THE HOPFLIKE PROPERTIES OF THE HYPERHECKE ALGEBRA

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Contents

In this essay I am going to rehearse the material of ([35] Chapter III and Appendix 3) in the "hyperHecke algebra context. If G is a group and k is an algebraically closed field write $R_+(G)$ (c.f. [30], [31]) for the free $\mathbb{Z}$-module whose basis consists of G-conjugacy classes of pairs (H, ϕ) where $\phi : H \longrightarrow k^*$ is a continuous character and H is a subgroup of G belonging to a suitable class closed under conjugation (e.g. compact open modulo the centre).

In ([34] §12) I got a little overzealous on the Hopf algebra properties of the hyperHecke algebra. The situation is weirder than the impression I gave. This preliminary essay goes into more details about the Hopf-like algebra structure of the general linear groups of a field.

In this version I have not got to the actual hyperHecke algebra yet!

If K is another field we define

$$R_+ = \oplus_{n=0}^{\infty} R_+(GL_nK).$$

Let $P_{m,n} \subset GL_{m+n}K$ be the traditional parabolic subgroup and let $U_{m,n} \subset P_{m,n}$ denote the usual unipotent subgroup such that $P_{m,n} = (GL_mK \times GL_nK)U_{m,n}$, $(GL_mK \times GL_nK) \bigcap U_{m,n} = \{1\}$ and $U_{m,n}$ is normal in $P_{m,n}$. Suppose that $\theta : U_{m,n} \longrightarrow k^*$ is a continuous character which is fixed under conjugation by $GL_mK \times GL_nK$ so that $\theta(zuz^{-1}) = \theta(u)$ for all $z \in P_{m.n}$ and $u \in U_{m,n}$.

Define a pairing

$$(m_\theta)_{m,n} : R_+(GL_mK) \otimes R_+(GL_nK) \longrightarrow R_+(GL_{m+n}K)$$

by the formula

$$m_\theta((H, \phi) \otimes (J, \mu))_{m,n} = ((H \times J)U_{m,n}, \rho_\theta(\phi, \mu))$$

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where $\rho_\theta(\phi,\mu))(u) = \theta(u)$ and $\rho_\theta(\phi,\mu))((h,j)) = \phi(h)\mu(j)$ for $h \in H, j \in J, u \in U_{m,n}$. This is a well-defined continuous character because

$$\rho_\theta(\phi,\mu))((h,j)u \cdot (h',j')u'))$$

$$= \rho_\theta(\phi,\mu))((h,j)(h',j') \cdot (h',j')^{-1}u(h',j')u'))$$

$$= \phi(hh')\mu(jj')\theta(u)\theta(u')$$

$$= \rho_\theta(\phi,\mu))((h,j)u)\rho_\theta(\phi,\mu))((h',j')u')).$$

In general we have a similar map, also denoted by $(m_\theta)_{m,n}$, of the form

$$(m_\theta)_{m,n} : R_+(GL_mK \times GL_nK) \longrightarrow R_+(GL_{m+n}K).$$

For $0 \leq a \leq s$ define a "coproduct"

$$(m^*_\theta)_{a,s-a} : R_+(GL_sK) \longrightarrow R_+(GL_aK \times GL_{s-a}K)$$

in the following manner.

Suppose that we have a character $\theta : U_{a,s-a} \longrightarrow k^*$ which is fixed under conjugation by elements of $P_{a.s-a}$. This is equivalent to being given a character on $P_{a,s-a}$ defined by $(x,y)u \mapsto \theta(u)$ for $x \in GL_aK, y \in GL_{s-a}K, u \in U_{a.s-a}$.

Given (H,ϕ) with $H \subset GL_sK$ and $\phi : H \longrightarrow k^*$ suppose that we have a double coset representative $z \in P_{a,s-a}\backslash GL_sK/H$ then the subgroup zHz^{-1} is well-defined and we define $(z^{-1})^*(\phi) : zHz^{-1} \longrightarrow k^*$ by $(z^{-1})^*(\phi)(zhz^{-1}) = \phi(h)$. Now consider the condition that

$$\theta = (z^{-1})^*(\phi) : U_{a,s-a} \bigcap zHz^{-1} \longrightarrow k^*.$$

This condition is independent of the double coset representative. For if we replace z by $z' = p'zh'$ with $p' \in P_{a,s-a}, h' \in H$ then $u'' = zh''z^{-1} \in U_{a,s-a} \bigcap zHz^{-1}$ implies that $p'u''(p')^{-1} = z'(h')^{-1}h''h'(z')^{-1} \in U_{a,s-a} \bigcap z'H(z')^{-1}$ and that $((z')^{-1})^*(\phi)((h')^{-1}h''h') = \phi(h'') = \theta(u'') = \theta(p'u''(p')^{-1})$.

Set $\pi_{a,s-a} : P_{a,s-a} \longrightarrow GL_a \times GL_{s-a}K$ equal to the surjection with kernel $U_{a,s-a}$. Hence, if $\theta = (z^{-1})^*(\phi)$ on $U_{a,s-a} \bigcap zHz^{-1}$ we have

$$\pi_{a,s-a}(P_{a,s-a} \bigcap zHz^{-1}) \subset GL_a \times GL_{s-a}K$$

and a character on this subgroup induced by the character $(z^{-1})^*(\phi)/\theta$ on $P_{a,s-a} \bigcap zHz^{-1}$.

Define $m^*_\theta((H,\phi))_{a,s-a} \in R_+(GL_aK \times GL_{s-a}K)$ by the formula

$$\sum_{\substack{z \in P_{a,s-a}\backslash GL_sK/H \\ \theta = (z^{-1})^*(\phi) \text{ on } U_{a,s-a} \bigcap zHz^{-1}}} (\pi_{a,s-a}(P_{a,s-a} \bigcap zHz^{-1}), (z^{-1})^*(\phi)/\theta).$$

1. The composition $(m^*_{\theta'})_{a,s-a} \cdot (m_\theta)_{b,s-b}$

Suppose that A is a subgroup of $GL_bK \times GL_{s-b}K$ and that $\lambda : A \longrightarrow k^*$ is a continuous character. Then $(m_\theta)(A,\lambda)_{b,s-b}$ is given by

$$(U_{b,s-b}A, \rho_\theta(\lambda)) \in R_+(GL_sK)$$

where $\rho_\theta(u) = \theta(u), \rho_\theta(x) = \lambda(a)$ for $u \in U_{b,s-b}, x \in A$. Under $(m^*_{\theta'})_{a,s-a}$ this is sent to

$$\sum_{\substack{z \in P_{a,s-a}\backslash GL_sK/U_{b,s-b}A \\ \theta'=(z^{-1})^*(\rho_\theta(\lambda)) \text{ on } U_{a,s-a}\bigcap zU_{b,s-b}Az^{-1}}} (\pi_{a,s-a}(P_{a,s-a}\bigcap zU_{b,s-b}Az^{-1}), (z^{-1})^*(\rho_\theta(\lambda))/\theta').$$

The choice of double coset representatives in this formula is given by the Bruhat Decomposition ([35] p.173) which asserts that the double cosets of any two parabolic subgroups in GL_sK are in one-one correspondence with the corresponding double cosets of "parabolic" subgroups of Σ_s. In particular

$$\Sigma_a \times \Sigma_{s-a}\backslash\Sigma_s/\Sigma_b \times \Sigma_{s-b} \stackrel{\cong}{\longrightarrow} P_{a,s-a}\backslash GL_sK/P_{b,s-b}$$

and $U_{b,s-b}A \subseteq P_{b,s-b}$ so that if w runs through the cosets

$$(GL_bK \times GL_{s-b}K)/A \stackrel{\cong}{\longrightarrow} P_{b,s-b}/U_{b,s-b}A$$

and y runs through permutation matrix representatives for $\Sigma_a \times \Sigma_{s-a}\backslash\Sigma_s/\Sigma_b \times \Sigma_{s-b}$ then $z = yw$ is a set of double coset representatives as featured in the above formula.

From ([35] p.171) the representatives of $\Sigma_a \times \Sigma_{s-a}\backslash\Sigma_s/\Sigma_b \times \Sigma_{s-b}$ are parametrised by matrices

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ & \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

with

$$x_{1,1} + x_{1,2} = b, x_{2,1} + x_{2,2} = s-b, x_{1,1} + x_{2,1} = a, x_{1,2} + x_{2,2} = s-a.$$

Therefore we have canonical inclusions

$$\Sigma_{x_{1,1}} \times \Sigma_{x_{1,2}} \times \Sigma_{x_{2,1}} \times \Sigma_{x_{2,2}} \subset GL_bK \times GL_{s-b}K$$

and

$$\Sigma_{x_{1,1}} \times \Sigma_{x_{2,1}} \times \Sigma_{x_{1,2}} \times \Sigma_{x_{2,2}} \subset GL_aK \times GL_{s-a}K.$$

The double coset representative y corresponding to the matrix of $x_{i,j}$'s may be taken as the matrix whose conjugation switches $\Sigma_{x_{1,2}} \times \Sigma_{x_{2,1}}$ to $\Sigma_{x_{2,1}} \times \Sigma_{x_{1,2}}$ in the standard manner - namely $1, \ldots, x_{1,1}$ and $x_{1,1} + x_{2,1} + x_{1,2} + 1, \ldots, s$ stay put and $x_{1,1} + 1, \ldots, x_{1,1} + x_{1,2}$ shift by $x_{2,1}$ places into $x_{1,1} + x_{2,1} + 1, \ldots, x_{1,1} + x_{2,1} + x_{1,2}$ and vice versa.

Now suppose that $A \subseteq GL_bK \times GL_{s-b}K$ is a cartesian product of the form $A = B \times C$ with $B \subseteq GL_bK$ and $C \subseteq GL_{s-b}K$. In that case $\lambda = \lambda_1\lambda_2$ where

$\lambda_1 : B \longrightarrow k^*$ and $\lambda_2 : C \longrightarrow k^*$ are continuous characters. Furthermore the set of coset representatives w for

$$(GL_bK \times GL_{s-b}K)/A \cong GL_bK/B \times GL_{s-b}K/C$$

may be chosen to be the set $\{(w', w'')\}$ where w' runs through coset representatives of GL_bK/B and w'' runs through those of $GL_{s-b}K/C$.

Given the $x_{i,j}$-matrix and a character θ_1 of $U_{x_{1,1},x_{1,2}} \subset GL_bK$ we have

$$m^*_{\theta_1}(-)_{x_{1,1},x_{1,2}} : R_+(GL_bK) \longrightarrow R_+(GL_{x_{1,1}}K) \times GL_{x_{1,2}}K)$$

and for a character θ_2 of $U_{x_{2,1},x_{2,2}} \subset GL_{s-b}K$ we have

$$m^*_{\theta_2}(-)_{x_{2,1},x_{2,2}} : R_+(GL_{s-b}K) \longrightarrow R_+(GL_{x_{2,1}}K) \times GL_{x_{2,2}}K).$$

Given the $x_{i,j}$-matrix I want to establish the conditions on the θ's which ensure that the following two compositions are comparable:

$$R_+(GL_bK) \otimes \underline{R}(GL_{s-b}K)$$

$$\downarrow\ m^*_{\theta_1}(-)_{x_{1,1},x_{1,2}} \otimes m^*_{\theta_2}(-)_{x_{2,1},x_{2,2}}$$

$$R_+(GL_{x_{1,1}}K \times GL_{x_{1,2}}K \times GL_{x_{2,1}}K \times GL_{x_{2,2}}K)$$

$$\downarrow\ (1 \times T \times 1)^*$$

$$R_+(GL_{x_{1,1}}K \times GL_{x_{2,1}}K \times GL_{x_{1,2}}K \times GL_{x_{2,2}}K)$$

$$\downarrow\ (m_{\theta'})_{x_{1,1},x_{2,1}} \times (m_{\theta''})_{x_{1,2},x_{2,2}}$$

$$R_+(GL_aK \times GL_{s-a}K)$$

and

$$R_+(GL_bK) \otimes R_+(GL_{s-b}K)$$

$$\downarrow\ (m_\theta)_{b,s-b}$$

$$R_+(GL_sK)$$

$$\downarrow\ m^*_{\theta'''}(-)_{a,s-a}$$

$$R_+(GL_aK \times GL_{s-a}K).$$

Starting with $(H_b, \phi_b) \in R_+(GL_bK)$ we have

$$m^*_{\theta_1}((H_b, \phi_b))_{x_{1,1},x_{1,2}}$$

$$\sum_{\substack{\tilde{z} \in P_{x_{1,1},x_{1,2}} \backslash GL_bK/H_b \\ \theta_1 = (\tilde{z}^{-1})^*(\phi_b) \text{ on } U_{x_{1,1},x_{1,2}} \bigcap \tilde{z}H_b\tilde{z}^{-1}}} \pi_{x_{1,1},x_{1,2}}(P_{x_{1,1},x_{1,2}} \bigcap \tilde{z}H_b\tilde{z}^{-1}), (\tilde{z}^{-1})^*(\phi_b)/\theta_1).$$

Starting with $(H_{s-b},\phi_{s-b}) \in R_+(GL_{s-b}K)$ we have

$$m^*_{\theta_2}((H_{s-b},\phi_{s-b}))_{x_{2,1},x_{2,2}}$$

$$\sum_{\substack{\underline{z}\in P_{x_{2,1},x_{2,2}}\backslash GL_{s-b}K/H_{s-b} \\ \theta_2=(\underline{z}^{-1})^*(\phi_{s-b}) \text{ on } U_{x_{2,1},x_{2,2}}\bigcap \underline{z}H_b\underline{z}^{-1}}} \pi_{x_{2,1},x_{2,2}}(P_{x_{2,1},x_{2,2}}\bigcap \underline{z}H_{s-b}\underline{z}^{-1}), (\underline{z}^{-1})^*(\phi_{s-b})/\theta_2).$$

For the second composition, starting with $(H_b,\phi_b)\otimes(H_{s-b},\phi_{s-b})$ we have

$$m_\theta((H_b,\phi_b)\otimes(H_{s-b},\phi_{s-b}))_{b,s-b} = ((H_b\times H_{s-b})U_{b,s-b},\rho_\theta(\phi_b,\phi_{s-b}))$$

where $\rho_\theta(\phi_b,\phi_{s-b}))(u) = \theta(u)$ and $\rho_\theta(\phi_b,\phi_{s-b}))((h,j)) = \phi_b(h)\phi_{s-b}(j)$ for $h\in H_b, j\in H_{s-b}, u\in U_{b,s-b}$.

Now let us examine the double coset conditions in the sum which gives $m^*_{\theta'''}(-)_{a,s-a}$ applied to $((H_b\times H_{s-b})U_{b,s-b},\rho_\theta(\phi_b,\phi_{s-b}))$. From the discussion at the beginning of this section the double coset representatives in the formula are the $z = yw$'s introduced earlier and $w = (w',w'') \in GL_bK/H_b \times GL_{s-b}/H_{s-b}$. Therefore we must examine

$$U_{a,s-a}\bigcap y(w',w'')U_{b,s-b}(H_b\times H_{s-b})(w',w'')^{-1}y^{-1}.$$

Since $H_b\times H_{s-b}\subseteq GL_b\times GL_{s-b}$ normalises $U_{b,s-b}$ this intersection equals

$$U_{a,s-a}\bigcap(yU_{b,s-b}y^{-1})y(w',w''))(H_b\times H_{s-b})(w',w'')^{-1}y^{-1}$$

$$= U_{a,s-a}\bigcap(yU_{b,s-b}y^{-1})y(w'H_b(w')^{-1}\times w''H_{s-b}(w'')^{-1})y^{-1}.$$

If y corresponds to the $x_{i,j}$-matrix then

$$U_{a,s-a}\bigcap(yU_{b,s-b}y^{-1}) = U_{x_{1,1},x_{2,1},x_{1,2},x_{2,2}}$$

the unitriangular unipotent subgroup of the parabolic group $P_{x_{1,1},x_{2,1},x_{1,2},x_{2,2}}$. Writing matrices in GL_sK as 4×4 matrices of $x_{1,1},x_{2,1},x_{1,2},x_{2,2}$-blocks we may write elements of $(w'H_b(w')^{-1}\times w''H_{s-b}(w'')^{-1})$ in the form

$$\begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix}$$

and the permutation y conjugates this to

$$\begin{pmatrix} a_{1,1} & 0 & a_{1,2} & 0 \\ 0 & a_{3,3} & 0 & a_{3,4} \\ a_{2,1} & 0 & a_{2,2} & 0 \\ 0 & a_{4,3} & 0 & a_{4,4} \end{pmatrix}.$$

In the terminology of ([35] p.168) $U_{a,s-a}$ is decomposable with respect to $(yU_{b,s-b}y^{-1}), y(w'H_b(w')^{-1} \times w''H_{s-b}(w'')^{-1})y^{-1})$ which means that

$$U_{a,s-a} \bigcap (yU_{b,s-b}y^{-1})y(w'H_b(w')^{-1} \times w''H_{s-b}(w'')^{-1})y^{-1}$$

$$= (U_{a,s-a} \bigcap (yU_{b,s-b}y^{-1})) \cdot (U_{a,s-a} \bigcap y(w'H_b(w')^{-1} \times w''H_{s-b}(w'')^{-1})y^{-1}).$$

The matrices in 4×4 block form which lie in the subgroup $(w'H_b(w')^{-1} \times w''H_{s-b}(w'')^{-1})$ and are conjugated by y to lie in $U_{a,s-a}$ must have $a_{2,1} = 0$ and $a_{4,3} = 0$. In addition each of $a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}$ must be equal to the appropriate identity matrix. Therefore matrices in $y(w'H_b(w')^{-1} \times w''H_{s-b}(w'')^{-1})y^{-1}$ have the form

$$\begin{pmatrix} I & 0 & a_{1,2} & 0 \\ 0 & I & 0 & a_{3,4} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and

$$U_{a,s-a} \bigcap (yU_{b,s-b}y^{-1})y(w'H_b(w')^{-1} \times w''H_{s-b}(w'')^{-1})y^{-1} = U_{x_{1,1},x_{2,1},x_{1,2},x_{2,2}}.$$

Also $(w'H_b(w')^{-1} \times w''H_{s-b}(w'')^{-1})$ must have the form

$$\begin{pmatrix} I & a_{1,2} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & a_{3,4} \\ 0 & 0 & 0 & I \end{pmatrix} \in U_{x_{1,1},x_{1,2}} \times U_{x_{2,1},x_{2,2}}.$$

Now let us recapitulate the conditions on the θ's.

$$k^* \stackrel{\theta_1}{\longleftarrow} U_{x_{1,1},x_{1,2}} \subset P_{x_{1,1},x_{1,2}} \subset GL_b$$

satisfies

$$\theta_1 = ((w')^{-1})^* \phi_b \text{ on } U_{x_{1,1},x_{1,2}} \bigcap w'H_b(w')^{-1}.$$

Also

$$k^* \xleftarrow{\theta_2} U_{x_{2,1},x_{2,2}} \subset P_{x_{2,1},x_{2,2}} \subset GL_b$$

satisfies

$$\theta_2 = ((w'')^{-1})^* \phi_{s-b} \text{ on } U_{x_{2,1},x_{2,2}} \bigcap w'' H_b (w'')^{-1}.$$

The character $\theta' : U_{x_{1,1},x_{2,1}} \longrightarrow k^*$ is normalised by $P_{x_{1,1},x_{2,1}}$ and $\theta'' : U_{x_{1,2},x_{2,2}} \longrightarrow k^*$ is normalised by $P_{x_{1,2},x_{2,2}}$. The character $\theta : U_{b,s-b} \longrightarrow k^*$ is normalised by $P_{b,s-b}$.

Finally θ''' has the form

$$k^* \xleftarrow{\theta'''} U_{a,s-a} \subset P_{a,s-a} \subset GL_s$$

and the condition on the summation in the formula for $(m^*_{\theta'''})_{a,s-a}$ is

$$\theta''' = ((y(w', w''))^{-1})^* (\rho_\theta(\phi_b \otimes \phi_{s-b})) \text{ on } U_{x_{1,1},x_{2,1},x_{1,2},x_{2,2}}.$$

Notice that $U_{x_{1,1},x_{2,1},x_{1,2},x_{2,2}}$ consists of matrices of the block-shape

$$\begin{pmatrix} I & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & I & a_{2,3} & a_{2,4} \\ 0 & 0 & I & a_{3,4} \\ 0 & 0 & 0 & I \end{pmatrix}$$

so that

$$U_{x_{1,1},x_{2,1},x_{1,2},x_{2,2}} = U_{a,s-a} \cdot (U_{x_{1,1},x_{2,1}} \times U_{x_{1,2},x_{2,2}}),$$

the product of two subgroups which intersect only at the identity matrix.

Returning to the compositions on p.4, the first one is zero unless there exists

$$(\theta_1, \theta_2) : U_{x_{1,1},x_{1.2}} \times U_{x_{2,1},x_{2,2}} \longrightarrow (k^*, k^*)$$

which extends $(((w')^{-1})^*(\phi_b), ((w'')^{-1})^*(\phi_{s-b}))$ on

$$(U_{x_{1,1},x_{1.2}} \bigcap w' H_b (w')^{-1}) \times (U_{x_{2,1},x_{2,2}} \bigcap w'' H_{s-b} (w'')^{-1}) \longrightarrow (k^*, k^*).$$

This is equivalent to the existence of a character

$$\theta_1 \otimes \theta_2 : U_{x_{1,1},x_{1.2}} \times U_{x_{2,1},x_{2,2}} \longrightarrow k^*$$

which extends $(((w')^{-1})^*(\phi_b) \otimes ((w'')^{-1})^*(\phi_{s-b}))$ on

$$(U_{x_{1,1},x_{1.2}} \bigcap w' H_b (w')^{-1}) \times (U_{x_{2,1},x_{2,2}} \bigcap w'' H_{s-b} (w'')^{-1}) \longrightarrow k^*.$$

Since the double coset representative y associated to the matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

is $1 \times T \times 1$ this is equivalent to the existence of $(1 \times T \times 1)^*(\theta_1 \otimes \theta_2)$ extending $((y(w', w''))^{-1})^*(\phi_b \otimes \phi_{s-b})$ on $U_{a,s-a} \bigcap y(w' H_b (w')^{-1} \times w'' H_{s-b} (w'')^{-1}) y^{-1}$

because $(1 \times T \times 1)(U_{x_{1,1},x_{1.2}} \times U_{x_{2,1},x_{2,2}})$ consists of all the matrices of the block-form

$$\begin{pmatrix} I & 0 & a_{1,2} & 0 \\ 0 & I & 0 & a_{3,4} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \in U_{a,s-a}$$

whereas, as we have seen, $U_{a,s-a} \bigcap y(w'H_b(w')^{-1} \times w''H_{s-b}(w'')^{-1})y^{-1}$ consists of some of the matrices of block-form

$$\begin{pmatrix} I & 0 & a_{1,2} & 0 \\ 0 & I & 0 & a_{3,4} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Suppose that $\theta : U_{b,s-b} \longrightarrow k^*$ extends $(\theta' \otimes \theta'')(1 \times T \times 1)$ on the subgroup of matrices of the block-form

$$\begin{pmatrix} I & 0 & a_{1,2} & 0 \\ 0 & I & 0 & a_{3,4} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \in U_{b,s-b}$$

which is $(1 \times T \times 1)(U_{x_{1,1},x_{2,1}} \times U_{x_{1,2},x_{2,2}})$. This is reasonable since the latter is normalised by a subgroup of the parabolic subgroup $P_{b,s-b}$ which normalises the θ.

Suppose also that $\theta''' : U_{a,s-a} \longrightarrow k^*$ extends $(\theta_1 \otimes \theta_2)(1 \times T \times 1)$ on the subgroup of matrices of the block-form

$$\begin{pmatrix} I & 0 & a_{1,2} & 0 \\ 0 & I & 0 & a_{3,4} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \in U_{a,s-a}$$

which is $(1 \times T \times 1)(U_{x_{1,1},x_{1,2}} \times U_{x_{2,1},x_{2,2}})$, which is also reasonable.

With these assumptions the two compositions on p.4 coincide.

Theorem 1.1.

Suppose that the positive integers in the matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$$

satisfy

$$x_{1,1} + x_{1,2} = b, x_{2,1} + x_{2,2} = s - b, x_{1,1} + x_{2,1} = a, x_{1,2} + x_{2,2} = s - a.$$

Suppose that the characters $\theta : U_{b,s-b} \longrightarrow k^*$ and $\theta''' : U_{a,s-a} \longrightarrow k^*$ extend the characters $(\theta' \otimes \theta'')(1 \times T \times 1)$ and $(\theta_1 \otimes \theta_2)(1 \times T \times 1)$, respectively. Then the compositions

$$R_+(GL_b K) \otimes \underline{R}(GL_{s-b} K)$$

$$\downarrow \; m^*_{\theta_1}(-)_{x_{1,1},x_{1,2}} \otimes m^*_{\theta_2}(-)_{x_{2,1},x_{2,2}}$$

$$R_+(GL_{x_{1,1}} K \times GL_{x_{1,2}} K \times GL_{x_{2,1}} K \times GL_{x_{2,2}} K)$$

$$\downarrow \; (1 \times T \times 1)^*$$

$$R_+(GL_{x_{1,1}} K \times GL_{x_{2,1}} K \times GL_{x_{1,2}} K \times GL_{x_{2,2}} K)$$

$$\downarrow \; (m_{\theta'})_{x_{1,1},x_{2,1}} \times (m_{\theta''})_{x_{1,2},x_{2,2}}$$

$$R_+(GL_a K \times GL_{s-a} K)$$

and

$$R_+(GL_b K) \otimes R_+(GL_{s-b} K)$$

$$\downarrow \; (m_\theta)_{b,s-b}$$

$$R_+(GL_s K)$$

$$\downarrow \; m^*_{\theta'''}(-)_{a,s-a}$$

$$R_+(GL_a K \times GL_{s-a} K).$$

are equal.

Remark 1.2. When we have a family of θ's and θ''''s which depend in a coherent manner with respect to the $x_{i,j}$-matrices then we obtain from Theorem 1.1 a Hopf-like property for the multiplication m.

Example 1.3. Suppose that θ and θ''' are always trivial. Then

$$R_+(GL_bK) \otimes R_+(GL_{s-b}K) \xrightarrow{m} R_+(GL_sK) \xrightarrow{m^*} \oplus_{a=0}^{s} R_+(GL_aK \times GL_{s-a}K)$$

is equal to

$$R_+(GL_bK) \otimes R_+(GL_{s-b}K) \overset{m^*\otimes m^*}{\longrightarrow}$$

$$\oplus_{i=0}^{b} \oplus_{j=0}^{s-b} R_+(GL_iK \times GL_{b-i}K) \otimes R_+(GL_jK \times GL_{s-b-j}K)$$

$$\overset{(1\times T\times 1)m}{\longrightarrow} \oplus_{a=0}^{s} R_+(GL_aK \times GL_{s-a}K).$$

This happens, for instance, when K is a p-adic local field and k is $\overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$.

Question 1.4. Is it possible to construct a family of non-trivial θ's which satisfy the conditions of the theorem for all $x_{i,j}$-matrices?

"Yes" seems plausible.

Let us pause to examine the groups. As remarked above $U_{x_{1,1},x_{2,1},x_{1,2},x_{2,2}}$ consists of matrices of the block-shape

$$\begin{pmatrix} I & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & I & a_{2,3} & a_{2,4} \\ 0 & 0 & I & a_{3,4} \\ 0 & 0 & 0 & I \end{pmatrix}$$

and the multiplication looks like

$$\begin{pmatrix} I & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & I & a_{2,3} & a_{2,4} \\ 0 & 0 & I & a_{3,4} \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & b_{1,2} & b_{1,3} & b_{1,4} \\ 0 & I & b_{2,3} & b_{2,4} \\ 0 & 0 & I & b_{3,4} \\ 0 & 0 & 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & a_{1,2}+b_{1,2} & a_{1,3}+b_{1,3}+a_{1,2}b_{2,3} & a_{1,4}+b_{1,4}+a_{1,2}b_{2,4}+a_{1,3}b_{3,4} \\ 0 & I & a_{2,3}+b_{2,3} & a_{2,4}+b_{2,4}+a_{2,3}b_{3,4} \\ 0 & 0 & I & a_{3,4}+b_{3,4} \\ 0 & 0 & 0 & I \end{pmatrix}$$

Therefore $U_{x_{1,1}+x_{2,1},x_{1,2}+x_{2,2}}$ consists of matrices of the form

$$\begin{pmatrix} I & 0 & a_{1,3} & a_{1,4} \\ 0 & I & a_{2,3} & a_{2,4} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and it is an abelian group which is the direct sum of $(x_{1,1}+x_{2,1})(x_{1,2}+x_{2,2})$copies of K. Also $U_{x_{1,1},x_{2,1}} \times U_{x_{1,2},x_{2,2}}$ consists of matrices of the form

$$\begin{pmatrix} I & a_{1,2} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & a_{3,4} \\ 0 & 0 & 0 & I \end{pmatrix}$$

which is an abelian group which is the direct sum of $x_{1,1}x_{2,1} + x_{1,2}x_{2,2}$copies of K. Composition with $1 \times T \times 1$ looks like the conjugation

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & a_{1,2} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & a_{3,4} \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & a_{1,2} & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & a_{3,4} \\ 0 & 0 & 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 & a_{1,2} & 0 \\ 0 & I & 0 & a_{3,4} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

Therefore any character on $U_{x_{1,1},x_{2,1}} \times U_{x_{1,2},x_{2,2}}$ always extends when composed with T to $U_{x_{1,1}+x_{2,1},x_{1,2}+x_{2,2}}$.

Therefore it looks easy using an additive character $K \longrightarrow k^*$ to about the θ's necessary for Theorem 1.1.

However things will become more complicated when we generalise Theorem 1.1 by replacing general linear groups by products of general linear groups. Although the analogous result to Theorem 1.1 is clearly true.

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