GALOIS DESCENT OF DETERMINANTS IN THE RAMIFIED CASE

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ABSTRACT. In the local, unramified case the determinantal functions associated to the group-ring of a finite group satisfy Galois descent. This note examines the obstructions to Galois determinantal descent in the ramified case.

1. INTRODUCTION

This note arose out of an attempt to answer a question posed to me by Otmar Vejakob in the autumn of 2009. I refer the reader to [3] for background literature and a description of the related context of non-commutative Iwasawa theory, which is currently a very active research topic in algebraic number theory.

Let M be a p-adic local field and O_M its valuation ring. Let G be a finite group and $O_M[G]^*$ the units in its O_M group-ring. The determinant gives a homomorphism into Galois equivariant, unit-valued functions on $R(G)$, the complex representation ring of G (see §2). The determinantal functions are those in the image of this homomorphism, $\mathrm{Det}(O_M[G]^*)$.

If $M/\mathbb{Q}_p$ is an unramified extension then there is an isomorphism of the form

$$\mathrm{Det}(O_M[G]^*)^{\mathrm{Gal}(M/\mathbb{Q}_p)} \cong \mathrm{Det}(\mathbb{Z}_p[G]^*)$$

. This Galois descent isomorpism was first proved by M.J. Taylor [2] using the Oliver-Taylor logarithm. Using Explicit Brauer Induction [5] I gave a simpler construction of the group-ring logarithm which, in turn, simplified the proof of unramified Galois descent for determinantal functions.

Otmar Venjakob's question was whether determinantal Galois descent held when M/K was ramified; a question motivated by the case of p-adic non-commutative Iwasawa algebras.

In the case when M/K is unramified the determinantal Galois descent hinges on the determination of $\hat{\alpha}_G(\mathrm{Det}(1 + A_M(G))$ where $\hat{\alpha}_G$ is defined in Proposition 3.3. This in turn hinges on an integrality result concerning the image of the group-ring logarithm. In Proposition 2.6 I generalise this integrality result to the case where M/K is arbitrary. The result gives the first signs of the difficulties which obstruct determinantal Galois descent - difficulties which are manifested in [3]. In the general case the logarithmic

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image considered in Proposition 2.6 involves in an essential wayan element $h_M \in O_M$, which may not even be $\mathrm{Gal}(M/K)$-fixed. When $M/\mathbb{Q}_p$ is unramified Galois descent depends crucially on the fact that $h_M = p$, which is fixed by the Galois action.

A second problem in the general case is that my version of the group-ring logarithm depends on a choice of lifted Frobenius. This dependence makes the equivariance of the group-ring logarithm more complicated (see Proposition 2.8).

After all these extra complications in the general case, in §3 I can only offer a very modest example of ramified determinantal Galois descent, which I have included to illustrate the intended structure of the determinantal Galois descent proof.

Written in January 2010, this note was not needed in [3] and accordingly I undertook to post it independently.

2. Determinantal Congruences

Throughout this section let p be a prime and let G be a finite group of order n. Let $N/\mathbb{Q}_p$ be a finite Galois extension containing all the n-th roots of unity and let M/K be a Galois subextension. Let O_M and $\pi_M O_M$ denote the valuation ring of M and its maximal ideal, respectively. We shall consider the group of Galois-equivariant, unit-valued functions $\mathrm{Hom}_{\mathrm{Gal}(N/M)}(R(G), O_N^*)$ where the complex representation ring $R(G)$ is identified with $R_N(G) = K_0(N[G])$ and is therefore generated by representations of the form $T : G \longrightarrow GL_u(N)$.

We have a determinantal homomorphism

$$Det : O_M[G]^* \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(N/M)}(R(G), O_N^*)$$

given by the formula

$$Det(\sum_\gamma \lambda_\gamma \gamma)(T) = det(\sum_\gamma \lambda_\gamma T(\gamma)) \in O_N^*.$$

Choose $F \in \mathrm{Gal}(M/K)$ which is a lift of the Frobenius automorphism of the residue fields. If $O_K/\pi_K = \mathbb{F}_{p^q}$ then $F(z) \equiv z^{p^q}$ (modulo $\pi_M O_M$) for all $z \in O_M$. If $\sum_\gamma \lambda_\gamma \gamma \in O_M[G]$ we may therefore define

$$F(\sum_\gamma \lambda_\gamma \gamma) = \sum_\gamma F(\lambda_\gamma)\gamma \in O_M[G],$$

so that F is a ring automorphism of $O_M[G]$. The following result generalises ([5] Theorem 4.3.10).

Theorem 2.1.

Let $z \in O_M[G]^*$. Then, for all $T \in R(G)$,

$$Det(F(z))(\psi^{p^q}(T))/(Det(z)(T))^{p^q} \in 1 + \pi_M O_N.$$

Here ψ^m denotes the m-th Adams operation on $R(G)$.

Proof

There exist integers, $\{n_i\}$ and one-dimensional representations $\{\phi_i : H_i \longrightarrow N^*\}$ such that, in $R(G)$,

$$T = \sum_i n_i Ind_{H_i}^G(\phi_i) \text{ and } \psi^{p^q}(T) = \sum_i n_i Ind_{H_i}^G(\phi_i^{p^q}).$$

By multiplicativity we have

$$\frac{Det(F(z))(\psi^{p^q}(T))}{(Det(z)(T))^{p^q}} = \prod_i \frac{Det(F(z))(Ind_{H_i}^G(\phi_i^{p^q}))^{n_i}}{(Det(z)(Ind_{H_i}^G(\phi_i)))^{p^q n_i}} \in O_N^*$$

so that we are reduced to the comparison (modulo $\pi_M O_N$) of the expressions

$$Det(\sum_\gamma \lambda_\gamma \gamma)(Ind_{H_i}^G(\phi_i))^{p^q} \text{ and } Det(\sum_\gamma F(\lambda_\gamma)\gamma)(Ind_{H_i}^G(\phi_i^{p^q}))$$

in O_N^*, where $z = \sum_\gamma \lambda_\gamma \gamma \in O_M[G]^*$. Let us abbreviate (H_i, ϕ_i) to (H, ϕ). Choose coset representatives, $x_1, \ldots, x_d \in G$, for G/H. There is a homomorphism, $\sigma : G \longrightarrow \Sigma_d$, such that for $g \in G$ $gx_i = x_{\sigma(g)(i)} h(i, g)$ with $h(i, g) \in H$. Realising these two representations on the vector space $N[G] \otimes_{N[H]} N$, with this notation we find that

$$Ind_H^G(\phi)(z) = \sum_\gamma \lambda_\gamma \sigma(\gamma) \cdot diag[\phi(h(1, \gamma)), \ldots, \phi(h(d, \gamma))]) = X$$

where $diag[u_1, \ldots, u_d]$ is the diagonal matrix whose (i, i)-th entry is equal to u_i. Also we have

$$Ind_H^G(\phi^{p^q})(F(z)) = \sum_\gamma F(\lambda_\gamma)\sigma(\gamma) \cdot diag[\phi^{p^q}(h(1, \gamma)), \ldots, \phi^{p^q}(h(d, \gamma))]) = Y.$$

The (u, v)-th entries of X and Y are given by

$$X_{u,v} = \sum_{\gamma,\ \sigma(\gamma)_{u,v}=1} \lambda_\gamma \phi(h(v, \gamma)) \text{ and } Y_{u,v} = \sum_{\gamma,\ \sigma(\gamma)_{u,v}=1} F(\lambda_\gamma)\phi(h(v, \gamma))^{p^q}$$

respectively. Therefore

$$\begin{aligned} X_{u,v}^{p^q} &\equiv \textstyle\sum_{\gamma,\ \sigma(\gamma)_{u,v}=1} \lambda_\gamma^{p^q} \phi(h(v, \gamma))^{p^q} \qquad (\text{modulo } pO_N) \\ &\equiv \textstyle\sum_{\gamma,\ \sigma(\gamma)_{u,v}=1} F(\lambda_\gamma)\phi(h(v, \gamma))^{p^q} \quad (\text{modulo } \pi_M O_N) \\ &= Y_{u,v} \end{aligned}$$

and so

$$\begin{aligned} det(X)^{p^q} &= (\textstyle\sum_{\beta\in\Sigma_d} \ \mathrm{sign}(\beta) X^l_{1,\beta(1)} \ldots X_{d,\beta(d)})^{p^q} \\ &\equiv \textstyle\sum_{\beta\in\Sigma_d} \ \mathrm{sign}(\beta) X^{p^q}_{1,\beta(1)} \ldots X^{p^q}_{d,\beta(d)} && (\text{modulo } pO_N) \\ &\equiv \textstyle\sum_{\beta\in\Sigma_d} \ \mathrm{sign}(\beta) Y_{1,\beta(1)} \ldots Y_{d,\beta(d)} && (\text{modulo } \pi_M O_N) \\ &= det(Y) && (\text{modulo } \pi_M O_N) \end{aligned}$$

which implies the result, since $det(X)$ and $det(Y)$ both lie in the units of O_N. □

Definition 2.2. In the situation of Theorem 2.1 we may define a homomorphism, which depends on the choice of Frobenius lift F,

$$\mathrm{Log}_F : O_M[G]^* \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(N/M)}(R(G), \pi_M O_N)$$

by the formula

$$\mathrm{Log}_F(z)(T) = log(Det(F(z))(\psi^{p^q}(T))/(Det(z)(T))^{p^q})$$

where, for $x \in \pi_M O_N$, $log(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^n/n \in \pi_M O_N$ as usual. We shall also denote by Log_F the composition

$$\mathrm{Log}_F : O_M[G]^* \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(N/M)}(R(G), \pi_M O_N) \subset \mathrm{Hom}_{\mathrm{Gal}(N/M)}(R(G), N).$$

Denote by $M\{G\}$ the M-vector space whose basis consists of the conjugacy classes of elements of G. Recall that there is an isomorphism of M-vector spaces ([5] Proposition 4.5.14)

$$\psi : M\{G\} \stackrel{\cong}{\longrightarrow} \mathrm{Hom}_{\mathrm{Gal}(N/M)}(R(G), N)$$

given by $\psi(\sum_\gamma \ m_\gamma \gamma)(T) = \sum_\gamma \ m_\gamma \mathrm{Trace}(T(\gamma))$.

The composition $\psi^{-1} \cdot \mathrm{Log}_F$ defines a logarithmic homomorphism, depending on the choice of Frobenius lift F, of the form

$$\alpha_G : O_M[G]^* \longrightarrow M\{G\}.$$

The Jacobson radical, $J \subseteq O_M[G]$ ([4] p.636) is the left ideal which is equal to the intersection of all the maximal left ideals of $O_M[G]$. In fact, J, is a two-sided ideal and $O_M[G]/J$ is semi-simple. Hence, by Wedderburn's theorem ([4] p.629) $O_M[G]/J$ is a product of matrix rings over division rings. However, in this finitely generated, local situation some power of J lies in $pO_M[G]$ ([1] vol. I, §5.22, p.112). Hence the division algebras have characteristic p. Since the Brauer group of a finite field vanishes, there is an isomorphism of the form $O_M[G]/J$ is isomorphic to a finite product of rings of matrices with entries in finite fields of characteristic p.

If $r \in J$ then $r^t \in pO_M[G]$ for some positive integer t ([1] vol.I, §5.22, p. 112) and the series for $(1-r)^{-1}$ converges p-adically so that $1 - r \in O_M[G]^*$.

Therefore $log(Det(1-r)(T))$ and $log(Det(F(1-r))(\psi^{p^q}(T))$ both converge p-adically in O_N and (c.f. [5] Lemma 4.3.21)

$$\mathrm{Log}_F(1-r)(T) = log(Det(F(1-r))(\psi^{p^q}(T))) - log((Det(1-r)(T))^{p^q})).$$

Define $L_{F,0}(1-r) \in M[G]$ by the p-adically convergent series

$$L_{F,0}(1-r) = p^q \sum_{n=1}^{\infty} r^n/n - \sum_{n=1}^{\infty} \hat{F}(r^n)/n$$

where $\hat{F}(\sum_{\gamma \in G} \lambda_\gamma \gamma) = \sum_{\gamma \in G} F(\lambda_\gamma)\gamma^{p^q}$.

Proposition 2.3. If $c : M[G] \longrightarrow M\{G\}$ sends $\gamma \in G$ to its conjugacy class then, in the notation of Definition 2.2,

$$\alpha_G(1-r) = c(L_{F,0}(1-r)) \in M\{G\}.$$

Proof

Let $r \in J$ and $T \in R(G)$ be as in Definition 2.2. Suppose that T is a representation and let $\lambda_1, \ldots, \lambda_u$ denote the eigenvalues of $T(r)$. Each λ_i lies in the maximal ideal of O_N ([5] Lemma 4.3.21) and therefore the following series converges:

$$\begin{aligned} log((Det(1-T(r))^{p^q})) &= p^q log(Det(1-T(r))) \\ &= p^q log(\textstyle\prod_{i=1}^{u}(1-\lambda_i)) \\ &= -p^q \textstyle\sum_{i=1}^{u}\sum_{m=1}^{\infty} \lambda_i^m/m \\ &= -p^q \textstyle\sum_{m=1}^{\infty}\sum_{i=1}^{u} \lambda_i^m/m \\ &= -p^q \textstyle\sum_{m=1}^{\infty} \ \mathrm{Trace}(T(r^m))/m. \end{aligned}$$

Similarly, since the eigenvalues of $\psi^{p^q}(T)(r)$ are $\{\lambda_i^{p^q}\}$, we find that

$$log(Det(1-F(r))(\psi^{p^q}(T))) = -\sum_{m=1}^{\infty} \mathrm{Trace}(T(\hat{F}(r^m)))/m.$$

Therefore $\psi(\alpha_G(1-r)) = \psi(c(L_{F,0}(1-r)))$ which completes the proof, since ψ is an isomorphism. $\square$

Definition 2.4.

Define an O_M-submodule Λ_G of $M\{G\}$ by

$$\Lambda_G = O_M[G]/(\sum_{x,y \in G} O_M(xy-yx)),$$

which embeds into $M\{G\}$ via c, the homomorphism of Proposition 2.3. For $r \in J$ the element $c(L_{F,0}(1-r))$ lies in Λ_G.

Define $h_{M/K} \in M$ (up to multiplication by O_M^*) by the formula

$$h_M O_M = O_M \pi_M \bigcup O_M \pi_M^p / p \bigcup O_M \pi_M^{p^2}/p^2 \bigcup \ldots \bigcup O_M \pi_M^{p^k}/p^k \bigcup \ldots .$$

If the ramification index of $M/\mathbb{Q}_p$ is equal to e, so that $pO_M = \pi_M^e O_M$, then $h_M = \pi_M^{\min_{k\geq 0}(p^k - ke)}$. For example, if$M/\mathbb{Q}_p$ is unramified then $h_M = p$.

Lemma 2.5.

For $x \in O_M$ and $k \geq 0$

$$F(x^{p^k}) \equiv x^{p^{q+k}} \text{ (modulo } \mathrm{H}(\pi_M^{p^k}, p\pi_M^{p^{k-1}}, p^2\pi_M^{p^{k-2}}, \ldots, p^k\pi_M)O_M)$$

where, in the notation of Definition 2.4,

$$\mathrm{H}(\pi_M^{p^k}, p\pi_M^{p^{k-1}}, p^2\pi_M^{p^{k-2}}, \ldots, p^k\pi_M) = p^k \pi_M^{\min_{0\leq i\leq k}(p^i - ie)}.$$

Proof

By definition of the lifted Frobenius F this is true for $k = 0$. By induction suppose that $F(x^{p^k}) = x^{p^{q+k}} + \mathrm{H}(\pi_M^{p^k}, p\pi_M^{p^{k-1}}, p^2\pi_M^{p^{k-2}}, \ldots, p^k\pi_M) \cdot z$ for some $z \in O_M$. Denoting $\mathrm{H}(\pi_M^{p^k}, p\pi_M^{p^{k-1}}, p^2\pi_M^{p^{k-2}}, \ldots, p^k\pi_M)$ by λ we have

$$F(x^{p^{k+1}}) = (x^{p^{q+k}} + \lambda z)^p = x^{p^{q+k+1}} + \lambda^p z^p + \sum_{j=1}^{p-1} \binom{p}{j} x^{jp^{q+k}} \lambda^{p-j} z^{p-j}$$

so that $F(x^{p^{k+1}}) - x^{p^{q+k+1}}$ is congruent to zero modulo

$$\mathrm{H}(\pi_M^{p^{k+1}}, p^p\pi_M^{p^k}, p^{2p}\pi_M^{p^{k-1}}, \ldots, p^{kp}\pi_M^p, p\pi_M^{p^k}, p^2\pi_M^{p^{k-1}}, p^3\pi_M^{p^{k-2}}, \ldots, p^{k+1}\pi_M)$$

which is $\mathrm{H}(\pi_M^{p^{k+1}}, p\pi_M^{p^k}, p^2\pi_M^{p^{k-1}}, p^3\pi_M^{p^{k-2}}, \ldots, p^{k+1}\pi_M)$, as required. $\square$

Proposition 2.6.

Let G be any finite group. If $r \in J$ then $c(L_{F,0}(1-r))$ lies in $h_M \cdot \Lambda_G$.

Proof

Consider the series

$$L_{F,0}(1-r) = p^q \sum_{m=1}^{\infty} r^m/m - \sum_{m=1}^{\infty} \hat{F}(r^m)/m \in M[G].$$

If p^q does not divide m then $p^q r^m/m \in pO_M[G]$ and the $c(p^q r^m/m) \in p\Lambda_G$. Now consider the remaining terms in the series

$$\sum_{m=1}^{\infty} p^q(r^{p^q m}/p^q m) - \hat{F}(r^m)/m = \sum_{m=1}^{\infty} (r^{p^q m} - \hat{F}(r^m))/m.$$

Suppose that $m = p^s u$ with $HCF(u,p) = 1$ then we may set $t = r^u$ so that

$$(r^{p^q m} - \hat{F}(r^m))/m = (t^{p^{q+s}} - \hat{F}(t^{p^s}))/p^s u.$$

Therefore it will suffice to show that, if $t \in J$ and $s \geq 0$,

$$c(t^{p^{q+s}} - \hat{F}(t^{p^s})) \in c(h_M p^s O_M[G]) = h_M p^s \Lambda_G.$$

If $G = \{g_1, \ldots, g_n\}$ suppose that $t = \sum_{i=1}^{n} a_i g_i$ so that

$$t^{p^{q+s}} = \sum_{\underline{j}} a_{j_1} \ldots a_{j_{p^{q+s}}} g_{j_1} \ldots g_{j_{p^{q+s}}}$$

where $\underline{j}$ ranges over all possible p^{q+s}-tuples. The cyclic group C of order p^{q+s} acts on the set of $\underline{j}$'s by cyclic permutation. The products $g_1 \ldots g_v$ and $g_2 \ldots g_v g_1$ are conjugate in G so that each term in the subsum of terms which are cyclically conjugate to $\underline{j}$ will have the same image under c. Similarly

$$\hat{F}(t^{p^s}) = \sum_{\underline{k}} F(a_{k_1} \ldots a_{k_{p^s}}) g_{k_1} \ldots g_{k_{p^s}} g_{k_1} \ldots g_{k_{p^s}} \ldots g_{k_1} \ldots g_{k_{p^s}}$$

where the product $g_{k_1} \ldots g_{k_{p^s}}$ is repeated p^q times.

Note that if $\underline{j} = (k_1, \ldots, k_{p^s}, k_1, \ldots, k_{p^s}, \ldots k_1, \ldots, k_{p^s})$, as for example in the above expression for $\hat{F}(t^{p^s})$, then p^q divides the stabiliser order of $\underline{j}$ in C.

Suppose that the stabiliser of $\underline{j}$ in C has order p^w with $0 \leq w \leq q-1$ then the C-orbit of $\underline{j}$ has order at least p^{s+1} and the subsum consisting of these terms has image under c which lies in $p^{s+1}\Lambda_G$. There are no terms in $\hat{F}(t^{p^s})$ corresponding to such a $\underline{j}$.

Now suppose that the stabiliser of $\underline{j}$ in C has order p^w with $q \leq w \leq q+s$. In this case $\underline{j} = (k_1, \ldots, k_{p^{q+s-w}}, k_1, \ldots, k_{p^{q+s-w}}, \ldots, k_1, \ldots, k_{p^{q+s-w}})$ where $k_1, \ldots, k_{p^{q+s-w}}$ is repeated p^w times. Associated to the term

$$a_{\underline{j}} g_{\underline{j}} = a_{k_1} \ldots a_{k_{p^{q+s-w}}} \ldots, a_{k_1}, \ldots, a_{k_{p^{q+s-w}}} g_{k_1} \ldots g_{k_{p^{q+s-w}}} \ldots g_{k_1} \ldots g_{k_{p^{q+s-w}}}$$

is the term

$$b_{\underline{j}} = -F(a_{k_1} \ldots a_{k_{p^{q+s-w}}} \ldots a_{k_1} \ldots a_{k_{p^{q+s-w}}}) g_{k_1} \ldots g_{k_{p^{q+s-w}}} g_{k_1} \ldots g_{k_{p^{q+s-w}}}$$

in which $a_{k_1} \ldots a_{k_{p^{q+s-w}}}$ and $g_{k_1} \ldots g_{k_{p^{q+s-w}}}$ are repeated p^{w-q} times. The cyclic group C' of order p^s acts on the p^s-tuple

$$(k_1, \ldots, k_{p^{q+s-w}}, k_1, \ldots, k_{p^{q+s-w}}, \ldots, k_1, \ldots, k_{p^{q+s-w}})$$

with stabiliser of order p^{w-q}. Therefore the image under c of the C-orbit of $a_{\underline{j}} g_{\underline{j}}$ is equal to p^{q+s}/p^w copies of $c(a_{\underline{j}} g_{\underline{j}})$ and the image under c of the C'-orbit of $b_{\underline{j}}$ is p^s/p^{w-q} copies of $c(b_{\underline{j}})$. Therefore the image under c of the sum over these two orbits, associated to $\underline{j}$, is equal to

$$p^{q+s-w} c((a_{k_1} \ldots a_{k_{p^{q+s-w}}})^{p^w} - F((a_{k_1} \ldots a_{k_{p^{q+s-w}}})^{p^{w-q}}) g_{k_1} \ldots g_{k_{p^{q+s-w}}} g_{k_1} \ldots g_{k_{p^{q+s-w}}}$$

which, by Lemma 2.5, lies in

$$p^{q+s-w} \mathrm{H}(\pi_M^{p^{w-q}}, p\pi_M^{p^{w-q-1}}, p^2\pi_M^{p^{w-q-2}}, \ldots, p^{w-q}\pi_M)\Lambda_G \subseteq p^s h_M \Lambda_G$$

as required. □

2.7. *The* Gal(M/K)*-equivariance*

Let $\sigma \in \mathrm{Gal}(M/K)$. Then σ acts on $z = \sum_\gamma \lambda_\gamma \gamma \in O_M[G]$ by the formula $\sigma(z) = \sum_\gamma \sigma(\lambda_\gamma)\gamma$. Also σ acts on $f \in \mathrm{Hom}_{\mathrm{Gal}(N/M)}(R(G), N)$ by the formula $(\sigma \cdot f)(T) = \tilde{\sigma}(f(\tilde{\sigma}^{-1}(T)))$, where $\tilde{\sigma} \in \mathrm{Gal}(N/K)$ is any lifting of σ. If $F \in \mathrm{Gal}(M/K)$ is one choice of lifted Frobenius, as in Definition 2.2 then $\sigma F \sigma^{-1}$ is another. The following result describes the relation between the homomorphisms Log_F and $\mathrm{Log}_{\sigma F \sigma^{-1}}$.

Proposition 2.8. In the notation of §2.7 $Log_{\sigma F\sigma^{-1}}(\sigma(z)) = \sigma \cdot (Log_F(z))$.

Proof

If $\tilde{\sigma} \in \mathrm{Gal}(N/K)$ is a lift of σ, as in §2.7, we have

$$\sigma \cdot (Log_F)(z))(T)$$

$$= \tilde{\sigma}(log(Det(F(z))(\psi^{p^q}(\tilde{\sigma}^{-1}(T)))/(Det(z)(\tilde{\sigma}^{-1}(T)))^{p^q}))$$

$$= log(\frac{Det(\sum_\gamma \ \tilde{\sigma}(F(\lambda_\gamma))\tilde{\sigma}(\tilde{\sigma}^{-1}(T(\gamma^{p^q}))))}{Det(\sum_\gamma \ \tilde{\sigma}(\lambda_\gamma)\tilde{\sigma}(\tilde{\sigma}^{-1}(T(\gamma))))^{p^q}})$$

$$= log(\frac{Det(\sum_\gamma \ \tilde{\sigma}(F(\lambda_\gamma))T(\gamma^{p^q}))}{Det(\sum_\gamma \ \tilde{\sigma}(\lambda_\gamma)T(\gamma))^{p^q}})$$

$$= log(\frac{Det(\sum_\gamma \ \tilde{\sigma}(F(\lambda_\gamma))T(\gamma^{p^q}))}{Det(\sum_\gamma \ \tilde{\sigma}(\lambda_\gamma)T(\gamma))^{p^q}})$$

$$= Log_{\sigma F\sigma^{-1}}(\sigma(z))(T),$$

as required. □

3. A 2-group example

Definition 3.1. In this section we shall suppose that we are in the situation of §1 and that G is a finite p-group. Define $A_M(G)$ to be equal to the kernel of the natural map from $O_M[G]$ to $O_M[G^{ab}]$

$$A_M(G) = \mathrm{Ker}(O_M[G] \longrightarrow O_M[G^{ab}]).$$

Therefore $A_M(G)$ is contained in J, the Jacobson radical of $O_M[G]$, which was introduced in 2.2. We are going to study the subgroup $Det(1 + A_M(G))$ of $\mathrm{Hom}_{\mathrm{Gal}(N/M)}(R(G), O_N^*)$.

Firstly we recall a result of [6].

Proposition 3.2. $Det(1 + A_M(G))$ is torsion free subgroup of $\mathrm{Hom}_{\mathrm{Gal}(N/M)}(R(G), O_N^*)$.

Proposition 3.3. There is a well-defined, injective homomorphism

$$\hat{\alpha}_G : Det(1 + A_M(G)) \longrightarrow M\{G\}$$

given by the formula $\hat{\alpha}_G(Det(u)) = \alpha_G(u)$, where α_G is the homomorphism introduced in Definition 2.2.

Proof

This result was proved in ([5] §4.5) when $M/\mathbb{Q}_p$ was unramified, using the fact that α_G was Gal$(M/\mathbb{Q}_p)$-equivariant. In the current, more general, situation this is no longer true because M/K need not be linearly disjoint from the cyclotomic p-power extension $K(\mu_{p^\infty})/K$. Accordingly, we shall use instead the partial Galois equivariance of Proposition 2.8. When G is abelian there is nothing to prove; therefore we will assume that G is non-abelian. Let $u \in 1 + A_M(G) \subset O_M[G]^*$ and suppose that $\alpha_G(u) = 0$. By Proposition 2.8, for all $j \geq 0$,

$$0 = F^j \cdot Log_F(u) = Log_F(F^j(u)).$$

This occurs if and only if the homomorphism

$$T \mapsto log(\frac{Det(F^{j+1}(u))\psi^{p^q}(T)}{(Det(F^j u)(T))^{p^q}}) = Log_F(F^j(u))(T)$$

is zero for all $T \in R(G)$. Therefore, since $R(G)$ is finitely generated, there exists a positive integer m such that for all T and $j \geq 0$

$$Det(F^{j+1}(u))(\psi^{p^q}(T))^{p^m} = Det(F^j u)(T)^{p^{m+q}}.$$

Hence we have

$$\begin{aligned} & Det(F^j(u))(\psi^{jp^q}(T))^{p^m} \\ & = Det(F^{j-1}(u))(\psi^{(j-1)p^q}(T))^{p^{m+q}} \\ & = Det(F^{j-2}(u))(\psi^{(j-2)p^q}(T))^{p^{m+2q}} \\ & = \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ & = Det(u)(T)^{p^{m+jq}} \end{aligned}$$

Now suppose that $\#(G) = jp^q$ with HCF$(j,p) = 1$ so that $\psi^{jp^q}(T) = dim(T) \in R(G)$. This means that, if $\epsilon : O_M[G] \longrightarrow O_M[\{1\}] = O_M$ is the augmentation map, then

$$Det(u^{p^{m+jq}})(T) = Det(\epsilon(F^j(u)))^{dim(T)} = 1,$$

since $\epsilon(1 + A_M(G)) = \{1\}$. Therefore $Det(u)^{p^{m+jq}} = 1$ and so $Det(u) = 1$, by Prposition 3.2, which shows that $\hat{\alpha}_G$ is injective, provided that it is well-defined.

To show well-definedness suppose that $Det(u)(T) = Det(T(u)) = 1$ for all $T \in R(G)$. If $u = \sum\ m_\gamma \gamma$ then $F(u) = \sum\ F(m_\gamma)\gamma$ and $T(F(u)) = \sum\ F(m_\gamma)T(\gamma)$. Let $\tilde{F} \in$ Gal(N/K) be a lift of F and suppose that $T = \tilde{F}(T')$ then

$$Det(T(F(u)) = Det(\sum\ F(m_\gamma)\tilde{F}(T')(\gamma)) = \tilde{F}(Det(T'(u))) = 1.$$

Therefore $Log_F(u) = 0$ and $\alpha_G(u) = 0$. If $Det(u') = Det(u'')$ then $Det(u'(u'')^{-1}) = 1$ and so $0 = \alpha_G(u'(u'')^{-1}) = \alpha_G(u') - \alpha_G(u'')$ and $\hat{\alpha}_G$ is well-defined. $\square$

With the results obtained so far we do not have a complete description of the image of $Det(1 + A_M(G))$ under α_G, except in the unramified case, because Proposition 2.6 is more complicated in the presence of ramification. However, our results are sufficient for a very unambitious example of Galois descent for determinantal functions.

Example 3.4. *An easy case when* $p = 2$

Let G be a finite 2-group containing a central element z of order 2 which is a commutator and such that $H = G/\langle z\rangle$ is abelian. In the notation of Definition 2.2, in particular $p = 2$ and the residue degree equals 2^q, we shall prove next that

$$c(L_{F,0}(1 - (1 - z)O_M[G])) = 2^q \cdot c((1 - z)O_M[G]).$$

If $x \in O_M[G]$ then, since z is central, $\hat{F}((1 - z)x) = 0$ because $z^2 = 1$. Therefore

$$\begin{aligned} L_{F,0}(1 - (1 - z)x) &= 2^q(\textstyle\sum_{n=1}^{\infty}(1 - z)^n x^n/n) \\ &\equiv 2^q(1 - z)(x + x^2) \text{ (modulo } 2^q(1 - z)^2x^2) \end{aligned}$$

so that $L_{F,0}(1 - (1 - z)x) \in 2^q(1 - z)O_M[G]$ and therefore

$$c(L_{F,0}(1 - (1 - z))O_M[G]) \subseteq 2^q c((1 - z))O_M[G]).$$

If $x \in J$ then one sees that $2^q c((1 - z)x) \in c(L_{F,0}(1 - (1 - z)O_M[G]))$, by means of a standard approximation argument. This implies that

$$2^q c((1 - z)J) \subseteq c(L_{F,0}(1 - (1 - z)O_M[G])) \subseteq 2^q \cdot c((1 - z)O_M[G]).$$

On the other hand we claim that

$$c((1 - z)O_M[G]) \subseteq c((1 - z)J)$$

which will complete the proof. Write $z = a^{-1}b^{-1}ab$ for $a, b \in G$ and let $v = \sum m_\gamma \gamma \in O_M[G]$. We have to show that $c((1 - z)v) \in c((1 - z)J)$. Rewrite v as $v = \sum m_\gamma(\gamma - a) + \sum m_\gamma a$ so that

$$\begin{aligned} c((1 - z)v) &= c((1 - z)(\textstyle\sum m_\gamma(\gamma - a) + \sum m_\gamma a)) \\ &= c((1 - z)\textstyle\sum m_\gamma(\gamma - a)), \end{aligned}$$

since $c(a - za) = c(a - b^{-1}ab) = 0$. However, $\gamma - a \in Ker(O_M[G] \longrightarrow O_M)$, the kernel of the augmentation, and since G is an 2-group the kernel of the augmentation lies inside J.

Since, in this example, $(1 - z)O_M[G] = A_M(G)$, we have established the following result.

Proposition 3.5.
In Example 3.4

$$\hat{\alpha}_G(Det(1 + A_M(G))) = 2^q \cdot c(A_M(G)).$$

We conclude this section by proving Galois descent for determinantal functions in the situation of Example 3.4, following the argument given in ([5] §4.5) for the unramified case.

Proposition 3.6.
Let M/K be as in §1 with $p = 2$ and let G be as in Example 3.4. Then

$$Det(O_M[G]^*)^{\mathrm{Gal}(M/K)} \cong Det(O_K[G]^*).$$

Proof

For completeness, although it follows ([5] §4.5), we shall give the complete proof. Consider the following diagram, whose rows are easily seen to be short exact.

$$\begin{array}{ccccc}
1 + A_M(G) & \rightarrow & O_M[G]^* & \rightarrow & O_M[G^{ab}]^* \\
\downarrow \mathrm{Det} & & \downarrow \mathrm{Det} & & \cong\downarrow \mathrm{Det} \\
Det(1 + A_M(G)) & \rightarrow & Det(O_M[G]^*) & \rightarrow & Det(O_M[G^{ab}]^*) \\
\downarrow & & \downarrow & & \downarrow \\
\{1\} & & \{1\} & & \{1\}
\end{array}$$

in which the vertical maps are induced by the determinant, which is an isomorphism for abelian groups. Let $U = G(M/K)$ then we may compare the bottom row for K with the U-invariants of the bottom row for M.

$$\begin{array}{ccccc}
Det(1 + A_K(G)) & \rightarrow & Det(O_K[G]^*) & \rightarrow & Det(O_K[G^{ab}]^*) \\
\cong\downarrow \beta_1 & & \downarrow \beta_2 & & \cong\downarrow \beta_3 \\
Det(1 + A_M(G))^U & \rightarrow & Det(O_M[G]^*)^U & \rightarrow & Det(O_M[G^{ab}]^*)^U
\end{array}$$

The map, β_3, is an isomorphism because $(O_M[G^{ab}]^*)^U \cong O_K[G^{ab}]^*$ and therefore the lower sequence is short exact.

At this point the argument of ([5] §4.5) concludes by observing, since α_G and $\hat{\alpha}_G$ are Galois equivariant in that unramified case, that β_1 may be identified with the isomorphism

$$2^q \cdot c(A_K(G)) \cong 2^q \cdot (c(A_M(G)))^U$$

so that β_2 is an isomorphism, by the five-lemma, which completes the proof.

However, in the general situation $\hat{\alpha}_G$ is *not* Galois equivariant in the naive sense; instead we must use Proposition 2.8. Specifically, in Example 3.4, we saw that $1 + A_M(G) = 1 + (1 - z)O_M[G]$ and that $L_{F,0}(1 - (1 - z)x)$ is independent of the choice of F so that Proposition 2.8 (or the explicit formulae of Example 3.4) imply that, in the present case, $\hat{\alpha}_G$ is Galois equivariant, as required. □

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