(\sigma,AC)\rangle$$

$$(\langle(\sigma,1)\rangle,\tau)<(\langle(\sigma,1),V_4\rangle,\tau\mu):\ 15\times 2\times\tfrac{2}{120}\times <\tau\mu,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,1)\rangle,\tau)<(\langle(\sigma,1),V_4\rangle,\tau):\ 15\times 2\times\tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,1)\rangle,\tau)<(C_2\times D_6,\tau):\ 10\times 4\times\tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=2/3$$

$$(\langle(\sigma,1)\rangle,1)<(C_2\times D_6,\phi):\ 10\times\tfrac{2}{120}\times <\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

$$(\langle(\sigma,A)\rangle,\tau)<(C_2\times D_6,\phi):\ 10\times 3\times\tfrac{2}{120}\times <\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,1)\rangle,\tau)<(C_2\propto A_4,\tau):\ 5\times 6\times\tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

2-chains:

$$(\langle(\sigma,1)\rangle,\tau) < (C_2\times C_2,\tau) < (\langle(\sigma,1),V_4\rangle,\tau): \\ 15\times 2\times \tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,1)\rangle,\tau) < (C_2\times C_2,\tau) < (\langle(\sigma,1),V_4\rangle,\tau\mu): \\ 15\times 2\times \tfrac{2}{120}\times <\tau\mu,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,1)\rangle,\tau) < (C_2\times C_2,\tau) < (C_2\times D_6,\tau): \\ 10\times 3\times 2\times \tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1$$

$$(\langle(\sigma,1)\rangle,1) < (C_2\times C_2,\phi) < (C_2\times D_6,\phi): \\ 10\times 3\times \tfrac{2}{120}\times <\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,A)\rangle,\tau) < (C_2\times C_2,\phi) < (C_2\times D_6,\phi): \\ 10\times 3\times \tfrac{2}{120}\times <\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,1)\rangle,\tau) < (C_2\times C_2,\tau) < (C_2\propto A_4,\tau): \\ 5\times 3\times 2\times \tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,1)\rangle,\tau) < (C_2\times C_3,\tau) < (C_2\times D_6,\tau): \\ 10\times \tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

$$(\langle(\sigma,1)\rangle,1) < (C_2\times C_3,1) < (C_2\times D_6,\phi): \\ 10\times \tfrac{2}{120}\times <\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/6$$

$$(\langle(\sigma,A)\rangle,\tau) < (\langle(\sigma,A),C\rangle,\tau) < ((C_2\times D_6),\tau): \\ 10\times 3\times \tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,A)\rangle,\tau) < (\langle(\sigma,A),C\rangle,\tau) < ((C_2\times D_6),\phi): \\ 10\times 3\times \tfrac{2}{120}\times <\phi,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

$$(\langle(\sigma,A)\rangle,\tau) < (\langle(\sigma,A),Y\rangle,\tau) < (C_2\propto A_4,\tau): \\ 5\times 4\times 3\times \tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1$$

$$(\langle(\sigma,1)\rangle,\tau) < (\langle(\sigma,1),V_4\rangle,\tau) < (C_2\propto A_4,\tau): \\ 5\times 3\times 2\times \tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

3-chains:

$$(\langle(\sigma,1)\rangle,\tau) < (C_2\times C_2,\tau) < (C_2\propto V_4,\tau) < (C_2\propto A_4,\tau): \\ 5\times 3\times 2\times \tfrac{2}{120}\times <\tau,\mathrm{Res}(\tilde{\nu}_4)>=1/2$$

There is no inclusion of a conjugate of $C_2 \times D_6$ into a conjugate of $C_2 \propto A_4$ because the former is its own normaliser.

$$\begin{aligned} b_{11} &= 1/6 - (1/4 + 1/4 + 1/6 + 1/6) + (1/2 + 1/6) \\ &= 0. \end{aligned}$$

$$\begin{aligned} b_{12} &= 1/2 - (1 + 1/4 + 1/4 + 1/6 + 1/6 + 1/6 + 1 \\ &\qquad +1/2 + 1/2 + 2/3 + 1/2 + 1/2) \\ &+(1/2 + 1/2 + 1 + 1/2 + 1/2 + 1/6 \\ &\qquad +1/2 + 1/2 + 1 + 1/2) - 1/2 \\ &= 1/2 - (1 + 1/2 + 1/2 + 1 + 1/2 + 1/2 + 2/3 + 1/2 + 1/2) \\ &+(1/2 + 1/2 + 1 + 1/2 + 1/2 + 2/3 + 1/2 + 1 + 1/2) - 1/2 \\ &= -(4 + 1/2 + 2/3) + (1 + 1 + 1 + 2/3 + 1/2 + 1) \\ &= 0. \end{aligned}$$

10.13. *Evaluation of* b_4

0-chains:

$$(V_4, 1): \; 5 \times \tfrac{4}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_4) >= 1/6$$

1-chains:

$$(V_4, 1) < (C_2 \propto V_4, \tau): \; 15 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/2$$

$$(V_4, 1) < (A_4, 1): \; 5 \times \tfrac{4}{120} \times < 1, \mathrm{Res}(\tilde{\nu}_4) >= 1/6$$

$$(V_4, 1) < (C_2 \propto V_4, \tau): \; 5 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/6$$

2-chains:

$$\begin{gathered} (V_4, 1) < (C_2 \propto V_4, \tau) < (C_2 \propto A_4, \tau): \\ 5 \times 3 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/2 \end{gathered}$$

$$\begin{gathered} (V_4, 1) < (A_4, 1) < (C_2 \propto A_4, \tau): \\ 5 \times \tfrac{4}{120} \times < \tau, \mathrm{Res}(\tilde{\nu}_4) >= 1/6. \end{gathered}$$

Therefore $b_4 = 1/6 - (1/2 + 1/6 + 1/6) + (1/2 + 1/6) = 0$.

10.14. *The ultimately simplified formulae*
The simplified formulae become:

$$
\begin{aligned}
&a_G(\tilde{\nu}_5)\\
&= (A_4,\phi)^G + (\langle(\sigma,B),X_\xi\rangle,\tau^2)^G + (\langle(\sigma,1),A,C\rangle,1)^G\\
&\quad +(\langle(\sigma,1),V_4\rangle,1)^G + (\langle(\sigma,1),V_4\rangle,\tau)^G + (\langle(\sigma,1),V_4\rangle,\mu)^G\\
&\quad +(\langle(\sigma,1),C\rangle,\phi)^G + (\langle(\sigma,1),C\rangle,\tau\phi)^G\\
&\quad +(C_5,\phi)^G - (V_4,1)^G - (C_3,\phi)^G - (C_2,\phi)^G\\
&\quad +(\{1\},1)^G - (\langle(\sigma,1)\rangle,1)^G - (\langle(\sigma,1)\rangle,\tau)^G - (\langle(\sigma,1),A\rangle,1)^G\\
&\quad +(\langle(\sigma,1),A\rangle,\phi)^G - (C_4,\phi)^G - (C_4,\phi^2)^G
\end{aligned}
$$

and

$$
\begin{aligned}
&a_G(\tilde{\nu}_4)\\
&= (\langle(\sigma,1),A_4\rangle,\tau)^G + (C_2\times D_6,\tau)^G + (C_2\times D_6,\phi)^G\\
&\quad +(C_5,\phi)^G + (\langle(\sigma,1),V_4\rangle,\tau\mu)^G + (\langle(\sigma,1),C\rangle,\tau\phi)^G\\
&\quad -(C_2,\phi)^G - (\langle(\sigma,1),A\rangle,\tau)^G + (C_4,\phi)^G - (\langle(\sigma,1),C\rangle,\tau)^G.
\end{aligned}
$$

11. The proof of Lemma 7.2

Recall that Lemma 7.2 is the following:

Lemma 11.1. (i) The induction homomorphism

$$\mathrm{Ind}_{D_6}^{PGL_2\mathbb{F}_4} : R(D_6) \longrightarrow R(PGL_2\mathbb{F}_4)$$

is injective.
(ii) The kernel of the induction homomorphism

$$\mathrm{Ind}_{C_2\times D_6}^{C_2\propto PGL_2\mathbb{F}_4} : R(C_2\times D_6) \longrightarrow R(C_2\propto PGL_2\mathbb{F}_4)$$

is equal to $\langle(1-\tau)\otimes(1-\chi)\rangle$ where τ,χ are the non-trivial quadratic characters of C_2, D_6, respectively.

Proof
The dihedral group D_6 has three irreducibles $1,\chi,\nu$ of dimensions $1,1,2$ respectively. The character values are shown in the following table, in which

the elements are labelled as in the notation of Appendices §9 and §10:

	1	χ	ν
1	1	1	2
C	1	1	-1
A	1	-1	0

To prove Part (i) we calculate the character values on $1, A$ and C. Recall that the induced character function is given by the formula

$$\mathrm{Ind}_{D_6}^{PGL_2\mathbb{F}_4}(\rho)(g) = \frac{1}{6} \sum_{y \in PGL_2\mathbb{F}_4, ygy^{-1} \in D_6} \chi_\rho(ygy^{-1}).$$

When $g = C$ then $ygy^{-1} \in D_6$ if and only if $y \in D_6$ giving three C's and three C^2's. When $g = A$ then $ygy^{-1} \in D_6 B^\epsilon$ for $\epsilon = 0, 1$. When $g = 1$ then any y will suffice. Hence we find the results of the following table of induced character values.

	$\mathrm{Ind}_{D_6}^{PGL_2\mathbb{F}_4}(\nu)$	$\mathrm{Ind}_{D_6}^{PGL_2\mathbb{F}_4}(\chi)$	$\mathrm{Ind}_{D_6}^{PGL_2\mathbb{F}_4}(1)$
C	-1	1	1
A	0	-2	2
1	20	10	10

Therefore $\mathrm{Ind}_{D_6}^{PGL_2\mathbb{F}_4}(\alpha \cdot \nu + \beta \cdot \chi + \gamma \cdot 1) = 0$ implies

$$0 = -\alpha + \beta + \gamma, \; 0 = -2\beta + 2\gamma, \; 0 = 2\alpha + \beta + \gamma$$

which implies $\alpha = 0$ and therefore $0 = \beta = \gamma$, as required.

To establish Part (ii) we calculate the character values on $(1,1), (1,A), (1,C), (\sigma,1)$ and (σ, C). The induced character function is given by the formula

$$\mathrm{Ind}_{C_2 \times D_6}^{C_2 \propto PGL_2\mathbb{F}_4}(\rho)(g) = \frac{1}{12} \sum_{y \in C_2 \propto PGL_2\mathbb{F}_4, ygy^{-1} \in C_2 \times D_6} \chi_\rho(ygy^{-1}).$$

Letting ρ run through $1, \chi, \nu, \tau \otimes 1, \tau \otimes \chi, \tau \otimes \nu$ we obtain the following table:

ρ	1	χ	ν	$\tau \otimes 1$	$\tau \otimes \chi$	$\tau \otimes \nu$
$(1,1)$	10	10	20	10	10	20
$(1,A)$	2	-2	0	2	-2	0
$(1,C)$	1	1	-1	1	1	-1
$(\sigma,1)$	2	-1	0	-2	1	0
(σ,C)	1	1	-1	-1	-1	1

This table is easily compiled using the following observations. In order for $y(1,A)y^{-1}$ to lie in $C_2 \times D_6$ it must equal one of $(1,A), (1,AC), (1,AC^2)$. The centraliser of $(1,A)$ is $C_2 \propto V_4$. In order for $y(1,C)y^{-1}$ to lie in $C_2 \times D_6$ it must be $(1,C)$ or $(1,C^2) = A(1,C)A$, which happens if and only if $y \in C_2 \times D_6$ giving 6 $(1,C)$'s and $(1,C^2)$'s.The centraliser of $(\sigma,1)$ is $C_2 \times D_6$ and $B(\sigma,1)B = (\sigma, B\sigma(B)) = (\sigma, A)$ so we obtain 24 conjugates 6 $(\sigma,1)$'s, 6 (σ, A)'s, 6 (σ, AC)'s and 6 (σ, CA)'s. Finally, the normaliser of $C_2 \times C_3$, which is generated by (σ, C), is $C_2 \times D_6$ giving 6 (σ, C)'s and 6 (σ, C^2)'s.

From the table one finds that

$$\mathrm{Ker}(\mathrm{Ind}_{C_2 \times D_6}^{C_2 \propto PGL_2 \mathbb{F}_4}) = \langle (1 + 2\chi + 3\nu)(1 - \tau) \rangle \cong \mathbb{Z}.$$

□

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